A Sufficient Condition for Further Reduction of Unknowns in the Einstein-Maxwell Field Equations in the Case of the Characteristic [(11)11].

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In a paper by Hlavatý (see [2]) it was shown that for the characteristic [(11)11] the number N of unknowns on which $g_{\lambda\mu}$ depends can be reduced to 5 while the number C of Cauchy data can be reduced to 8.

The purpose of this paper is to establish a sufficient condition for further reduction such that $N = 4$ and $C = 6$. The results obtained by Hlavatý in [1] and [2] will be used freely throughout this paper. We shall also adopt the same notation unless otherwise specified.

For the benefit of the reader we repeat some agreements regarding the symbolism used in the paper. The Greek indices ξ, η, ζ, τ have always the range I, II, III while any other Greek index has the range I, II, III, IV. Latin non-primed indices have the range 1, 2, 3, 4 unless stated explicitly otherwise. Primed Latin indices have the range 1, 2 and double primed Latin indices have the range 3, 4. With the exception of x and y , all indices follow the usual summation convention. Any exception to the rules mentioned above will be stated explicitly.

We consider the fields equations

$$(1) \quad \text{a) } R_{\lambda}^{\nu} - \frac{1}{2}R\delta_{\lambda}^{\nu} = P_{\lambda}^{\nu} \quad \text{b) } P_{\lambda}^{\nu} = \mu u_{\lambda}u^{\nu} + F_{\lambda}^{\nu} + p \delta_{\lambda}^{\nu}$$

where μ is the rest mass, u^{ν} the time-like unit velocity vector, F_{λ}^{ν} the momentum energy tensor of $k_{\lambda\mu}$ and p the pressure. The tensor P_{λ}^{ν} is the momentum energy tensor of matter and electromagnetism.

For the characteristic [(11)11] the tensor P_{λ}^{ν} may be expressed in terms of eigenvectors $P_{\lambda}^{\nu} = \pi_{\lambda}^{\alpha} I_{\alpha}^{\nu}$ where $\pi = \begin{smallmatrix} \alpha \\ 1 \\ 2 \end{smallmatrix}$. The following lemma concerning the structure of the metric tensor $g_{\lambda\mu}$ was proved in [2].

Lemma 1. The eigenvectors e^{ν} , e^{ν} are mutually perpendicular with respect to the unknown metric tensor $g_{\lambda\mu}$. Moreover, the bivectors $e^{(\lambda}e^{\nu)}$, $e^{(\lambda}e^{\nu)}$ are totally perpendicular with respect to $g_{\lambda\mu}$.

Attempting to reduce further the number N mentioned above, we raise the following question.

Question 1. What are necessary and sufficient conditions that e^{λ} and e^{λ} may be thought of as being conjugate with respect to the unknown metric tensor $g_{\lambda\mu}$ which is a solution of (1)?

In order to establish a necessary condition we make the following assumption.

Assumption 1. The eigenvectors e^{λ} and e^{λ} are conjugate with respect to the metric tensor $g_{\lambda\mu}$ which is a solution of (1).

Lemma 2. If e^{λ} and e^{λ} satisfy Assumption 1, then e^{λ} is conjugate to e^{λ} with respect to the unknown metric tensor g^{λ} .

Proof. It was shown by Hlavatý in [2] that the metric tensor must be of the form

$$(2) \quad a) \quad g_{\lambda\mu} = g_{a'b'}e_{\lambda}e_{\mu} + \lambda_{a''}E_{\lambda\mu} \quad b) \quad g^{\lambda\nu} = g^{a'b'}e^{\lambda}e^{\nu} + \lambda^{a''}E^{\lambda\nu}$$

where $g_{a'b'}$, $g^{a'b'}$, $\lambda_{a''}$, $\lambda^{a''}$ are scalars satisfying $g_{a'b'}g^{a'c'} = \delta_{b'}^{c'}$ and $\lambda_{a''}\lambda^{x''} = 1$ (no summation on x'').

We know that $g_{\lambda\mu}e^{\lambda}e^{\mu} = g_{12}$ and $g^{\lambda\nu}e_{\lambda}e_{\nu} = g^{12}$. Since Assumption 1 implies $g_{12} = 0$, it follows immediately that $g^{12} = 0$.

We may now simplify our notation by introducing $\lambda_1 = g_{11}$, $\lambda_2 = g_{22}$, $\lambda = g^{11}$, $\lambda = g^{22}$. Using this notation we state the following

Theorem 1. Under Assumption 1, the metric tensor must be of the form

$$(3) \quad a) \quad g_{\lambda\mu} = \lambda_a E_{\lambda\mu} \quad b) \quad g^{\lambda\nu} = \lambda_a E^{\lambda\nu} \quad (a = 1, 2, 3, 4)$$

where λ_a , λ_a are scalars satisfying $\lambda_a \lambda^a = 1$ (no summation on x). (See Theorem (16.2) in [2].)

The proof follows immediately from (2) and Lemma 2.

Theorem 2. Under Assumption 1, the contracted curvature tensor $R_{\lambda\mu}$ may be expressed as follows

$$(4) \quad R_{\lambda\mu} = \frac{1}{2}\gamma_{\xi\eta}[-g^{IV\ IV}\delta_{\lambda}^{\xi}\delta_{\mu}^{\eta} + 2g^{IV\ \eta}\delta_{(\lambda}^{\xi}\delta_{\mu)}^{IV} - g^{\xi\eta}\delta_{\lambda}^{IV}\delta_{\mu}^{IV}] + \alpha_{\lambda\mu},$$

where

$$(5) \quad \gamma_{\xi\eta} = \gamma_{\xi\eta} = \partial_{IV\ IV}\lambda_{\xi\eta}^a,$$

and $\alpha_{\lambda\mu}$ is a well defined function of the

(a) λ 's and their first two derivatives (with the exception of $\partial_{IV\ IV}\lambda$) and

(b) the vectors e^{ν} , e_{λ}^a and their first two derivatives. (See Theorem (8.1) in [1] and Theorem (18.1) in [2].)

Proof. It is known (see Lichnerowicz [3]) that $R_{\lambda\mu}$ may be expressed by

$$(6) \quad R_{\lambda\mu} = g^{\alpha\beta}[\partial_{\mu|\alpha}g_{\lambda|\beta} + \partial_{\beta|\lambda}g_{\alpha|\mu}] + \Phi_{\lambda\mu},$$

where $\Phi_{\lambda\mu}$ is a well defined function of $g_{\lambda\mu}$ and its first derivatives. Substituting (3)a into (6) and computing coefficients of $g^{IV\ IV}$, $g^{IV\ \eta}$ and $g^{\xi\eta}$ yields (4) and (5).

Remark. This theorem reduces the number 10 of unknown $g_{\lambda\mu}$ to the 4 unknowns λ ($a = 1, 2, 3, 4$).

Hlavatý has shown (see [1]) that the most general Einstein-Maxwell field equations based on (1) are

$$(7) \quad C_{\lambda}^{\nu} = R_{\lambda}^{\nu} - Q_{\lambda}^{\nu},$$

where $Q_{\lambda}^{\nu} = P_{\lambda}^{\nu} - \frac{1}{2}(C_{\alpha}^{\alpha} + P_{\alpha}^{\alpha})\delta_{\lambda}^{\nu}$ and C_{λ}^{ν} is an arbitrary divergence free tensor, $\nabla_{\lambda}C_{\mu}^{\lambda} = 0$. Taking into account Theorem (18.3) from Hlavatý [2] we get the following

Theorem 3. The equations (7) may be split into two sets: six equations

$$(8) \quad (\partial_{IV\ IV}\lambda)E_{\xi\eta}^a = 2h(\alpha_{\xi\eta} - Q_{\xi\eta})$$

and four equations

$$C_{\xi}^{IV} = \alpha_{\xi}^{IV} - Q_{\xi}^{IV},$$

$$C_{IV}^{IV} = hg^{\xi\ IV}(\alpha_{\xi}^{IV} - Q_{\xi}^{IV}) - g^{\xi\eta}(\alpha_{\xi\eta} - Q_{\xi\eta}) + g^{IV\ IV}(\alpha_{IV\ IV} - Q_{IV\ IV})$$

(see also Theorem (8.3) in [1]).

In the next theorem we shall use the abbreviation

$$\psi_b^a \stackrel{\text{def}}{=} E_{\xi\eta}^a E_b^{\xi\eta}(a, b = 1, 2, 3, 4).$$

We shall also assume the following

Agreement 1. Throughout this paper we shall assume that the components $e_a^{\text{IV}}, e_{\text{IV}}^a$ are not restricted by any further requirements than $e_1^{\text{IV}} e_2^{\text{IV}} e_3^{\text{IV}} e_4^{\text{IV}} \neq 0$.

Theorem 4. The rank of the matrix $\left(\left(\begin{smallmatrix} a \\ \psi \\ b \end{smallmatrix}\right)\right)$ is equal to 4. (see Theorem (18.4) in [2]).

Proof. It follows easily from the structure of $\left(\left(\begin{smallmatrix} a \\ \psi \\ b \end{smallmatrix}\right)\right)$ that we can write $\text{Det}\left(\left(\begin{smallmatrix} a \\ \psi \\ b \end{smallmatrix}\right)\right) = 1 + \Phi(e_a^{\text{IV}}, e_{\text{IV}}^b)$ where Φ depends only on e_a^{IV} and e_{IV}^b . By Agreement 1 it follows that $\text{Det}\left(\left(\begin{smallmatrix} a \\ \psi \\ b \end{smallmatrix}\right)\right) \neq 0$.

Let us denote by $\left(\left(\begin{smallmatrix} b \\ \theta \\ c \end{smallmatrix}\right)\right)$ the inverse matrix of $\left(\left(\begin{smallmatrix} a \\ \psi \\ b \end{smallmatrix}\right)\right)$ such that $\psi_{\frac{a}{b}}^{\frac{b}{c}} \theta = \delta_c^a$. It follows immediately from Theorem 4 that $E_{\xi\eta}^a$ and $E_b^{\xi\eta}$ are matrices of rank 4.

Theorem 5. The equations (8) are equivalent to

$$\begin{aligned} (9) \quad (a) \quad \partial_{\text{IV IV}} \lambda_a^b &= 2h(\alpha_{\xi\eta} - Q_{\xi\eta}) E_b^{\xi\eta} \theta_a^b \\ (b) \quad (\alpha_{\xi\eta} - Q_{\xi\eta}) Y_x^{\xi\eta} &= 0 \quad (x = 5, 6) \end{aligned}$$

where $Y_x^{\xi\eta}$ are two linearly independent solutions of

$$(10) \quad E_{\xi\eta}^a Y^{\xi\eta} = 0 \quad (\text{see Theorem (8.5)a in [1]}).$$

Proof. Multiplying (8) by $E_b^{\xi\eta} \theta_c^b$ we get (9)a. The existence of two linearly independent solutions $Y_x^{\xi\eta}$ ($x = 5, 6$) of (10) follows from the fact that the matrix $E_{\xi\eta}^a$ is of rank 4. Multiplying (8) by $Y_x^{\xi\eta}$ and making use of (10) we are led to (9)b.

Next, we show that (9)a,b imply (8). Multiplying (9)a by $E_{\xi\tau}^a E_c^{\xi\tau}$ and rearranging we get

$$(11) \quad [\partial_{\text{IV IV}} \lambda_a^c E_{\xi\tau}^a - 2h(\alpha_{\xi\tau} - Q_{\xi\tau})] E_c^{\xi\tau} = 0 \quad (c = 1, 2, 3, 4).$$

From (9)b and (10) we get

$$(12) \quad [\partial_{IV} \lambda \overset{a}{E}_{\tau} - 2h(\alpha_{\tau} - Q_{\tau})] Y^{\tau} = 0 \quad (x = 5, 6).$$

Equations (11) and (12) may be viewed as a system of six equations with a matrix of coefficients consisting of $\overset{a}{E}_{\tau}$ and Y^{τ} . Since the matrix of coefficients has rank 6 it follows that (11) and (12) possesses the unique solution $\partial_{IV} \lambda \overset{a}{E}_{\tau} - 2h(\alpha_{\tau} - Q_{\tau}) = 0$.

Hlavatý has shown (see [1]) that the assumption $C^{\text{IV}}_{\tau} \doteq 0$ leads to the conclusion that C^{τ}_{λ} is a zero tensor. It follows that under Assumption 1 equation (7) reduces to (9)a, b and

$$(13) \quad \begin{aligned} \text{a)} \quad & \alpha^{\text{IV}}_{\xi} - Q^{\text{IV}}_{\xi} \doteq 0 \\ \text{b)} \quad & -g^{\xi\eta}(\alpha_{\xi\eta} - Q_{\xi\eta}) + g^{\text{IV IV}}(\alpha_{\text{IV IV}} - Q_{\text{IV IV}}) \doteq 0. \end{aligned}$$

Agreement 2. We shall assume that the field equations (1) possess at least one solution $g_{\lambda\mu}$.

It follows that the set of solutions of (9)a, b and (13)a, b is not empty.

It also follows that the set of solutions of (9)a with the boundary conditions (13)a, b contains always a solution which satisfies

$$(14) \quad \begin{aligned} \text{a)} \quad & \Lambda_{\mu} \omega^{\mu}_{x'} \doteq 0 \\ \text{b)} \quad & \partial_{IV} L_{y''} \doteq -\Lambda_{\mu} \epsilon^{\mu}_{x''} / \pi_{x''}, \quad y'' \neq x''. \end{aligned}$$

(For derivation of (14)a, b see Theorem (17.1)c in Hlavatý [2].) We shall next find the unknown functions λ satisfying equations (9)a and the boundary conditions (13)a, b.

Definition. The set of 6 functions

$$(15) \quad \lambda_a, \quad \partial_{IV} \lambda_1, \quad \partial_{IV} \lambda_2$$

will be termed an appropriate set if the functions (15) satisfy the following conditions:

- (1) They are real and analytic.
- (2) They are given along the hypersurface H ($x^{\text{IV}} = 0$).
- (3) They satisfy the boundary conditions (13)a, b.
- (4) $\lambda_1 \lambda_2 > 0$, $\lambda_3 > 0$ and $\lambda_4 < 0$ along H .

Theorem 6. The appropriate data (15) are Cauchy data for the differential equations (9)a and the boundary conditions (13)a, b, (14)a, b (see Theorem (9.2)a in [1]).

Proof. From the given λ along H we construct $\partial_{\xi} \lambda_a$. Making use of (14)b

we find $\partial_{\text{IV}}\lambda$ and $\partial_{\text{IV}}\lambda$. We know at this stage all first derivatives of λ along H . Next we construct $\partial_{\alpha\beta}\lambda$ along H . We determine $\partial_{\text{IV}}\partial_{\text{IV}}\lambda$ by making use of equation (9)a. We now have all second derivatives of λ along H . Next we construct $\partial_{\alpha\beta\gamma}\lambda$ along H . In order to determine $\partial_{\text{IV}}\partial_{\text{IV}}\partial_{\text{IV}}\lambda$ we differentiate (9)a with respect to x^{IV} and make use of the resulting equations. Proceeding in this fashion we determine all derivatives of λ along H which are necessary for the construction of a solution.

We now state two theorems the proof of which follows directly from the fact that equations (9)a, b, (13)a, b and (14)a, b are equivalent to (1).

Theorem 7. *The appropriate data (15) are Cauchy data for (1) if and only if the solution S satisfying (9)a, (13)a, b and (14)a, b also satisfies (9)b (see Theorem (9.2)b in [1]).*

Theorem 8. *A necessary and sufficient condition for (1) to admit a solution (3) is that the Cauchy data mentioned in Theorem 6 be also Cauchy data for (1).*

The answer to Question 1 is provided by the following theorem.

Theorem 9. *A necessary and sufficient condition that Assumption 1 holds is that the Cauchy data mentioned in Theorem 6 be also Cauchy data for (1).*

Proof. If Assumption 1 holds, then the metric tensor $g_{\lambda\mu}$ has the form (3). By Theorem 8, it follows that if (1) admits (3) as a solution, then the Cauchy data mentioned in Theorem 6 are also Cauchy data for (1). On the other hand, if we assume that the Cauchy data mentioned in Theorem 6 are also Cauchy data for (1), then by Theorem 8 it follows that (1) admits a solution (3). An easy calculation shows that $g_{\lambda\mu}e^{\lambda}_{12}e^{\mu} = 0$ when $g_{\lambda\mu}$ is given by (3). This shows that Assumption 1 holds.

Theorem 10. *A sufficient condition that for the Characteristic [(11)11] the number N and C can be reduced to $N = 4$ and $C = 6$ is that the Cauchy data mentioned in Theorem 6 be also Cauchy data for (1).*

The proof follows directly from Theorem 9.

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