Spectral Measures and Flows

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If F_t is a spectral resolution (*i.e.*, resolution of the identity) generating a flow, then it will be shown that F_t can be extended from L^2 to L^1 and that for any functions f, g in L^2 and real numbers s, t we have $(F_t f)(F_s g) = F_{t+s}(fF_s g + gF_t f - fg)$. From this we shall then obtain a generalization of a convolution property due to Koopman [8] and, for ergodic flows on a set of finite measure, a generalization of a result due to Sinai [10]. But first we need a representation for the spectral resolution F_t .

By Stone's theorem (Riesz and Sz-Nagy [9]) a one parameter group $\{U_t : -\infty < t < \infty\}$ of unitary operators on a Hilbert space H with inner product denoted by $(\cdot, \cdot)$ for which $(U_t f, g)$ is continuous in t for all f, g in H has a spectral representation

$$U_t = \int e^{its} F(ds)$$

where F is a spectral measure with support on the real line. Define the corresponding spectral resolution F_t by $F(\{s : -\infty < s < t\})$. It is known (Hopf [6]) that the spectral mass of a point is given by the strong convergence of

$$(1) \quad F(\{t\}) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{-irt} U_r \, dr.$$

Now to determine F_t . Let f, g be elements of H and consider the function of bounded variation $(F_t f, g)$ (Akhiezer and Glazman [1]). By an argument completely analogous to the proof of Fejer's theorem in Katznelson [7], page 18, it can be shown that for all s on the real line

$$0 \leq \frac{1}{2\pi} \int_{-T}^T \left(1 - \frac{|r|}{T}\right) \left(\frac{e^{-irt_1} - e^{-irt_2}}{ir}\right) e^{irs} \, dr \leq 1,$$

for all positive T since $(e^{-irt_1} - e^{-irt_2})/2\pi ir$ is the Fourier transform of the characteristic function of the interval $[t_1, t_2]$. Thus the integral

$$\frac{1}{2\pi} \int_{-T}^T \frac{(e^{-irt_1} - e^{-irt_2})}{ir} e^{irs} \, dr$$

is uniformly bounded in T and s , and, for the special case $s = 0$ and $t_1 = -t_2$, we get the uniform boundedness in T of the integral

$$\int_0^T \frac{(e^{-ir t_1} - e^{-ir t_2})}{ir} dr$$

which has the limit $-\pi$, 0 , and π as $T \rightarrow \infty$ for the case $t_1 > 0$, $t_1 = 0$, and $t_1 < 0$ respectively (Taylor [11]).

Hence for any real t

$$\begin{aligned} 2(F_t f, g) &= \int \left(\int_0^\infty \frac{(e^{+ir(t-w)} - e^{-ir(t-w)})}{- \pi ir} dr \right) (F(dw)f, g) - (f, g) - (F(\{t\})f, g) \\ &= \int_0^\infty \frac{(e^{irt}(U_{-r}f, g) - e^{-irt}(U_r f, g))}{- \pi ir} dr - (f, g) - (F(\{t\})f, g). \end{aligned}$$

From this it follows that in the sense of strong convergence

$$(2) \quad F_t = \lim_{T \rightarrow \infty} \int_0^T \frac{(e^{-irt}U_r - e^{irt}U_{-r})}{2\pi ir} dr - \frac{1}{2}F(\{t\}) - \frac{1}{2}I$$

where I is the identity. The strong convergence follows if we take the inner product of $\int_0^T (e^{-irt}U_r - e^{irt}U_{-r})f/2\pi ir dr$ with itself for an arbitrary f in H , then use Stone's theorem, and by the properties discussed above, Lebesgue's dominated convergence theorem as $T \rightarrow \infty$.

By equation (2) one concludes that for a one parameter group of unitary operators on the Hilbert space L^2 of square integrable functions mapping real functions into real functions one has that for any Borel subset B of the real line

$$(3) \quad F(-B)\bar{f} = \overline{F(B)f}$$

where the bar represents complex conjugation. Such is the case for a one parameter group of unitary operators induced by a flow (*i.e.*, a one parameter group of invertible measure preserving transformations) which possesses the additional multiplicative property that if f, g are two functions in L^2 , then $U_t(fg) = (U_t f) \cdot (U_t g)$ (Halmos [5]). Obviously the domain of such operators can be extended from L^2 to the space of integrable functions L^1 , and thus by equation (2) so can the spectral measure, as will be shown. In fact from now on the spectral measure will be defined by equation (2), and the next theorem will show that on L^1 there is weak convergence and hence the limit must lie in L^1 (Taylor [12]). Equivalently on the essentially bounded functions L^∞ the limit must lie in L^∞ .

Theorem. *If F_t is the spectral resolution for a flow, then for any functions f, g in L^2 and $-\infty < t, s < \infty$*

$$F_t f \cdot F_s g = F_{t+s}(f F_s g) + F_{t+s}(g F_t f) - F_{t+s}(fg).$$

Proof. Let h be an arbitrary essentially bounded function and denote by (k, h) the integral of the function kh where k is an arbitrary function in L^1 .

Hence, using equation (2) for the case of continuous spectrum we get for $t' < t$ and $s' < s$

$$\begin{aligned}
 &(((F_t - F_{t'})f)((F_s - F_{s'})g), h) \\
 &= \lim_{\substack{T \rightarrow \infty \\ S \rightarrow \infty}} \int_{-T}^T \int_{-S}^S \frac{(e^{-ir t'} - e^{-ir t})}{2\pi i r} \frac{(e^{-i w s'} - e^{-i w s})}{2\pi i w} (U_r f \cdot U_w g, h) dr dw \\
 &= \lim_{\substack{T \rightarrow \infty \\ S \rightarrow \infty}} \int_{-T}^T \int_{-S}^S p(r, w) (U_r (f U_{w-r} g), h) dr dw \\
 &\quad + \lim_{\substack{T \rightarrow \infty \\ S \rightarrow \infty}} \int_{-T}^T \int_{-S}^S q(r, w) (U_w (g U_{r-w} f), h) dr dw
 \end{aligned}$$

where

$$p(r, w) = \frac{(e^{-ir(t'+s')} - e^{-ir(t+s')})e^{-i(w-r)s'}}{2\pi i r 2\pi i (w-r)} + \frac{(e^{-ir(t+s)} - e^{-ir(t'+s)})e^{-i(w-r)s}}{2\pi i r 2\pi i (w-r)}$$

and

$$q(r, w) = \frac{e^{-i(r-w)t'}(e^{-i w(s'+t')} - e^{-i w(t'+s)})}{2\pi i w 2\pi i (r-w)} + \frac{e^{-i(r-w)t}(e^{-i w(t+s)} - e^{-i w(t'+s')})}{2\pi i w 2\pi i (r-w)}.$$

By equation (2) the second integral becomes $F_{t+s}(g(F_t f - f'))$ as $s', t' \rightarrow -\infty$ for some f' in L^2 . As for the first integral, the order of integration can be changed for the following reasons. Let C_1 , C_2 , and C_3 be the sets consisting of the strip $\{(r, w) : -b < w - r < b, -\infty < r < \infty\}$ for some constant $b > 0$, the strip $\{(r, w) : -b < r < b, -\infty < w < \infty\}$ less its intersection with the previous one, and the complement of the union of the previous two sets, respectively. In the first case use the fact that the set of functions $(e^{-iwx} U_s f - e^{+iwx} U_{-s} f)/v$ for $-b < v < b$ is uniformly bounded in L^2 norm as can be seen by taking the inner product of the above functions with themselves followed by an application of Stone's theorem and the fact that $(F_t f, f)$ is a function of bounded variation. Using this and Schwarz' inequality we get a uniform bound for

$$\frac{1}{w-r} (e^{-i(r-w)x} (f U_{r-w} g, U_r h) - e^{-i(w-r)x} (f U_{w-r} g, U_r h))$$

and hence considering the function obtained by multiplying this by the characteristic function for the complement of a closed interval with center at the origin on the r -axis, we see that by Plancherel's theorem ([9]) the order of integration can be changed. For the remainder of the r -axis we can use Fubini's theorem. The sets C_2 and C_3 can be treated in a similar manner. Thus the first integral becomes $F_{t+s}(f(F_s g - g'))$ as $s', t' \rightarrow -\infty$, g' some function in L^2 . Hence

$$F_t f \cdot F_s g = F_{t+s}(f(F_s g - g')) + F_{t+s}(g(F_t f - f')).$$

If $s, t \rightarrow \infty$, then we get $f' + g' = fg$ from which follows the theorem. For discontinuous spectrum apply right continuity of the spectral resolution. Q.E.D.

Corollary. *If F_t is the spectral resolution for a flow, then for any f, g in L^2 such that $F_t f = f$ and $F_s g = g$ for some fixed real numbers s and t there is the relationship $F_{s+t}(fg) = fg$.*

Corollary. *If F_t is the spectral resolution for a flow, then for any essentially bounded function h and f, g in L^2 , $(F_t f \cdot F_s g, h)$ is a function of bounded variation.*

Proof. Let K be the essential supremum of h . It is sufficient to show that for any two partitions of the real line determined by the points $\{t_i : i = 1, 2, 3, \dots\}$ and $\{s_k : k = 1, 2, 3, \dots\}$ and any two real numbers u, v one has for $f' = F_u f$ and $g' = F_v g$

$$\sum_{i,k} |((F_{t_{i+1}} - F_{t_i})f' \cdot (F_{s_{k+1}} - F_{s_k})g', h)| \leq K(f, f')^{1/2}(g, g')^{1/2}.$$

By the previous corollary $F_{t_i} f' \cdot F_{s_k} g' = F_{u+v}(F_{t_i} f' \cdot F_{s_k} g')$. Writing w for $u + v$ we see by the above inequality that it is sufficient to show that for any real number w we get

$$\sum_{i,k} |((F_{t_{i+1}} - F_{t_i})f' \cdot (F_{s_{k+1}} - F_{s_k})g', F_w h)| \leq K(f', f')^{1/2}(g', g')^{1/2}.$$

The left hand side can be written as (use equation (3))

$$\sum_{i,k} |(f', (F_{t_{i+1}} - F_{t_i})((F_{-s_{k+1}} - F_{-s_k})\overline{g'} \cdot F_w h))|.$$

Applying the theorem to $(F_{-s_{k+1}} - F_{-s_k})\overline{g'} \cdot F_w h$ and then operating by $F_{t_{i+1}} - F_{t_i}$ on the result, we see that the left hand side of the above inequality can be written as

$$\sum_k |((F_{t'_{k+1}} - F_{t'_k})f' \cdot (F_{s_{k+1}} - F_{s_k})g', h)|$$

where $\{t'_k : k = 1, 2, 3, \dots\}$ are points partitioning the real line, $t'_k < t'_{k+1}$. By Schwarz' inequality this sum is bounded by

$$K(\sum_k ((F_{t'_{k+1}} - F_{t'_k})f', f') \sum_k ((F_{s_{k+1}} - F_{s_k})g', g'))^{1/2}.$$

which is exactly $K(f', f')^{1/2}(g', g')^{1/2}$. Q.E.D.

By a slight variation of the proof of the previous corollary it can be shown that for any sequence $\{h_{i,k} : i, k = 1, 2, 3, \dots\}$ of functions essentially bounded by 1, we have for any f, g in L^2

$$\sum_{i,k} |((F_{t_{i+1}} - F_{t_i})f \cdot (F_{s_{k+1}} - F_{s_k})g, h_{i,k})| \leq (f, f)^{1/2}(g, g)^{1/2}$$

from which follows that for any two sequences $\{B_i' : i = 1, 2, 3, \dots\}$ and $\{B_k'' : k = 1, 2, 3, \dots\}$ of disjoint Borel subsets of the real line the sum $\sum_{i,k} F(B_i')f \cdot F(B_k'')g$ converges strongly in L^1 (to a function in L^1). Such sums

determine a L^1 valued vector measure $Q(\cdot)$ on the Borel subsets of the plane (Dinculeanu [3]) which of course depends on f and g . To emphasize that Q is determined by the vector valued set function $F(\cdot)f \cdot F(\cdot)g$ we shall often write

$$\iint_{\{(r,w) \in B\}} F(dr)f \cdot F(dw)g$$

for the Q measure of a Borel subset B of the plane.

Given a bounded Borel measurable function $a(r, w)$ on the plane, define its Q integral in the usual way ([3]). If $a(r, w) = p(r)q(w)$ for bounded functions p, q on the real line, then

$$\iint p(r)q(w)F(dr)f \cdot F(dw)g = \left(\int p(r)F(dr)f \right) \left(\int q(w)F(dw)g \right).$$

Also if C is the characteristic function of the strip $\{(r, w) : r + w \in B\}$ for some Borel subset B of the real line, then since Q is continuous from above at the nullset

$$\iint CF(dr)f \cdot F(dw)g = \int F(B - s)f \cdot F(ds)g$$

where the integral on the right is taken in the usual Riemann sense. Finally by the preceding paragraph if $\{B_j : j = 1, 2, 3, \dots\}$ is a sequence of disjoint Borel subsets of the plane, then for any f, g in L^2 we have $\sum_j \|Q(B_j)\|_1 \leq (f, f)^{1/2}(g, g)^{1/2}$ where $\|\cdot\|_1$ denotes L^1 norm.

Lemma. *The operators (1) and (2) corresponding to a flow converge strongly on L^1 .*

Proof. Since an L^1 function is the product fg of L^2 functions we can use the multiplicative property and then Stone's theorem to get

$$\begin{aligned} & \left\| \frac{1}{T_2} \int_{-T_2}^{T_2} e^{-is} U_s(fg) ds - \frac{1}{T_1} \int_{-T_1}^{T_1} e^{-is} U_s(fg) ds \right\|_1 \\ & \leq \iint \left| \left(\frac{1}{T_2} \int_{-T_2}^{T_2} - \frac{1}{T_1} \int_{-T_1}^{T_1} \right) (e^{-is(t-r-w)}) \right| \|Q(dr dw)\|_1 \end{aligned}$$

which by Lebesgue's dominated convergence theorem tends to zero. The operator (2) is treated the same way using the fact that $\int (e^{+ir(t-s)} - e^{-ir(t-s)})/ir dr$ is uniformly bounded, a fact that we already made use of at the beginning. Q.E.D.

We now use the lemma to deduce a result first stated by Koopman [8] and proved by Foias [4] but generalized here to the case where the product fg need not lie in L^2 .

Theorem. *The spectral measure F corresponds to a flow if and only if for any Borel subset B of the real line and functions f, g in L^2*

$$(4) \quad F(B)(fg) = \int F(B - s)f \cdot F(ds)g.$$

Proof. The proof involves simple ideas but very tedious calculations which can be found in great detail in [2]. First consider the case when B is a point. With equation (1) we make use of the multiplicative property, Stone's theorem, and then Lebesgue's dominated convergence theorem. Next, when B is an interval $[t_1 ; t_2]$ use equation (2) and repeat the steps above. The result then follows for an arbitrary Borel set B .

Conversely, assuming the result it is sufficient to show multiplicativity for the spectral measure to correspond to a flow (Halmos [5]), and this is done as follows (now it is enough to suppose fg lies in L^2). If $\{U_t : -\infty < t < \infty\}$ is the one parameter group of unitary operators generated by the spectral measure F , then by Stone's theorem and equation (4)

$$\begin{aligned} U_t(fg) &= \int e^{+itr} F(dr)(fg) \\ &= \int e^{+itr} \int F(dr - s)f \cdot F(ds)g \\ &= \iint e^{+i(r-s)t} e^{+ist} F(dr - s)f \cdot F(ds)g \\ &= \int (U_t f) e^{+ist} F(ds)g \\ &= (U_t f)(U_t g). \end{aligned}$$

Q.E.D.

Since the operator (2) converges weakly on L^1 and hence on L^∞ it follows that for any Borel subsets B' and B'' of the real line $(F(B')F(B'')) = F(B' \cap B'')$ on L^1 . To see this note that for any f in L^1 and essentially bounded function h , $(F(B')F(B'')f, h) = (f, F(B'')F(B')h) = (f, F(B' \cap B'')h) = (F(B' \cap B'')f, h)$. Thus for any Borel subset B of the real line $F(B)$ is a projection on L^1 . Furthermore it is bounded by 1 in norm as is now shown. Every L^1 function can be written as the product fg of two L^2 functions. Hence applying the theorem followed by an argument similar to that in the proof of the previous corollary we get $\|F(B)(fg)\|_1 \leq (f, f)^{1/2}(g, g)^{1/2}$. For a function k in L^1 with some square root $k^{1/2}$ let $f = g = k^{1/2}$. Then by the above inequality we get $\|F(B)k\|_1 \leq (k^{1/2}, k^{1/2}) = \|k\|_1$. From this we conclude that a non trivial (*i.e.*, non zero) projection $F(B)$ has norm 1 on L^1 .

By definition a flow is ergodic if the only sets it leaves invariant are ones of measure zero and their complements. It follows from this that for any function f in L^1 , $F(\{0\})f$ is constant almost everywhere if F is a spectral measure corresponding to an ergodic flow (Hopf [6] and equation (1)). Let $\int \overline{F(dr)}f \cdot F(dr)g$ denote the limit in L^1 of the sums $\sum_i \overline{F(B_i)}f \cdot F(B_i)g$ as the mesh of the partition $\{B_i : j = 1, 2, 3, \dots\}$ of the real line into half open intervals tends to zero. If

$f = g$, then write the limit as $\int |F(dr)f|^2$. Obviously we can extend these integrals to ones over Borel subsets of the real line.

Theorem. *If a flow on a space of finite measure X with spectral measure F is ergodic, then for any Borel subset B of the real line and functions f, g in $L^2(X)$ one has for almost all x in X*

$$\left(\int_B \overline{F(dr)f} \cdot F(dr)g \right)(x) = (F(B)f, g).$$

In particular, for almost all x

$$\left(\int_B |F(dr)f|^2 \right)(x) = (F(B)f, f).$$

Proof.

$$\begin{aligned} F(\{0\})(\bar{f} \cdot F(B)g) &= \iint_{\{r+w=0\}} F(dr)\bar{f} \cdot F(dw)(F(B)g) \\ &= \int \overline{F(dr)f} \cdot F(dr)(F(B)g). \end{aligned}$$

The first equality is a result of the previous theorem, and the second follows from the discussion preceding the previous theorem and equation (3). If B is an interval, then it follows from the definition of the last integral and the fact that $F(B')F(B)g = 0$ if $B' \cap B$ is empty for any Borel subset B' , that

$$\int_B \overline{F(dr)f} \cdot F(dr)(F(B)g) = \int_B \overline{F(dr)f} \cdot F(dr)g.$$

The standard outer measure argument shows that this holds when B is an arbitrary Borel set. Also, by ergodicity $F(\{0\})(\bar{f} \cdot F(B)g)$ is a constant almost everywhere. Hence

$$\begin{aligned} \int_B \overline{F(dr)f} \cdot F(dr)g &= F(\{0\})(\bar{f} \cdot F(B)g) \\ &= \frac{1}{|X|} \int F(\{0\})(\bar{f} \cdot F(B)g) \, dx \\ &= \frac{1}{|X|} \int \left(\int_B \overline{F(dr)f} \cdot F(dr)g \right) dx \end{aligned}$$

where $|X|$ denotes the measure of X . Let B be an interval. Then if $|B_i|$ represents the mesh of the partition $\{B_i\}$ of B , we get from above

$$\begin{aligned} \int_B \overline{F(dr)f} \cdot F(dr)g &= \frac{1}{|X|} \int \lim_{|B_i| \rightarrow 0} \left(\sum_i \overline{F(B_i)f} \cdot F(B_i)g \right) dx \\ &= \lim_{|B_i| \rightarrow 0} \frac{1}{|X|} \int \left(\sum_i \overline{F(B_i)f} \cdot F(B_i)g \right) dx \end{aligned}$$

$$\begin{aligned}
&= \lim_{|B_i| \rightarrow 0} \sum_i (F(B_i)g, F(B_i)f) \\
&= \lim_{|B_i| \rightarrow 0} \sum_i (F(B_i)g, f) \\
&= (F(B)f, g).
\end{aligned}$$

If B is not an interval but an arbitrary Borel subset of the unit circle, then applying the usual measure theoretic argument we get

$$\int_B \overline{F(dr)} f \cdot F(dr) g = (F(B)g, f). \quad \text{Q.E.D.}$$

The theorem above is a generalization of a result due to Sinai [10], since it holds for all functions f, g in L^2 and all Borel subsets B of the real line.

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