Generating Forms of Lagrangian Submanifolds

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1. Introduction. In this paper the existence of generating forms for a class of Lagrangian submanifolds of special symplectic manifolds is established and the composition properties of generating forms of symplectic relations are derived.

Lagrangian submanifolds have been studied extensively in a recent publication by Weinstein [5]. A Lagrangian submanifold of a symplectic manifold is a submanifold for which the restriction of the symplectic form vanishes, and which in addition satisfies certain maximality conditions. Special symplectic manifolds are manifolds isomorphic to cotangent bundles. Symplectic relations are Lagrangian submanifolds of products of symplectic manifolds. Their composition is understood in the set-theoretical sense.

This work is motivated by physical applications. The formulation of a physical theory in terms of symplectic geometry is considered to be an essential step towards its quantization. The study of generating forms could be useful in formulating a Schrödinger type quantum theory of a physical system.

The notions of symplectic relations, their generating forms and their composition can also be useful in the study of differential equations. In a recent paper by Hörmander [1] the composition of symplectic relations is used in connection with the study of Fourier integral operators.

2. Symplectic manifolds. All manifold studies in this paper are C^∞ manifolds modelled on reflexive Banach spaces.

2.1. Definition. A symplectic manifold is a pair (P, ω) , where P is a manifold and ω is a closed 2-form in P , such that the mapping $TP \rightarrow T^*P: u \rightarrow u \lrcorner \omega$ is an isomorphism.

2.2. Example. Let T^*X be the cotangent bundle space of a manifold X , and let $\pi_X: T^*X \rightarrow X$ be the cotangent bundle projection. The Liouville form θ_X in T^*X is defined by

$$\langle u, \theta_X(p) \rangle = \langle T\pi_X(u), p \rangle,$$

for each $u \in T_p T^*X$. The pair $(T^*X, d\theta_x)$ is a symplectic manifold.

This example is typical in the sense that symplectic manifolds found in most applications are isomorphic to cotangent bundles.

2.3. Definition. A special symplectic manifold is a system $(P, X, \pi, \theta, \alpha)$, where (P, X, π) is a differentiable fibre bundle, θ is a 1-form in P , and α is a diffeomorphism from P to T^*X , such that $\pi = \pi_X \circ \alpha$ and $\theta = \alpha^* \theta_X$.

If $(P, X, \pi, \theta, \alpha)$ is a special symplectic manifold, then $(P, d\theta)$ is a symplectic manifold isomorphic to $(T^*X, d\theta_X)$.

2.4. Examples.

2.4.1. Let (X, g) be a Riemannian manifold, and let $\alpha: TX \rightarrow T^*X$ be defined by $\langle u, \alpha(v) \rangle = g(u, v)$. Then $(TX, X, \tau_X, \alpha^* \theta_X, \alpha)$, where $\tau_X: TX \rightarrow X$ is the tangent bundle projection, is a special symplectic manifold. The symplectic form $\alpha^* d\theta_X$ in TX is used to define the geodesic spray of g in X [2].

2.4.2. Let $\alpha: T^*X \rightarrow T^*X$ be defined by $\alpha(p) = p - e\pi_X^* A(p)$, where e is a constant and A is a 1-form in X . The system $(T^*X, X, \pi_X, \theta_X - e\pi_X^* A, \alpha)$ is a special symplectic manifold. When X is a 4-dimensional manifold representing space-time, the symplectic form $d\theta_X - e\pi_X^* dA$ is used in the canonical description of the dynamics of particles with charge e in an electromagnetic field represented by dA [3].

3. Lagrangian submanifolds.

3.1. Definition. Let (P, ω) be a symplectic manifold, and p any point in P . An isotropic subspace of $T_p P$ is a linear subspace E of $T_p P$ such that, for each $u, v \in E$, $\langle u \wedge v, \omega_p \rangle = 0$.

3.2. Definition. An isotropic submanifold of a symplectic manifold (P, ω) is a submanifold N of P such that, for each $p \in N$, $T_p N$ is an isotropic subspace of $T_p P$.

An equivalent characterization of an isotropic submanifold is given by the following obvious proposition.

3.3. Proposition. A submanifold N of (P, ω) is isotropic if and only if $i_N^* \omega = 0$, where $i_N: N \rightarrow P$ is the imbedding of N in P .

3.4. Definition. A Lagrangian submanifold of a symplectic manifold (P, ω) is an isotropic submanifold N such that, for each $p \in N$, $T_p N$ has an isotropic complement in $T_p P$.

3.5. Proposition. Let $(P, X, \pi, \theta, \alpha)$ be a special symplectic manifold, and N an isotropic submanifold of $(P, d\theta)$ such that $K = \pi(N)$ is a submanifold of X , the restriction of π to N induces a submersion $\sigma: N \rightarrow K$, and the fibres of σ are connected. Then there exists a unique closed 1-form γ in K such that $i_N^* \theta = \sigma^* \gamma$.

Proof. Let $u \in TN$ be such that $T\sigma(u) = 0$, then $T\pi \circ Ti_N(u) = 0$, and

$$\begin{aligned}
\langle u, i_N^* \theta \rangle &= \langle Ti_N(u), \theta \rangle \\
&= \langle Ti_N(u), \alpha^* \theta_x \rangle \\
&= \langle T\alpha \circ Ti_N(u), \theta_x \rangle \\
&= \langle T\pi_X \circ T\alpha \circ Ti_N(u), \alpha \circ \tau_{TP}(u) \rangle \\
&= \langle T\pi \circ Ti_N(u), \alpha \circ \tau_{TP}(u) \rangle \\
&= 0.
\end{aligned}$$

Hence, $i_N^* \theta$ induces a mapping $\eta: N \rightarrow T^*K$ such that for each $p \in N$ and each $w \in T_{\sigma(p)}K$, $\langle w, \eta(p) \rangle = \langle u, i_N^* \theta(p) \rangle$, where u is any vector in $T_p N$ satisfying $T\sigma(u) = w$.

Since $\sigma: N \rightarrow K$ is a submersion, for every $p \in N$, there exists a chart (U, ϕ) at p and a chart (V, ψ) at $x = \sigma(p)$ such that

- (i) $\phi(U) = W_1 \times W_2$, where W_1 and W_2 are open subsets of some Banach spaces E_1 and E_2 , and W_1 is connected;
- (ii) $\psi(V) = W_2$;
- (iii) $\psi \circ \sigma \circ \phi^{-1}: W_1 \times W_2 \rightarrow W_2$ is the projection onto the second factor.

In the trivialisation of T^*N given by (U, ϕ) the principal part of $i_N^* \theta$ is a mapping $f: W_1 \times W_2 \rightarrow E_1^* \times E_2^*$. The vanishing of $\langle u, i_N^* \theta \rangle$ whenever $T\sigma(u) = 0$ implies $f(W_2 \times W_2) \subset \{0\} \times E_2^*$. Hence f factors through the mapping $h: W_1 \times W_2 \rightarrow E_2^*$ so that the following diagram commutes

$$\begin{array}{ccc}
W_1 \times W_2 & \xrightarrow{f} & E_1^* \times E_2^* \\
h \downarrow & & \uparrow \text{inclusion} \\
E_2^* & \xrightarrow{0 \times id} & \{0\} \times E_2^*.
\end{array}$$

The continuity and differentiability of h follows from the continuity and differentiability of f . The mapping h is the principal part of η in the trivialisation of T^*K given by (V, ψ) . For each vector field v in N such that $T\sigma \circ v = 0$, $\mathcal{L}_v(i_N^* \theta) = d\langle v, i_N^* \theta \rangle + v \lrcorner di_N^* \theta = 0$. This fact and the connectedness of W_1 imply that h is constant on the first factor in $W_1 \times W_2$. Therefore h factors through the mapping $g: W_2 \rightarrow E_2^*$, such that the following diagram commutes

$$\begin{array}{ccc}
W_1 \times W_2 & & \\
pr_2 \downarrow & \searrow h & \\
W_2 & \xrightarrow{g} & E_2^*
\end{array}$$

Clearly g is continuous and differentiable.

For a given $p \in N$, the set of points p' at which $\eta(p') = \eta(p)$ is closed in the fibre of σ through p since η is continuous. This set is also open as follows from

the preceding paragraph. By hypothesis the fibres of σ are connected; therefore η is constant on the fibres of σ . Hence η factors through the mapping $\gamma: K \rightarrow T^*K$. Clearly, $\pi_K \circ \gamma = id$, and the principal part of γ in the chart (V, ψ) is given by the differentiable mapping $g: W_2 \rightarrow E_2^*$. Hence γ is a differential 1-form in K . It follows from the construction that $\sigma^*\gamma = i_N^*\theta$. Since $i_N^*\theta$ is closed γ is closed. The uniqueness of γ is obvious. Q.E.D.

Let $(P, X, \pi, \theta, \alpha)$ be a special symplectic manifold, K a submanifold of X , and γ a closed 1-form in K . Further, let M denote the inverse image of K by π , $M = \pi^{-1}(K)$, and let $\rho: M \rightarrow T^*K$ denote the mapping defined by $\langle u, \rho(p) \rangle = \langle Ti_K(u), \alpha \circ i_M(p) \rangle$, for each $p \in M$ and each $u \in TK$ such that $i_K \circ \tau_{TK}(u) = \pi \circ i_M(p)$. Clearly ρ is a submersion.

3.6. Proposition. *$N = i_M(\rho^{-1}(\gamma(K)))$ is the maximal submanifold of P such that π restricted to N induces a submersion $\sigma: N \rightarrow K$, and $i_N^*\theta = \sigma^*\gamma$.*

Proof. Let $\tilde{N}$ be a submanifold of P such that π restricted to $\tilde{N}$ induces a submersion $\tilde{\sigma}: \tilde{N} \rightarrow K$, and $i_{\tilde{N}}^*\theta = \tilde{\sigma}^*\gamma$. We shall show that $\tilde{N} \subset N$. Since $\pi \circ i_{\tilde{N}} = i_K \circ \tilde{\sigma}$, $\tilde{N}$ is contained in M . Let $\tilde{\nu}$ denote the imbedding of $\tilde{N}$ in M . For each $p \in \tilde{N}$ and each $u \in T_p\tilde{N}$,

$$\begin{aligned} \langle u, i_N^*\theta(p) \rangle &= \langle Ti_N(u), \theta(i_N(p)) \rangle \\ &= \langle Ti_N(u), \alpha^*\theta_X(i_N(p)) \rangle \\ &= \langle T\alpha \circ Ti_N(u), \theta_X(\alpha \circ i_N(p)) \rangle \\ &= \langle T\pi_X \circ T\alpha \circ Ti_N(u), \alpha \circ i_N(p) \rangle \\ &= \langle T\pi \circ Ti_N(u), \alpha \circ i_N(p) \rangle \\ &= \langle Ti_K \circ T\tilde{\sigma}(u), \alpha \circ i_N(p) \rangle \\ &= \langle Ti_K \circ T\tilde{\sigma}(u), \alpha \circ i_M \circ \tilde{\nu}(p) \rangle \\ &= \langle T\tilde{\sigma}(u), \rho \circ \tilde{\nu}(p) \rangle. \end{aligned}$$

Similarly,

$$\langle u, \tilde{\sigma}^*\gamma(p) \rangle = \langle T\tilde{\sigma}(u), \gamma \circ \tilde{\sigma}(p) \rangle.$$

Hence,

$$\rho \circ \tilde{\nu}(p) = \gamma \circ \tilde{\sigma}(p),$$

and

$$\tilde{\nu}(p) \in \rho^{-1}(\gamma(K)), \quad \text{or} \quad i_N(p) \in i_M(\rho^{-1}(\gamma(K))),$$

which proves $\tilde{N} \subset N$.

It has been shown that each submanifold $\tilde{N}$ of P such that π restricted to $\tilde{N}$ induces a submersion $\tilde{\sigma}: \tilde{N} \rightarrow K$ and $i_{\tilde{N}}^*\theta = \tilde{\sigma}^*\gamma$, is contained in N . However, N is a submanifold of P satisfying these conditions. Therefore N is the maximal

submanifold of P such that π restricted to N induces a submersion $\sigma: N \rightarrow K$ and $i_N^*\theta = \sigma^*\gamma$. Q.E.D.

3.7. Proposition. *The submanifold N of P in Proposition 3.6 is a Lagrangian submanifold of $(P, d\theta)$.*

Proof. N is isotropic since γ is closed and $i_N^*d\theta = di_N^*\theta = d\sigma^*\gamma = \sigma^*d\gamma = 0$.

Given $p \in N$, there exist a submanifold $\tilde{K}$ of X , transverse to K at $x = \sigma(p)$, and a closed 1-form $\tilde{\gamma}$ in $\tilde{K}$ such that $\tilde{\rho}(p) = \tilde{\gamma}(x)$, where $\tilde{\rho}: \tilde{M} = \pi^{-1}(\tilde{K}) \rightarrow T^*\tilde{K}$ is defined in the same way as $\rho: M \rightarrow T^*K$. The construction of $\tilde{K}$ and $\tilde{\gamma}$ can be given in terms of a local chart in X at the point x . Clearly, $\tilde{N} = i_M(\tilde{\rho}^{-1}(\tilde{\gamma}(\tilde{K})))$ is an isotropic submanifold of $(P, d\theta)$. It remains to show that $T_pP \simeq T_p\tilde{N} \oplus T_pN$.

Since $\tilde{K}$ is transverse to K at x , $T_xX \simeq T_xK \oplus T_x\tilde{K}$ and $T_x^*X \simeq T_x^*K \oplus T_x^*\tilde{K}$. Further, $\alpha(\rho^{-1}(\gamma(x)))$ is the coset of $T_x^*\tilde{K}$ in T_x^*X which contains $\alpha(p)$. Therefore $T_{\alpha(p)}\alpha(\rho^{-1}(\gamma(x)))$ is isomorphic to $T_x^*\tilde{K}$. Similarly $T_{\alpha(p)}\alpha(\tilde{\rho}^{-1}(\tilde{\gamma}(x)))$ is isomorphic to T_x^*K .

Hence,

$$\begin{aligned} T_pN \oplus T_p\tilde{N} &\simeq T_{\alpha(p)}\alpha(N) \oplus T_{\alpha(p)}\alpha(\tilde{N}) \\ &\simeq (T_{\alpha(p)}\alpha(\rho^{-1}(\gamma(x))) \oplus T_xK) \oplus (T_{\alpha(p)}\alpha(\tilde{\rho}^{-1}(\tilde{\gamma}(x))) \oplus T_x\tilde{K}) \\ &\simeq (T_x^*\tilde{K} \oplus T_xK) \oplus (T_x^*K \oplus T_x\tilde{K}) \\ &\simeq (T_x^*\tilde{K} \oplus T_x^*K) \oplus (T_xK \oplus T_x\tilde{K}) \\ &\simeq T_x^*X \oplus T_xX \\ &\simeq T_{\alpha(p)}T^*X \\ &\simeq T_pP. \end{aligned}$$

Therefore N is isotropic, and, for each $p \in N$, there exists an isotropic complement $T_p\tilde{N}$ of T_pN in T_pP . And hence N is a Lagrangian submanifold of $(P, d\theta)$. Q.E.D.

3.8. Definition. The submanifold $N = i_M(\rho^{-1}(\gamma(K)))$ is called the Lagrangian submanifold generated by γ , and γ is called the generating form of N .

4. Symplectic relations.

4.1. Definition. A symplectic relation from a symplectic manifold (P_1, ω_1) to (P_2, ω_2) is an isotropic submanifold of the symplectic manifold $(P_1 \times P_2, pr_1^*\omega_1 - pr_2^*\omega_2)$, where pr_1 and pr_2 are projections from $P_1 \times P_2$ to P_1 and P_2 respectively.

4.2. Examples.

4.2.1. If χ is a symplectic diffeomorphism from (P_1, ω_1) to (P_2, ω_2) , then graph χ is a symplectic relation.

4.2.2. Let M be a submanifold of a symplectic manifold (P, ω) . Suppose there exist a symplectic manifold $(\tilde{P}, \tilde{\omega})$ and a submersion $\rho: M \rightarrow \tilde{P}$ such that $\rho^*\tilde{\omega} = i_M^*\omega$. Then, graph χ , imbedded into $P \times \tilde{P}$, is a symplectic relation from (P, ω) to $(\tilde{P}, \tilde{\omega})$. This relation appears in the transition from the phase space, (P, ω) , of a canonical system, (P, M, ω) , to the reduced phase space isomorphic to $(\tilde{P}, \tilde{\omega})$ [4].

If $(P_1, X_1, \pi_1, \theta_1, \alpha_1)$ and $(P_2, X_2, \pi_2, \theta_2, \alpha_2)$ are special symplectic manifolds, then $P_1 \times P_2$ has the structure of a special symplectic manifold given by $(P_1 \times P_2, X_1 \times X_2, \pi_1 \times \pi_2, pr_1^*\theta_1 - pr_2^*\theta_2, \alpha_1 \times -\alpha_2)$. If N_{12} is a Lagrangian submanifold of $P_1 \times P_2$, generated by a closed 1-form γ_{12} in a submanifold K_{12} of $X_1 \times X_2$, then we say that the symplectic relation N_{12} is generated by γ_{12} .

To simplify the notation, we identify the tangent bundle spaces of product manifolds with the products of the tangent bundle spaces of the factors, e.g., $T(P_1 \times P_2)$ is identified with $TP_1 \times TP_2$. Further submanifolds are identified with their imbeddings. With these identifications the tangent bundle space TN_{12} of a symplectic relation from (P_1, ω_1) to (P_2, ω_2) can be treated (in the sense of point set theory) as a relation from TP_1 to TP_2 . Following the notation in point set theory, the symbol $\circ$ is used to denote the composition of relations.

4.3. Proposition. *Let N_{12} and N_{23} be symplectic relations from (P_1, ω_1) to (P_2, ω_2) , and from (P_2, ω_2) to (P_3, ω_3) , respectively, such that:*

- (i) $N_{13} = N_{23} \circ N_{12}$ is a submanifold of $P_1 \times P_3$;
- (ii) for each (p, q, r) such that $(p, q) \in N_{12}$ and $(q, r) \in N_{23}$,

$$T_{(p,r)}N_{13} = T_{(q,r)}N_{23} \circ T_{(p,q)}N_{12}.$$

Then N_{13} is a symplectic relation from (P_1, ω_1) to (P_3, ω_3) .

Proof. For any two vectors (u_1, u_3) and (v_1, v_3) in the same fibre of TN_{13} there exist u_2 and v_2 in the same fibre of TP_2 such that (u_1, u_2) and (v_1, v_2) are in TN_{12} , while (u_2, u_3) and (v_2, v_3) are in TN_{23} . Therefore,

$$\begin{aligned} \langle (u_1, u_3) \wedge (v_1, v_3), pr_1^*\omega_1 - pr_3^*\omega_3 \rangle &= \langle u_1 \wedge v_1, \omega_1 \rangle - \langle u_3 \wedge v_3, \omega_3 \rangle \\ &= \langle u_1 \wedge v_1, \omega_1 \rangle - \langle u_2 \wedge v_2, \omega_2 \rangle \\ &\quad + \langle u_2 \wedge v_2, \omega_2 \rangle - \langle u_3 \wedge v_3, \omega_3 \rangle \\ &= \langle (u_1, u_2) \wedge (v_1, v_2), pr_1^*\omega_1 - pr_2^*\omega_2 \rangle \\ &\quad + \langle (u_2, u_3) \wedge (v_2, v_3), pr_2^*\omega_2 - pr_3^*\omega_3 \rangle \\ &= 0 \end{aligned}$$

since N_{12} and N_{23} are symplectic relations. Hence the relation N_{13} is symplectic.

Q.E.D.

Let $(P_i, X_i, \pi_i, \theta_i, \alpha_i)$, $i = 1, 2, 3$, be special symplectic manifolds, and let for each $i < j$, N_{ij} be a symplectic relation from $(P_i, d\theta_i)$ to $(P_j, d\theta_j)$ generated by a closed 1-form γ_{ij} in a submanifold K_{ij} of $X_i \times X_j$.

4.4. Proposition. If $N_{13} = N_{23} \circ N_{12}$ and, for each (p, q, r) such that $(p, q) \in N_{12}$ and $(q, r) \in N_{23}$, $T_{(p,r)}N_{13} = T_{(q,r)}N_{23} \circ T_{(p,q)}N_{12}$, then:

(i) $(x, z) \in K_{13}$ if and only if there exists $y \in X_2$ satisfying the following conditions: $(x, y) \in K_{12}$, $(y, z) \in K_{23}$, and, for all $v \in T_v X_2$ for which $(0, v) \in T_{(x,v)}K_{12}$ and $(v, 0) \in T_{(v,z)}K_{23}$,

$$\langle (0, v), \gamma_{12}(x, y) \rangle + \langle (v, 0), \gamma_{23}(y, z) \rangle = 0;$$

(ii) for each $(x, z) \in K_{13}$ and each $(u, w) \in T_{(x,z)}K_{13}$,

$$\langle (u, w), \gamma_{13}(x, y) \rangle = \langle (u, v), \gamma_{12}(x, y) \rangle + \langle (v, w), \gamma_{23}(y, z) \rangle,$$

where y is any point in X_2 satisfying the conditions of (i) and $v \in T_v X_2$ is any vector such that $(u, v) \in TK_{12}$ and $(v, w) \in TK_{23}$.

Proof. For each $i, j = 1, 2, 3, i < j$, M_{ij} denotes the inverse image of K_{ij} by $\pi_i \times \pi_j$, and $\rho_{ij} : M_{ij} \rightarrow T^*K_{ij}$ denotes a mapping such that

$$\begin{aligned} \langle (u_i, u_i), \rho_{ij}(p_i, p_i) \rangle &= \langle (u_i, u_i), (\alpha_i \times -\alpha_i)(p_i, p_i) \rangle \\ &= \langle u_i, \alpha_i(p_i) \rangle - \langle u_i, \alpha_i(p_i) \rangle. \end{aligned}$$

Let $(x, z) \in X_1 \times X_3$ be such that there exists $y \in X_2$ satisfying $(x, y) \in K_{12}$, $(y, z) \in K_{23}$, and, for all $v \in T_v X_2$ for which $(0, v) \in T_{(x,v)}K_{12}$ and $(v, 0) \in T_{(v,z)}K_{23}$,

$$\langle (0, v), \gamma_{12}(x, y) \rangle + \langle (v, 0), \gamma_{23}(y, z) \rangle = 0.$$

Then, there exists p in $\pi_1^{-1}(x)$, q' and q'' in $\pi_2^{-1}(y)$, and r in $\pi_3^{-1}(z)$ such that $(p, q') \in N_{12}$ and $(q'', r) \in N_{23}$. Moreover,

$$\begin{aligned} 0 &= \langle (0, v), \gamma_{12}(x, y) \rangle + \langle (v, 0), \gamma_{23}(y, z) \rangle \\ &= \langle (0, v), \rho_{12}(p, q') \rangle + \langle (v, 0), \rho_{23}(q'', r) \rangle \\ &= -\langle v, \alpha_2(q') \rangle + \langle v, \alpha_2(q'') \rangle \\ &= \langle v, \alpha_2(q'') - \alpha_2(q') \rangle. \end{aligned}$$

The tangent spaces $T_{(x,v)}K_{12}$ and $T_{(v,z)}K_{23}$ can be identified with the subspaces $T_{(x,v)}K_{12} \times \{0\}$ and $\{0\} \times T_{(v,z)}K_{23}$, respectively, of the vector space $T_x X_1 \times T_v X_2 \times T_z X_3$. Vectors $v \in T_v X_2$ such that $(0, v) \in T_{(x,v)}K_{12}$ and $(v, 0) \in T_{(v,z)}K_{23}$ form a vector space which can be identified with $(T_{(x,v)}K_{12} \times \{0\}) \cap (\{0\} \times T_{(v,z)}K_{23})$. Further, the covectors $(\alpha_1(p), -\alpha_2(q')) \in T_x^* X_1 \times T_v^* X_2$ and $(-\alpha_2(q''), \alpha_3(r)) \in T_v^* X_2 \times T_z^* X_3$ can be extended to the covectors $(\alpha_1(p), -\alpha_2(q'), 0)$ and $(0, -\alpha_2(q''), \alpha_3(r))$ in $T_x^* X_1 \times T_v^* X_2 \times T_z^* X_3$. The condition $\langle v, \alpha_2(q'') - \alpha_2(q') \rangle = 0$ whenever $(0, v) \in T_{(x,v)}K_{12}$ and $(v, 0) \in T_{(v,z)}K_{23}$ implies that the restrictions of the covectors $(\alpha_1(p), -\alpha_2(q'), 0)$ and $(0, -\alpha_2(q''), \alpha_3(r))$ to $(T_{(x,v)}K_{12} \times \{0\}) \cap (\{0\} \times T_{(v,z)}K_{23})$ are equal. Hence, there exists a covector $(\alpha_1(\tilde{p}), -\alpha_2(\tilde{q}), \alpha_3(\tilde{r})) \in T_x^* X_1 \times T_v^* X_2 \times T_z^* X_3$ such that the restrictions of $(\alpha_1(\tilde{p}), -\alpha_2(\tilde{q}), \alpha_3(\tilde{r}))$ and $(\alpha_1(p), -\alpha_2(q'), 0)$ to $T_{(x,v)}K_{12} \times \{0\}$ are equal, and the restrictions of $(\alpha_1(\tilde{p}), -\alpha_2(\tilde{q}), \alpha_3(\tilde{r}))$ and $(0, -\alpha_2(q''), \alpha_3(r))$ to $\{0\} \times T_{(v,z)}K_{23}$ are equal. Therefore, for each $(u, v) \in T_{(x,v)}K_{12}$,

$$\begin{aligned}
\langle (u, v), \rho_{12}(\tilde{p}, \tilde{q}) \rangle &= \langle u, \alpha_1(\tilde{p}) \rangle - \langle v, \alpha_2(\tilde{q}) \rangle \\
&= \langle (u, v, 0), (\alpha_1(\tilde{p}), -\alpha_2(\tilde{q}), \alpha_3(\tilde{r})) \rangle \\
&= \langle (u, v, 0), (\alpha_1(p), -\alpha_2(q'), 0) \rangle \\
&= \langle (u, v), \rho_{12}(p, q') \rangle \\
&= \langle (u, v), \gamma_{12}(x, y) \rangle,
\end{aligned}$$

which implies that $(\tilde{p}, \tilde{q}) \in N_{12}$. Similarly, $(\tilde{q}, \tilde{r}) \in N_{23}$, and hence $(\tilde{p}, \tilde{r}) \in N_{23}$. Since $x = \pi_1(\tilde{p})$ and $z = \pi_3(\tilde{r})$ it follows that $(x, z) \in K_{13}$.

Let (x, z) be in K_{13} and let $(p, r) \in N_{13}$ be such that $x = \pi_1(p)$ and $z = \pi_3(r)$. Since $N_{13} = N_{23} \circ N_{12}$, there exists a $q \in P_2$ such that $(p, q) \in N_{12}$ and $(q, r) \in N_{23}$. Let $y = \pi_2(q)$, then $(x, y) \in K_{23}$ and $(y, z) \in K_{23}$. For an arbitrary $(u, w) \in T_{(x, z)}K_{13}$ there exists a $v \in T_{y, X_2}$ such that $(u, v) \in TK_{23}$ and $(v, w) \in TK_{23}$. Hence

$$\begin{aligned}
\langle (u, w), \gamma_{13}(x, z) \rangle &= \langle (u, w), \rho_{13}(p, r) \rangle \\
&= \langle u, \alpha_1(p) \rangle - \langle w, \alpha_3(r) \rangle \\
&= \langle u, \alpha_1(p) \rangle - \langle v, \alpha_2(q) \rangle + \langle v, \alpha_2(q) \rangle - \langle w, \alpha_3(r) \rangle \\
&= \langle (u, v), \gamma_{12}(x, y) \rangle + \langle (v, w), \gamma_{23}(y, z) \rangle.
\end{aligned}$$

This proves part (ii) of the proposition.

Notice that the last equality above gives

$$\langle (0, v), \gamma_{12}(x, y) \rangle + \langle (v, 0), \gamma_{23}(y, z) \rangle = 0$$

for $u = w = 0$. This completes the proof of part (i) of the proposition.

4.5. Example. Let $\alpha: \mathbf{R} \times \mathbf{R} \rightarrow T^*\mathbf{R}$ be an identification of the cotangent bundle space of the real line $\mathbf{R}$ with $\mathbf{R} \times \mathbf{R}$ such that $\pi_{\mathbf{R}} \circ \alpha = pr_1$. Then $(\mathbf{R} \times \mathbf{R}, \mathbf{R}, pr_1, \alpha^*\theta_{\mathbf{R}}, \alpha)$ is a special symplectic manifold.

Let N_{12} be a relation from $\mathbf{R} \times \mathbf{R}$ to itself defined by $N_{12} = \{((x, p), (y, q)) \in (\mathbf{R} \times \mathbf{R}) \times (\mathbf{R} \times \mathbf{R}) \mid x^2 - y^2 = 1, py - qx = x\}$. It is a symplectic relation with respect to the special symplectic structure in $\mathbf{R} \times \mathbf{R}$ defined above. The projection K_{12} of N_{12} is a hyperbola, $K_{12} = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x^2 - y^2 = 1\}$, and the generating form γ_{12} of N_{12} is given by $\langle (u, v), \gamma_{12}(x, y) \rangle = v$, for any $(x, y) \in K_{12}$ and any $(u, v) \in T_{(x, y)}K_{12}$. Let N_{23} be another relation from $\mathbf{R} \times \mathbf{R}$ to itself defined by $N_{23} = \{((y, q), (z, r)) \in (\mathbf{R} \times \mathbf{R}) \times (\mathbf{R} \times \mathbf{R}) \mid y^3 - z = 0, q - 3ry^2 = 0\}$. It is a symplectic relation with $K_{23} = \{(y, z) \in \mathbf{R} \times \mathbf{R} \mid y^3 - z = 0\}$ and $\gamma_{23} = 0$. The composition $N_{13} = N_{23} \circ N_{12}$ is given by $N_{13} = \{((x, p), (z, r)) \in (\mathbf{R} \times \mathbf{R}) \times (\mathbf{R} \times \mathbf{R}) \mid x^2 - z^{2/3} = 1, z \neq 0, pz^{1/3} - 3rxz^{2/3} = x\}$. It is a symplectic relation with $K_{13} = \{(x, z) \in \mathbf{R} \times \mathbf{R} \mid x^2 - z^{2/3} = 1, z \neq 0\}$ and γ_{13} defined by $\langle (u, w), \gamma_{13}(x, z) \rangle = xz^{-1/3}u$, for each $(x, z) \in K_{13}$ and each $(u, w) \in T_{(x, z)}K_{13}$. The relations N_{12} , N_{23} and N_{13} satisfy the assumptions of Proposition 4.4. Consequently, K_{13} and γ_{13} satisfy the statements (i) and (ii) of this proposition. In particular, the last condition in (i) excludes from K_{13} the points $(-1, 0)$ and $(1, 0)$ which belong to $K_{23} \circ K_{12}$.

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