Penetrations into Shadow Regions and Unique Continuation Properties in Hyperbolic Mixed Problems

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0. This paper studies the solutions of hyperbolic mixed initial boundary value problems with time independent coefficients. The theory of semigroups of operators is used throughout. In Section 1 mixed problems for second order wave equations are considered from the semigroup point of view and the appropriate extension of the groups to spaces of distributions are obtained. Section 2 contains the main results of the paper. Viewing the solution as a signal or disturbance propagating in time, we prove

1. Every open set exterior to the initial disturbance is reached by the signal at some time $t \geq 0$ (Theorem 2).
2. Every open set is reached by the signal at some time t , $-\infty < t < \infty$ (Lemma 2).
3. For a suitable generalization of the Klein Gordon equation every open set is reached by the signal at some time $t \geq 0$ (Theorem 5).

The first of these results asserts that there are no perfect shadows and the second is a sort of unique continuation theorem. In Section 3 extensions to systems and to Maxwell's equations are discussed.

1. Let Ω be a connected region in $\mathbf{R}^n$ with smooth, compact boundary. Let $A = \sum_{|\alpha| \leq 2} a_\alpha(x) D^\alpha$ be a second order elliptic operator. Assume the coefficients a_α are smooth, real valued, and bounded on $\bar{\Omega}$, and assume there is a constant $C > 0$ such that $\sum_{|\alpha| = 2} a_\alpha(x) \xi^\alpha \geq C |\xi|^2$ for all $x \in \bar{\Omega}$, $\xi \in \mathbf{R}^n$. Here we take

$$D = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right),$$

the real differential operator.

We suppose the domain $\mathfrak{D}(A)$ is given by regular elliptic boundary conditions and there exists a strictly negative self adjoint operator A_0 with the same principal part as A , and $\mathfrak{D}(A) = \mathfrak{D}(A_0)$. For example, $\mathfrak{D}(A)$ could be given by Dirichlet or Neumann boundary conditions.

In such a case, the operator

$$T = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

considered as an unbounded operator on $\mathfrak{D}(A^{1/2}) \oplus L^2(\Omega) = \mathfrak{H}$, with domain $\mathfrak{D}(T) = \mathfrak{D}(A) \oplus \mathfrak{D}(A^{1/2})$ generates a strongly continuous one parameter group P_t . Indeed, if $\mathfrak{H}$ is given the "energy norm" $\| (u_0, u_1) \|_{\mathfrak{H}}^2 = \| (-A_0)^{1/2} u_0 \|^2 + \| u_1 \|^2$, where $\| \cdot \|$ is the $L^2(\Omega)$ norm, then the operator

$$T_0 = \begin{pmatrix} 0 & I \\ A_0 & 0 \end{pmatrix}, \quad \mathfrak{D}(T_0) = \mathfrak{D}(A_0) \oplus \mathfrak{D}(A_0^{1/2})$$

is skew adjoint, and T is a bounded perturbation of T_0 .

We might remind the reader that $\mathfrak{D}(A^{1/2}) = \mathfrak{D}(A_0^{1/2}) = [\mathfrak{D}(A), L^2]_{1/2}$, a space obtained by quadratic interpolation (see [5], [7]).

The group P_t has the important property that for $u = (\varphi_0, \varphi_1) \in \mathfrak{D}(T)$, $u(t) = P_t u = (u_0(t), u_1(t))$ satisfies the hyperbolic equation

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u_0 - A u_0 &= 0, \\ (1) \quad u_0(0) &= \varphi_0, \end{aligned}$$

$$\frac{\partial u_0}{\partial t}(0) = u_1(0) = \varphi_1,$$

$$u_0(t) \in \mathfrak{D}(A).$$

The boundary conditions satisfied by u_0 are hidden in the last line of 1. If $u \in \mathfrak{H}$ but $u \notin \mathfrak{D}(T)$, we can approximate u by $u_n \in \mathfrak{D}(T)$ and conclude that u is still a strong solution to the above mixed initial-boundary value problem.

We can also solve (1) with initial data more general distributions by using the following device. For k a positive integer, $\mathfrak{D}(T^k)$ and $\mathfrak{D}(T^{*k})$ are Banach spaces with the graph topologies. Let $\mathfrak{H}_k = \mathfrak{D}(T^k)$ and let $\mathfrak{H}_{-k}$ be the dual of $\mathfrak{D}(T^{*k})$. Then $\mathfrak{H}_k$ is invariant under P_t , and P_t is a group on each $\mathfrak{H}_k$. Similarly P_t^* is a strongly continuous group on $\mathfrak{D}(T^{*k})$. Now we can extend P_t to $\mathfrak{H}_{-k}$ by means of the formula

$$\langle u, P_t \varphi \rangle = \langle P_t^* u, \varphi \rangle, \quad u \in \mathfrak{D}(T^{*k}), \quad \varphi \in \mathfrak{H}_{-k},$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing between $\mathfrak{D}(T^{*k})$ and $\mathfrak{H}_{-k}$. Clearly P_t is weakly continuous, and hence, by classical results, P_t is a strongly continuous group on $\mathfrak{H}_{-k}$.

Since $\mathfrak{H}_k \subset \mathfrak{H} \subset \mathfrak{H}_{-k}$, we have extended P_t by this means. To see that there is only one such extension, we note the following.

Lemma 1. $\mathfrak{H}$ is dense in $\mathfrak{H}_{-k}$.

Proof. The inclusion map $J : \mathcal{H} \rightarrow \mathcal{H}_{-k}$ is given by $\langle F, J(\Phi) \rangle = (F, \Phi)_{\mathcal{H}}$ for $F \in \mathcal{D}(T^{*k})$. Thus to show that range J is dense it suffices to show that if $F \in (\mathcal{H}_{-k})'$ annihilates range J , then $F = 0$. That is, if $F \in \mathcal{D}(T^{*k})$ and $(F, \Phi)_{\mathcal{H}} = 0 \quad \forall \Phi \in \mathcal{H}$, then $F = 0$. This is clear.

A direct proof of Lemma 1 can be given using the group P_t . For $\Phi \in \mathcal{H}_{-k}$ let $\Phi_\epsilon = (1/\epsilon) \int_0^\infty \varphi(t/\epsilon) P_t \Phi dt$ where $\varphi \in C_0^\infty(\mathbb{R}^+)$ and $\int \varphi dt = 1$, then $\Phi_\epsilon \rightarrow \Phi$ in $\mathcal{H}_{-k}$ as $\epsilon \rightarrow 0$. In addition $\Phi_\epsilon \in \mathcal{H}$, for if $F \in \mathcal{D}(T^{*k})$, then

$$\begin{aligned} |\langle F, \Phi_\epsilon \rangle| &= \left| \left\langle \frac{1}{\epsilon} \int_0^\infty \varphi\left(\frac{t}{\epsilon}\right) P_t^* F dt, \Phi \right\rangle \right| \leq \frac{1}{\epsilon} \left\| \int_0^\infty \varphi\left(\frac{t}{\epsilon}\right) P_t^* F dt \right\|_{\mathcal{D}(T^{*k})} \|\Phi\|_{\mathcal{H}_{-k}} \\ &\leq C_\epsilon \|F\|_{\mathcal{H}}. \end{aligned}$$

The last inequality is well-known from the theory of semigroups. It follows that $\Phi_\epsilon \in \mathcal{H}' = \mathcal{H}$.

It remains to show that $\mathcal{H}_{-k}$ contains lots of distributions, and to see this, we must look at T^{*k} . An elementary calculation gives

$$\begin{aligned} T^* &= \begin{bmatrix} 0 & (-A_0)^{-1} A^* \\ -A_0 & 0 \end{bmatrix}, \\ (-1)^m T^{*2m} &= \begin{bmatrix} A_0^{-1} A^{*m} A_0 & 0 \\ 0 & A^{*m} \end{bmatrix}. \end{aligned}$$

The precise domains of these operators are not easy to specify. However $C_0^\infty(\Omega) \oplus C_0^\infty(\Omega) \subset \mathcal{D}(T^{*2m})$, and a simple continuity argument yields $\dot{H}_{2m+1}(\Omega) \oplus \dot{H}_{2m}(\Omega) \subset \mathcal{D}(T^{*2m})$. In addition, the local regularity theorem for elliptic operators shows that $\mathcal{D}(T^{*2m}) \subset H_{2m+1}^{1, \text{loc}}(\Omega) \oplus H_{2m}^{1, \text{loc}}(\Omega)$. It follows that any distribution with compact support in Ω acts continuously on $\mathcal{D}(T^{*k})$, for k sufficiently large. That is, if $\varphi_0, \varphi_1 \in \mathcal{S}'(\Omega)$, we can regard $(\varphi_0, \varphi_1) \in \mathcal{H}_{-k}$ for some k .

Note that $\mathcal{H}_{-k}$ will not in general be a space of distributions, but rather a space of equivalence classes of distributions. However, any $\Phi \in \mathcal{H}_{-k}$, when its action is restricted to test functions supported by any relatively compact open set $\Theta \subset \Omega$, gives rise to a distribution, and the resulting restriction map gives rise to a continuous map $\mathcal{H}_{-2m} \rightarrow H_{-2m-1}(\Theta) \oplus H_{-2m}(\Theta)$. This remark will be necessary in the proof of Theorem 3.

If $\varphi_0, \varphi_1 \in \mathcal{S}'(\Omega)$, then $P_t(\varphi_0, \varphi_1) = (u_0(t), u_1(t))$ is well-defined and $u_0(t)$ is a generalized solution of (1). To see that this agrees with the usual extensions of the notion of solution we notice that $\mathcal{D}(T)$ is dense in $\mathcal{H}_{-k}$ (same proof as Lemma 1) and choose $U_\epsilon \in \mathcal{D}(T)$, $U_\epsilon \rightarrow (\varphi_0, \varphi_1)$ in $\mathcal{H}_{-k}$. Then $P_t U_\epsilon \rightarrow P_t(\varphi_0, \varphi_1)$ in $\mathcal{H}_{-k}$ so $P_t(\varphi_0, \varphi_1)$ is the limit of genuine solutions.

2. In particular, the *Riemann function* with source at $y \in \Omega$ is given by $R(t)$ where

$$\left(R(t), \frac{d}{dt} R(t) \right) = P_t(0, \delta_y).$$

We can interpret $R(t)$ as the signal produced by a point source. This signal is propagated along until it hits the boundary, $\partial\Omega$, where it is reflected and diffracted, so that some of the signal gets behind the obstacle, into the region that geometrical optics would call the shadow. We now show that, indeed, this signal must affect every portion of Ω , no matter how convoluted the boundary.

Theorem 1. *There is no nonempty open set $\omega \subset \Omega$ such that $R(t) = 0$ on ω for all $t \geq 0$.*

Proof. For large k , $(0, \delta_\nu) \in \mathcal{H}_{-k}$. Then, for $\operatorname{Re} \lambda$ large,

$$(v, w) = \int_0^\infty e^{-\lambda t} P_t(0, \delta_\nu) dt$$

is an absolutely convergent integral in $\mathcal{H}_{-k}$. Choose $\Phi_\epsilon = (\varphi_0^\epsilon, \varphi_1^\epsilon) \in \mathcal{H}$ with $\Phi_\epsilon(0, \delta_\nu)$ in $\mathcal{H}_{-k}$ and define (v^ϵ, w^ϵ) analogously. Then $(\lambda - T)(v^\epsilon, w^\epsilon) = \Phi_\epsilon$ so $(\lambda^2 - A)v^\epsilon = \varphi_1^\epsilon$ and letting $\epsilon \rightarrow 0$ we have $(\lambda^2 - A)v = \delta_\nu$ in the sense of distributions. If $R(t) = 0$ on ω for $t \geq 0$, then $v = 0$ on ω . By the unique continuation theorem for second order elliptic equations ([1] or [2]) $(\lambda^2 - A)v = 0$ on $\Omega \setminus \{y\}$ yields $v = 0$ on $\Omega \setminus \{y\}$, so $\operatorname{supp} v \subset \{y\}$. Then $v = Q(D) \delta_\nu$ for some $Q(\xi)$ a non-zero polynomial and we have $\langle Q(D) \delta_\nu, (\lambda^2 - A)^* \varphi \rangle = \varphi(y) \quad \forall \varphi \in C_0^\infty(\Omega)$. If $Q^*(\lambda^2 - A)^* = \sum b_\alpha(x) D^\alpha$ let $\tilde{Q}(D) = \sum b_\alpha(y) D^\alpha$, then $(\tilde{Q}\varphi)(y) = \varphi(y)$ so $\tilde{Q}(\xi) \equiv 1$. However, $\tilde{Q}(\xi)$ is a polynomial of order at least 2, a contradiction.

The above proof breaks down for more general initial data, and indeed if Ω is unbounded there exist signals produced by initial data with compact support which do not reach everywhere. For example, if an obstacle is placed in the center of an outgoing spherical wave solution of $(\partial^2/\partial t^2 - \Delta)u = 0$, then the region inside the support of the initial data but exterior to the obstacle will never be influenced by a signal. However, the rest of space will, and we shall show that this is the worst that can happen. First we prove a unique continuation lemma, which is a strengthening of a theorem of Masuda [9].

Lemma 2. *Let A be a strongly elliptic second order operator whose principal part has real coefficients; let the coefficients be smooth on a connected region $\mathcal{U}$. Suppose $u \in \mathcal{D}'(\mathbf{R} \times \mathcal{U})$ satisfies the hyperbolic equation $(\partial^2/\partial t^2)u - Au = 0$ and suppose furthermore that u is a continuous function of t with values in some Sobolev space $H_{-m}(\mathcal{U})$; write $u(t) = u(t, \cdot)$. Finally suppose $\|u(t)\|_{H_{-m}(\mathcal{U})} \leq Ce^{k|t|}$ for some constants C and k . If there is a nonempty open set $\omega \subset \mathcal{U}$ such that $u(t) = 0$ on ω for all $t \in \mathbf{R}$, then $u \equiv 0$.*

Proof. Consider $u_\epsilon(t) = (1/(\pi\epsilon)^{1/2}) \int_{-\infty}^\infty e^{-(t-\tau)^2/\epsilon} u(\tau) d\tau$. Since $u_\epsilon \rightarrow u$ as $\epsilon \rightarrow 0$ it suffices to show that $u_\epsilon \equiv 0$. But note that, by virtue of the growth restriction $\|u(\tau)\| \leq Ce^{k|\tau|}$, $u_\epsilon(t)$ can be continued as an entire analytic function of $t \in \mathbf{C}$. Now, $u_\epsilon(t)$ is a solution of $(\partial^2/\partial t^2 - A)u_\epsilon = 0$ so $v(t) = u_\epsilon(it)$ satisfies the elliptic equation $(\partial^2/\partial t^2 + A)v = 0$. In addition $v = 0$ on $\mathbf{R} \times \omega$ so by the

unique continuation principle for second order elliptic equations $v = 0$ on $\mathbf{R} \times \mathfrak{U}$. Thus, $u_\epsilon(it) = 0$ for all real t so $u_\epsilon \equiv 0$ by unique continuation for analytic functions.

Remark. If the coefficients of A happen to be analytic, Holmgren's uniqueness theorem immediately gives the above lemma, with no growth restrictions. In general, the growth restriction given above can certainly be relaxed, but there is a limit to how far the above method can be extended, due to the well known limit to the speed at which a function holomorphic on a strip can vanish.

Theorem 2. Let $\Omega = \mathbf{R}^n \setminus K$ where K is compact. If $\Phi \in \mathcal{H}_{-k}$ has bounded support and is not identically zero, then there is no nonempty open set ω in the unbounded component $\mathcal{O}$ of $\mathbf{R}^n \setminus (K \cup \text{supp } \Phi)$ on which $P_t \Phi$ vanishes for all $t \geq 0$.

Proof. If $(u_0(t), u_1(t)) = P_t \Phi$ vanishes on ω for $t \geq 0$, then as in Theorem 1 let $v_\lambda = \int_0^\infty e^{-\lambda t} u_0(t) dt$ for $\text{Re } \lambda$ large. It follows that $v_\lambda = 0$ on ω and $(\lambda^2 - A)v_\lambda = \lambda\varphi_0 + \varphi_1$ if $\Phi = (\varphi_0, \varphi_1)$. The unique continuation property of $\lambda^2 - A$ implies that $v_\lambda = 0$ on $\mathcal{O}$, for all λ with $\text{Re } \lambda$ sufficiently large. By uniqueness of the Laplace transform it follows that $u_0(t) = 0$ on $\mathcal{O}$ for all $t \geq 0$. In particular $u_0(t, x) = 0$ for all $t \geq 0$ if $|x| \geq R$, for R sufficiently large. Thus u_0 , restricted to the ball $\mathcal{B} = \{x \in \mathbf{R}^n: |x| \leq R + 1\}$ is a solution of the equation

$$\frac{\partial^2}{\partial t^2} u_0 - A u_0 = 0 \quad \text{on } \mathcal{B} \setminus K,$$

$$u_0(0) = \varphi_0,$$

$$\frac{\partial}{\partial t} u_0(0) = \varphi_1.$$

Furthermore, u_0 satisfies Dirichlet boundary condition on $\partial \mathcal{B}$ and the given boundary condition on ∂K , in the generalized sense. It is easily seen that u_0 is given by the evolution operator on $\mathcal{B} \setminus K$ determined by A and these boundary conditions, so our theorem is a consequence of the following result.

Theorem 3. Suppose Ω is bounded. If $\Phi \in \mathcal{H}_{-k}$ is not identically zero, then there is no nonempty open set $\omega \subset \Omega$ such that $P_t \Phi = 0$ on ω for all $t \geq 0$.

Proof. Suppose $P_t \Phi$ vanishes on ω for all $t \geq 0$. Let

$$\Phi_\epsilon = (1/\epsilon) \int_0^\infty \varphi(t/\epsilon) P_t \Phi dt \in \mathcal{H},$$

where $\varphi \in C_0^\infty(\mathbf{R}^+)$ and $\int_0^\infty \varphi(t) dt = 1$. Then $P_t \Phi_\epsilon$ vanishes on ω for all $t \geq 0$ and $\Phi_\epsilon \rightarrow \Phi$ in $\mathcal{H}_{-k}$. It suffices to show $\Phi_\epsilon = 0$, so the proof is reduced to the case $\Phi \in \mathcal{H}$. If $\text{Re } \lambda$ is sufficiently large $(\lambda - T)^{-1} \Phi = \int_0^\infty e^{-\lambda t} P_t \Phi dt$. Also $(\lambda - T)^{-1} \Phi = - \int_{-\infty}^0 e^{-\lambda t} P_t \Phi dt$ if $\text{Re } \lambda$ is sufficiently negative. Now since Ω is bounded, Rellich's theorem implies that $(\lambda - T)^{-1}$ is compact for all $\lambda \in \rho(T)$. Thus the spectrum of T is a countable set with no finite limit point and in particular does not

separate the complex plane. Now $P_t\Phi = 0$ on ω for all $t \geq 0$ implies that $\langle (\lambda - T)^{-1}\Phi, u \rangle = 0$ for all $u \in C_0^\infty(\omega)$, if $\operatorname{Re} \lambda$ is sufficiently large. This identity is then true for all λ in the connected set $\rho(T)$, and in particular for all λ with $\operatorname{Re} \lambda$ sufficiently negative. Thus, by uniqueness of the Laplace transform, we have $\langle P_t\Phi, u \rangle = 0$ for $t \leq 0$ so $P_t\Phi = 0$ on ω for all $t \in \mathbf{R}$. Since any group P_t has at most exponential growth, this theorem is now an immediate consequence of Lemma 2.

Lemma 2 can also be used to derive results about "incoming waves" and about generalizations of the Klein-Gordon equation.

Definition. If $\Lambda \in \mathcal{H}_{-k}$ some k , the solution $P_t\Lambda$ is called an *incoming wave* if for every relatively compact set $\omega \subset \Omega$ there is a $\tau_0 > -\infty$ with $P_t\Lambda = 0$ on ω for $t < \tau_0$.

It is an immediate consequence of Lemma 2 that the disturbance from an incoming wave reaches everywhere in Ω .

Theorem 4. If $P_t\Lambda$ is an incoming wave and $\Lambda \neq 0$, there is no nonempty open set $\omega \subset \Omega$ on which $P_t\Lambda = 0$ for all t .

Definition. If the spectrum of T does not separate the half planes $\operatorname{Re} \lambda > \alpha$ and $\operatorname{Re} \lambda < -\alpha$ where α is so large that both sets are contained in $\rho(T)$, then $\partial^2/\partial t^2 - A$ is called a *generalized Klein-Gordon operator*.

Note that since T^2 is a linear operator on $\mathcal{H} = \mathcal{D}(A^{1/2}) \oplus L^2(\Omega)$ given by

$$T^2 = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \mathcal{D}(T^2) = \mathcal{D}(A^{3/2}) \oplus \mathcal{D}(A),$$

it follows that T^2 is similar to W , a linear operator on $L^2(\Omega) \oplus L^2(\Omega)$ given by

$$W = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \mathcal{D}(W) = \mathcal{D}(A) \oplus \mathcal{D}(A).$$

Indeed if $\operatorname{Re} \lambda$ is sufficiently large then $(\lambda - A)^{-1/2}$ is well defined,

$$S = \begin{bmatrix} (\lambda - A)^{-1/2} & 0 \\ 0 & I \end{bmatrix} : L^2(\Omega) \oplus L^2(\Omega) \rightarrow \mathcal{H} \text{ isomorphically,}$$

and $ST^2S^{-1} = W$. Thus $\sigma(T^2) = \sigma(W) = \sigma(A)$. It follows by the spectral mapping theorem that $\sigma(T) \subset \{\zeta \in \mathbf{C} : \zeta^2 \in \sigma(A)\}$. For example with Ω as in Theorem 2 and $A = \Delta - m^2$ with Dirichlet boundary conditions on $\partial\Omega$, then $\sigma(T) = \{\pm it : t \geq |m|\}$ and $\partial^2/\partial t^2 - A$ is a generalized Klein Gordon operator if and only if $m \neq 0$. The next theorem is a generalization of a result of Morawetz [11].

Theorem 5. If $\partial^2/\partial t^2 - A$ is a generalized Klein-Gordon operator on Ω , then there is no nonzero $\Phi \in \mathcal{H}_{-k}$ such that for some nonempty open set $\omega \subset \Omega$, $P_t\Phi = 0$ on ω for all $t \geq 0$.

Proof. As in the proof of Theorem 3, the hypotheses here imply that $P_t \Phi = 0$ on ω for all $t \in \mathbf{R}$, and then the conclusion follows from Lemma 2.

Let us finally mention that the results of this section hold, with few essential changes, for hyperbolic operators of the form $\partial^2/\partial t^2 - b(x)(\partial/\partial t) - A$.

3. Many of the results in Section 1 extend to hyperbolic systems of the form

$$\frac{\partial}{\partial t} - \sum_{i=1}^n A_i(x) \frac{\partial}{\partial x_i} + B(x)$$

in a region Ω . We suppose A_i and B are smooth $k \times k$ matrix valued functions and that $\partial\Omega$ is non-characteristic, that is, $\sum A_i(x)\nu_i(x)$ is nonsingular for $x \in \partial\Omega$ where ν is a normal to $\partial\Omega$. Homogeneous boundary conditions are prescribed on the boundary of Ω in the form $u(t, x) \in N(x)$ for $x \in \partial\Omega$, where N is a smoothly varying linear subspace of $\mathbf{C}^k$. We define an operator G on $L^2(\Omega)$ as the closure of the operator G_0 given by,

$$\mathfrak{D}(G_0) = \{v \in H_1(\Omega) \mid v(x) \in N(x) \text{ for } x \in \partial\Omega\},$$

$$G_0 v = \sum_{i=1}^n A_i(x) \frac{\partial}{\partial x_i} v + B(x)v \text{ for } v \in \mathfrak{D}(G_0).$$

There are essentially two general situations in which it is known that G is the generator of a strongly continuous semigroup of transformations on $L^2(\Omega)$. The first (see [6]) is when $A_i = A_i^*$ and the boundary condition is maximal non-positive. In this case G is a bounded perturbation of a dissipative operator. Many self-adjoint (energy-conserving) problems of physics fall into this class. The second class discovered by Kreiss is not as easily described, and no direct proof that G is a generator has been given. The only proof proceeds by constructing the semi-group and showing that G is the generator (see [13], [14]). Assuming that G does generate a strongly continuous one parameter semigroup, we now describe the modifications needed to mimic the results of Section 1.

First we extend the semi-group $P_t = e^{tG}$ to a suitable space of distributions. This can be done using the recent result of Massey and Rauch that P_t leaves $V_s = H^s(\Omega) \cap \mathfrak{D}(G^s)$ invariant and acts as a strongly continuous semigroup on V_s , for each positive integer s . Note that $v \in V_s$ if and only if $v \in H^s(\Omega)$ and $G^j v(x) \in N(x)$ for almost all $x \in \partial\Omega$, $0 \leq j \leq s-1$. Thus V_s is a closed linear subspace of $H^s(\Omega)$. We set $V_{-\infty} = (H^s(\Omega) \cap \mathfrak{D}(G^{*s}))'$. As in Section 1, if $u \in \mathcal{E}'(\Omega)$, or more precisely if u is a distribution with compact support in Ω , taking values in $\mathbf{C}^k$, then $u \in V_{-\infty}$ for s sufficiently large. The Riemann function is defined to be $R(t) = P_t \delta_v I$.

Theorem 6. *If G is elliptic and has the unique continuation property, then there is no nonempty open set $\omega \subset \Omega$ such that $R(t) = 0$ on ω for all $t \geq 0$.*

Proof. Identical to that for Theorem 1.

Obtaining an analogue of Theorem 2 is more difficult. Suppose G is elliptic and the boundary conditions determining $\mathfrak{D}(G)$ are coercive, and that, furthermore, G generates a one parameter group. The difficulty is that, in the proof of Theorem 2, boundary conditions must be imposed on $\{x : |x| = R + 1\}$ so that the resulting operator is coercive and generates a group. In the class of boundary conditions we have discussed this may be impossible. However, one can “double” the manifold $\{x : |x| \leq R + 1, x \notin K\}$, identifying the spherical boundaries. The coefficients of the differential operator are extended to the “double” by symmetry and smoothed in a small neighborhood of the glued edge. By this device the compactness needed in the proof of Theorem 2 is restored. In this way we obtain an analogue of Theorem 2 provided $i(d/dt) - G$ has the unique continuation property.

We remark that an analogue of Lemma 2 can be easily obtained provided the elliptic operator $i(\partial/\partial t) - G$ satisfies the unique continuation principle.

Note that Theorem 6 does not apply to Maxwell’s equations

$$\begin{aligned}\frac{\partial}{\partial t} E &= \operatorname{curl} B, \\ \frac{\partial}{\partial t} B &= -\operatorname{curl} E, \\ \operatorname{div} E &= \operatorname{div} B = 0,\end{aligned}$$

since the operator G we obtain is not elliptic. However, as is often the case, the condition that E and B be divergence free can be used to avoid this difficulty. Thus, for example, each component of E and B satisfies the wave equation, and so, by a slight modification of the arguments of Section 1, it follows that the conclusion of Theorem 6 and Theorem 2 are valid for Maxwell’s equations, provided appropriate boundary conditions are given.

Example. If $\Omega = \mathbf{R}^3 \setminus K$ and K is a perfect conductor, then Maxwell’s equations are supplemented by the boundary conditions $E \times \nu = 0$ where ν is normal to $\partial\Omega$. It follows that the solution E, B satisfies the following well posed mixed problems for second order hyperbolic equations.

$$\begin{aligned}(\ast) \quad & \frac{\partial^2}{\partial t^2} E - \Delta E = 0, & \frac{\partial^2}{\partial t^2} B - \Delta B = 0, \\ & E \times \nu = 0 \quad \text{at } \partial\Omega, & B \cdot \nu = 0 \quad \text{at } \partial\Omega, \\ & \operatorname{div} E = 0 \quad \text{at } \partial\Omega, & \nu \times \operatorname{curl} B = 0 \quad \text{at } \partial\Omega.\end{aligned}$$

Conversely, given $E(0), B(0)$ with $\operatorname{div} E(0) = \operatorname{div} B(0) = 0$ the solution of $(\ast)$ with initial data given by $E(0), B(0), (dE/dt)(0) = \operatorname{curl} B(0), (dB/dt)(0) = -\operatorname{curl} E(0)$ is identical with the solution of Maxwell’s equations. The problems $(\ast)$ can be treated by the methods of Section 1. This reduction also yields a straightforward approach to the study of scattering problems associated with

Maxwell's equations since the methods of Lax-Phillips can be directly applied to the problems (*).

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