Structure of Certain Nonnormal Operators II

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In this note we continue the study of certain nonnormal operators begun in [1]. As in [1] we use C^* -algebra methods to obtain information about single operators. The results of [1] are extended and in particular we give a decomposition and structure theory for nearly normal, nearly binormal and nearly quasinormal operators. Finally we present a new result about postliminal operators. Throughout H denotes a complex not necessarily separable Hilbert space. Let A be a bounded linear operator on H . We obtain information about A by studying the C^* -algebra $C^*(A)$, which is generated by A and 1. Our major tool in this investigation is the algebra $\mathfrak{K}(H)$ of all compact linear operators on H . If S is a subset of $\mathfrak{B}(H)$, the algebra of all bounded linear operators on H , the commutant of S will be denoted by S' . For $A, B \in \mathfrak{B}(H)$ we use $A \leftrightarrow B$ and $A \rightleftharpoons B$ for $AB - BA = 0$ and $AB - BA \in \mathfrak{K}(H)$. The real (imaginary) part of $A \in \mathfrak{B}(H)$ is denoted by $\operatorname{Re} A$ ($\operatorname{Im} A$). Among the properties of operators are some which are invariant under C^* -homomorphisms. Those we call C^* -properties. Examples are: A is normal, hyponormal or a contraction. If an operator A has a C^* -property modulo $\mathfrak{K}(H)$, we say it has nearly this property. Thus we speak of nearly normal or nearly quasinormal operators. Our basic reference for the theory of C^* -algebras will be the book by J. Dixmier [5].

The following lemma generalizes a result of Kaplansky [5, 4.7.20] and Lemma 4 of [1]. It is the key for the remainder.

Lemma 1. *Let $\mathfrak{A}$ be a C^* -algebra of operators on H with $1 \in \mathfrak{A}$. Then there exists an index set J and a family of orthogonal projections $\{P_0, P_\alpha\}_{\alpha \in J}$ with*

- (1) $P_0 + \sum P_\alpha = 1$, $P_0, P_\alpha \in \mathfrak{A}'' \cap \mathfrak{A}'$ for all $\alpha \in J$,
- (2) $P_0\mathfrak{A} \cap \mathfrak{K}(H) = (0)$ and
- (3) $P_\alpha H = K_\alpha \otimes \mathbb{C}^{n_\alpha}$, $P_\alpha\mathfrak{A} \cap \mathfrak{K}(H) = \mathfrak{K}(K_\alpha) \otimes 1$ and $n_\alpha < \infty$.

The P_α , with $\alpha \neq 0$, are minimal central projections of $\mathfrak{A}''$ and each algebra $P_\alpha\mathfrak{A}''$, with $\alpha \neq 0$, is a factor of type I with finite multiplicity n_α . $\mathfrak{A}''$ is of type I if $P_0\mathfrak{A}$ is of type I. If $\mathfrak{A}$ is separable J is countable and $(1 - P_0)H$ is separable.

Proof. The C^* -algebra $\mathfrak{B} = \mathfrak{A} \cap \mathfrak{K}(H)$ is a two sided closed ideal of $\mathfrak{A}$. Since $\mathfrak{B} \subset \mathfrak{K}(H)$, $\mathfrak{B}$ is a dual C^* -algebra. Thus $\mathfrak{B}$ is the weak C^* -sum of its minimal two sided ideals J_α , with $\alpha \in J = \mathfrak{B}$ [5, 4.7.20]. Each J_α is a simple dual C^* -algebra isomorphic to $\mathfrak{K}(K_\alpha)$, where K_α is a suitable Hilbert space. Let $H_\alpha = \overline{J_\alpha H}$ and let P_α be the corresponding projection. Then H_α is $\mathfrak{B}$ and $\mathfrak{A}$ invariant [5, 1.8.5 and 2.10.3]. Thus P_α and $P_0 = 1 - \sum P_\alpha$ belong to $\mathfrak{A}'$. Since each J_α is isomorphic to $\mathfrak{K}(K_\alpha)$ the identity representation of J_α on H_α has multiplicity n_α [5, 8.6.6]. Each n_α must be finite because $J_\alpha \subset \mathfrak{K}(H)$. Thus we may write $H_\alpha = K_\alpha \otimes \mathbb{C}^n$ and $J_\alpha = \mathfrak{K}(K_\alpha) \otimes 1$, where $\mathbb{C}^n$ denotes an n -dimensional complex Hilbert space. This implies $P_\alpha \in \mathfrak{B}'' \subset \mathfrak{A}''$ and $P_\alpha \mathfrak{B}'' = P_\alpha \mathfrak{A}'' = \mathfrak{B}(K_\alpha) \otimes 1$ for all $\alpha \in J$. Thus for $\alpha \neq 0$ the P_α are minimal central projections of $\mathfrak{A}''$ and $\mathfrak{A}''$ is of type I if $P_0 \mathfrak{A}''$ is of type I.

If $\mathfrak{A}$ is separable, $\mathfrak{B}$ is separable, and thus the decomposition of $\mathfrak{B}$ is countable. In addition each J_α is separable. Thus each H_α is separable also.

What we just proved is that the "compact" part $\mathfrak{B}$ of $\mathfrak{A}$ induces on $\mathfrak{A}$ a discrete decomposition. Thus the lemma is only useful if $\mathfrak{A}$ contains some non-trivial compact operators.

This lemma has the following application to operator theory.

Theorem 1. Let p_i , with $i \in I$ and I some index set, be a family of complex polynomials in two non commuting variables and let $A \in \mathfrak{B}(H)$ be an operator with

$$(4) \quad p_i(A, A^*) = C_i \in \mathfrak{K}(H) \text{ for all } i \in I \text{ and the } C^*\text{-properties } R_1, R_2, \dots$$

Then there exists a countable family of orthogonal projections $P_0, P_1, \dots$ with

$$(5) \quad P_0, P_1, \dots \in C^*(A)' \cap C^*(A)'' \quad \text{and} \quad \sum_{\alpha \geq 0} P_\alpha = 1.$$

$(1 - P_0)H$ is separable. For $\alpha = 0, 1, \dots$ let $A_\alpha = P_\alpha A$. Then

$$(6) \quad A = A_0 \oplus \sum_{\alpha \geq 1} \oplus A_\alpha,$$

$$(7) \quad p_i(A_0, A_0^*) = 0 \quad \text{and} \quad p_i(A_\alpha, A_\alpha^*) = P_\alpha C_i \quad \text{for all } i \in I.$$

Moreover $|P_\alpha C_i| \rightarrow 0$ as α tends to infinity. The A_α , with $\alpha \geq 0$ have the C^* -properties $R_1, R_2, \dots$ and the A_α , with $\alpha > 0$, are primary components of A of finite multiplicity. A is of type I if A_0 is of type I.

Proof. Apply Lemma 1 to the separable C^* -algebra $C^*(A)$. Since $C^*(A) \cap \mathfrak{K}(H)$ is a dual C^* -algebra $|C_i P_\alpha|$ converges to 0 as α tends to infinity.

A special case of Theorem 1 has been shown by F. Gilfeather by using W^* -algebra methods [1]. It is obvious that in the theorems the polynomials p_i can be replaced by more general functions of A and A^* , as long as these functions define elements of $C^*(A)$. Likewise it is easy to give a many operator analog of Theorem 1.

An important subclass of C^* -algebras is the class of postliminal algebras [5, ch. 4] and we call an operator postliminal if $C^*(A)$ is postliminal. This class contains, for example, all compact, n -normal or isometric operators. For our applications the following lemma may be quite useful.

Lemma 2. *Let $A \in \mathfrak{B}(H)$ be an operator with (4). Then A has a decomposition as in Theorem 1 and A is postliminal (nearly n -normal) if every irreducible operator B with*

$$(8) \quad p_i(B, B^*) = 0 \text{ for all } i \in I \text{ and the } C^*\text{-properties } R_1, R_2, \dots$$

is postliminal (n -normal).

Proof. a) By definition A is postliminal if $\mathfrak{A} = C^*(A)$ is postliminal. This is true if $\mathfrak{A} = \mathfrak{A}/\mathfrak{B}$ with $\mathfrak{B} = \mathfrak{A} \cap \mathfrak{K}(H)$ is postliminal [5, 4.3.4]. $\mathfrak{A}$ is postliminal if for every irreducible representation $\tilde{\Pi}$ of $\mathfrak{A}$ the algebra $\tilde{\Pi}(\mathfrak{A})$ is postliminal [5, 4.3.7 and 9.1]. Let $\tilde{A} = A + \mathfrak{B}$ and $B = \tilde{\Pi}(\tilde{A})$. Then B satisfies (8) and thus $C^*(B) = \tilde{\Pi}(\mathfrak{A})$ is postliminal.

b) With the same notation as in a) we have to show that any irreducible representation $\tilde{\Pi}$ of $\mathfrak{A}$ is at most n -dimensional. This is true because, by assumption the operator $\tilde{\Pi}(\tilde{A}) = B$ is n -normal.

In the above lemma n -normal may be replaced by unitary, selfadjoint or quasinormal. This lemma shows also that nearly n -normal operators are postliminal.

These results have an immediate application to operators A with (4), where the polynomial is $p(z, \bar{z}) = z\bar{z} - \bar{z}z$ ($z, z - \bar{z}, \bar{z}z - 1$ or $z^2 - z$). In that case A is postliminal and nearly normal (compact, nearly selfadjoint, nearly isometric or nearly idempotent). In the next theorem we discuss applications of Theorem 1 to nearly normal operators.

Theorem 2. *The operator $A \in \mathfrak{B}(H)$ is nearly normal and has a countable decomposition as in Theorem 1 with A_0 normal if one of the following conditions holds*

- i) $A \rightleftharpoons A^*(AA^*)^\gamma$, with $\gamma \in \mathbb{R}$ and $\gamma \geq 0$;
- ii) $A(A^*A)^\gamma$ is nearly normal, with γ as in (i);
- iii) $A \rightleftharpoons (AA^*)^\beta (A^*A)^\gamma (AA^*)^\beta = D$, with AA^* invertible, $\beta, \gamma \in \mathbb{R}$ and $\gamma > 0$;
- iv) $\operatorname{Re} A \rightleftharpoons AA^*$ and $\operatorname{Im} A \rightleftharpoons A^*A$;
- v) $\operatorname{Re} A \rightleftharpoons AA^*, A^*A$;
- vi) $AA^* \rightleftharpoons A^*A$ and $\operatorname{Re} A$ is definite;
- vii) $\operatorname{Re} A \rightleftharpoons AA^* - A^*A$ or $\operatorname{Im} A \rightleftharpoons AA^* - A^*A$;
- viii) $A \rightleftharpoons AA^* - A^*A$.

Proof. By Theorem 1 and Lemma 2 it suffices to show that the relations i)–viii) with $\rightleftharpoons$ replaced by $\leftrightarrow$ and A irreducible imply the normality of A . Then i) and ii) are obvious.

iii) Since $A \leftrightarrow D$ and $A^* \leftrightarrow D$ and because A is irreducible $D = \lambda 1$ with $\lambda > 0$. Thus $(AA^*)^{-2\beta}\lambda = (A^*A)^\gamma$ and A is invertible. The result follows with the aid of the spectral mapping theorem and the fact that AA^* and A^*A have identical spectra.

iv), v) and vi) This is a consequence of the results of M. Embry [6].

vii) and viii) The Kleinecke-Shirokov theorem [8] shows that $AA^* - A^*A$ is quasinilpotent, hence 0.

The use of the Kleinecke-Shirokov was suggested to the author by F. Gilfeather.

In the next two theorems we study conditions implying binormality.

Theorem 3. *Let p be a nontrivial complex polynomial of degree less than or equal to 2 and let $A \in \mathfrak{B}(H)$ such that $p(A)$ is normal. Then A is binormal.*

Proof. Because of Lemma 2 it suffices to show the theorem under the added assumption of irreducibility. $A \leftrightarrow p(A)$ implies by the Fuglede theorem $A^* \leftrightarrow p(A)$. Thus by the irreducibility of A we have $p(A) = \lambda 1$. But by [3] any operator satisfying a quadratic identity is binormal.

This theorem permits us to write down the most general solution of the equation: $A^2 + \lambda A = N$, with N normal. In this case A is the direct sum of its normal part A_1 and its completely nonnormal part A_2 , $A = A_1 \oplus A_2$. A_2 may be written as

$$A_2 = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

with $b > 0$, $a + c + \lambda = 0$ and where a, b and c commute. Theorem 3 is optimal, because for any polynomial of degree at least three, there exists an operator A of type III with $p(A) = 0$ [2]. Theorems 2 and 3 can be used to derive conditions for nearly binormal operators. In the next theorem we state further conditions, which cannot be obtained in this fashion.

Theorem 4. *An operator $A \in \mathfrak{B}(H)$ is nearly binormal and has a countable decomposition as in Theorem 1 with A_0 binormal, if one of the following conditions holds*

- i) $A \preceq AA^* + A^*A$;
- ii) $A \preceq AA^{*2}$ or $A \preceq A^{*2}A$;
- iii) $A \preceq \operatorname{Re}(A^2 + kA)$, where k is a complex number;
- iv) $A \preceq [A, A^*]^2$, $\operatorname{Re}[A^2, A^*]^2$, $\operatorname{Im}[A^2, A^*]^2$ and $\operatorname{Re}[A^2 + A, A^*]^2$.

Proof. Again we shall replace $\preceq$ by $\leftrightarrow$ and assume A to be irreducible.

i) This condition gives immediately $A^2A^* = A^*A^2$. Hence $A^2 = \lambda 1$ and the result follows from Theorem 3.

ii) We have $A \leftrightarrow A^2 A^{*2}$ and thus also $A^* \leftrightarrow A^2 A^{*2}$. Hence $A^2 A^{*2} = \lambda 1$. If $\lambda = 0$ we get $A^2 = 0$ and A is binormal. Otherwise we may assume $\lambda = 1$. Then $1 = AA^*A = A^2 A^{*2}$ and A is invertible with $A^{-2} = A^{*2}$. Again the result follows from Theorem 3.

iii) Since A is irreducible $\operatorname{Re} A^2 + kA = \lambda 1$. Thus $(A^2 + kA) + (A^2 + kA)^* = 2\lambda$ and $(A^2 + kA)$ is normal. Now apply Theorem 3.

iv) Apart from the trivial case, where A is normal, we have to study the conditions $[A, A^*]^2 = 1$, $[A^2, A^*]^2 = \mu 1$ and $\operatorname{Re} [A^2 + A, A^*]^2 = \nu \cdot 1$. Because of $[A, A^*]^2 = 1$ we may write

$$[A, A^*] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

With this the other conditions give $4a^2 = 4d^2 = \mu 1$ and $\operatorname{Re} (4a + 4a^2) = \operatorname{Re} (4d + 4d^2) = (\nu - 1) \cdot 1$, and this implies that $a = d$ is a scalar. The first condition finally shows $1 = bb^* - c^*c$ and $1 = b^*b - cc^*$. Hence b is invertible and by a unitary transformation U of the form

$$U = \begin{bmatrix} 1 & \\ & u \end{bmatrix}$$

we can achieve $b > 0$. Then the above conditions imply that c is normal and $b \leftrightarrow c$. This and the irreducibility of A show that b and c are complex numbers.

We remark that condition (iv) is also necessary [3]. Since the algebra of all $n \times n$ matrices satisfies certain polynomial identities [5, 3.6], n -normal operators can be characterized by a countable family of polynomials and Theorem 1 can also be applied to nearly n -normal operators.

Operators with $A \leftrightarrow AA^*$ are called quasinormal. Brown [4] has given a complete structure theory for these operators. Yoshino [14] has shown that quasinormal operators are of type I. The next theorem gives a structure theory for nearly quasinormal operators.

Theorem 5. *An operator $A \in \mathfrak{B}(H)$ is nearly quasinormal and has a decomposition as in Theorem 1 with A_0 quasinormal, if one of the following conditions holds*

- i) $A \rightleftharpoons AA^*$;
- ii) $AA^* \rightleftharpoons A^*A$ and $\operatorname{Re} A \rightleftharpoons AA^*$.

Nearly quasinormal operators are postliminal.

Proof. i) As before we may assume A to be irreducible and $A \leftrightarrow AA^*$. Then $AA^* = \lambda 1$, with $\lambda > 0$, and $\lambda^{-1/2}A$ is an irreducible coisometry. Thus $\lambda^{-1/2}A$ is unitary or a simple unilateral shift. In particular A is postliminal.

- ii) This is a consequence of [6].

Since postliminal operators are of type I, Theorem 5 generalizes the result by Yoshino [14].

Schwartz [10] and Suzuki [12] have shown how Theorem 1 can yield information about spectral operators with compact imaginary part.

Theorem 6. *Let A be a spectral operator with compact quasi nilpotent part. Then A is similar to a nearly normal operator $\tilde{A}$, which has a decomposition*

$$(9) \quad \tilde{A} = \tilde{A}_0 \oplus \sum_{i=1}^{\infty} \oplus (\lambda_i + N_i)$$

as in Theorem 1, with $\tilde{A}_0$ normal. The N_i are compact quasi nilpotent operators with finite multiplicity n_i and $|N_i|$ converges to 0 as i tends to infinity.

This theorem is shown by following the proof of Suzuki [12]. When does a spectral operator have a compact quasi nilpotent part? The following lemma is almost trivial.

Lemma 3. *Let ψ denote the canonical C^* -homomorphism $\psi: \mathfrak{B}(H) \rightarrow \mathfrak{B}(H)/\mathfrak{K}(H)$, where we consider $\mathfrak{B}(H)/\mathfrak{K}(H)$ as a C^* -algebra of operators on some Hilbert space K . Then the spectral operator $A = S + N$ has a compact quasi nilpotent part if $\psi(A)$ is a scalar type operator.*

Proof. For an arbitrary spectral operator A the operator $\psi(A) = \psi(S) + \psi(N)$ is again a spectral operator and $\psi(S)$ is an operator of scalar type. Since by assumption $\psi(A)$ is an operator of scalar type, the uniqueness of the decomposition of spectral operators shows $\psi(A) = \psi(S)$. Hence $\psi(N) = 0$.

Corollary. i) *Nearly normal and nearly quasinormal spectral operators have a compact quasi nilpotent part.*

ii) *Spectral operators, which are n^{th} roots of invertible nearly normal operators have a compact quasi nilpotent part.*

Proof. i) Apply the map ψ and observe that a quasi normal operator is a spectral operator only if it is normal.

ii) Apply ψ and use that n^{th} roots of invertible normal operators are scalar type operators [11].

Thus for example an invertible spectral operator A with $A^n \rightleftharpoons A^n A^{n*}$ or $A^n \rightleftharpoons A^n A^{n*} - A^{n*} A^n$ is similar to an operator A with (10).

Above we have defined the class of postliminal operators, and we have seen that many operators in analysis belong to this class. Postliminal operators are of type I, but the converse is not true. If A is postliminal, then any complex polynomial in A and A^* is postliminal [5, 4.3.5]. The restriction of a postliminal operator to an invariant subspace is generally not postliminal, because there are subnormal operators of type III [2]. Clearly sums and products of postliminal operators are in general not of type I even. However we can show:

Theorem 7. *Let A and B be postliminal operators in $\mathfrak{B}(H)$ with $A \preceq B$ and $A \preceq B^*$. Then any operator in $C^*(A, B)$, the C^* -algebra generated by A and B , is postliminal.*

Proof. Since any subalgebra of a postliminal C^* -algebra is postliminal [5, 4.3.5], it suffices to show that $C^*(A, B) = \mathfrak{A}$ is postliminal. This is true if $\tilde{\mathfrak{A}} = \mathfrak{A}/\mathfrak{B}$, with $\mathfrak{B} = \mathfrak{A} \cap \mathfrak{K}(H)$, is postliminal. Let $\tilde{A} = A + \mathfrak{B}$ and $\tilde{B} = B + \mathfrak{B}$. Then $\tilde{A} \leftrightarrow \tilde{B}$ and $\tilde{A} \leftrightarrow \tilde{B}^*$. Hence we may assume without loss of generality $A \leftrightarrow B$ and $A \leftrightarrow B^*$. Then the identity representations Π_1 and Π_2 of $C^*(A)$ and $C^*(B)$ respectively extend to a representation Π of the postliminal C^* -algebra $C^*(A) \otimes C^*(B)$ [13]. Hence $\Pi(C^*(A) \otimes C^*(B)) = \mathfrak{A}$ is postliminal.

By induction, Theorem 7 can be extended to finitely many operators. This theorem extends Theorem 3 of [1] and a result of Okayasu [9]. The following example shows that this theorem can hardly be weakened.

Let $\mathfrak{A}$ be a factor of type II or III on the separable Hilbert space H . Assume $\mathfrak{A}$ is generated by the positive invertible operator a and the projection p [2]. On $H \oplus H$ consider the operators

$$A = \begin{bmatrix} q & 0 \\ 0 & a^{-1}qa \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

with $q = 1 - 2p$. Then $A^2 = 1$, $(B - 1)^2 = 0$ and $AB = BA$. Thus A and B are commuting binormal operators. But AB and $A + B$ are operators of type II or III respectively, because they generate the factor $\mathfrak{A} \otimes M_2$. This is shown as in [2].

Now let a and p be as above and let r be a one dimensional projection in $\mathfrak{B}(H)$. Then the C^* -algebra $\mathfrak{B} = C^*(a, p, r)$ contains nontrivial compact operators and Lemma 1 may be applied to $\mathfrak{B}$. Since r is one dimensional $\mathfrak{B}$ decomposes into at most two summands, $\mathfrak{B} = P_0\mathfrak{B} \oplus P_1\mathfrak{B}$, and $P_1\mathfrak{B}$ has multiplicity $n_1 = 1$. Thus $\mathfrak{K}(H_1) \subset P_1\mathfrak{B}$ and $P_0\mathfrak{B}''$ is a factor of type II or of type III respectively. $P_1C^*(a, p)''$ is also a factor of type II or III respectively. Restricting to $P_1H = H_1$, if necessary, we may thus assume $C^*(a, p)'' = \mathfrak{A}$ and $\mathfrak{K}(H) \subset C^*(a, p, r)$. On $H \oplus H$ consider now the operators

$$C_t = \begin{bmatrix} p + itr & 0 \\ a & 0 \end{bmatrix} \quad \text{with} \quad t \in \mathbf{R}.$$

By construction C_0 is of type II or III respectively, whereas C_t , for $t \neq 0$, is irreducible. Thus the class of operators of type I, II or III is not stable under compact perturbation, and the class of operators of type I is not closed. Since $C_0 \notin C^*(C_t)$, for $t \neq 0$, the C^* -algebra generated by a type I operator may contain operators of type II or III. In this respect postliminal operators are better behaved than operators of type I. It is still unknown whether the set of postliminal operators is dense or closed in $\mathfrak{B}(H)$.

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