The Linearized Equations of Water Waves

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Introduction. Let $x' = (x, y) = (x_1, x_2, \dots, x_{n-1}, y)$ denote a variable point in $\mathbf{R}^n$. Let G be a domain contained in the half space $\{(x, y); x \in \mathbf{R}^{n-1}, y < 0\}$ whose boundary consists of two parts, Γ and Γ' . Γ is an $(n - 1)$ -dimensional subdomain of the plane $y = 0$ and is conceived of as the free surface of a fluid that fills G . Γ' is a fixed boundary through which no fluid can flow. Let

$$\Delta = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y^2},$$

and let n denote the outward normal to Γ' . Let subscripts denote differentiation. We seek a function $\mu(x', t)$ defined on $\bar{G} \times [0, \infty]$ satisfying the following equations:

$$(0.1) \quad \Delta \mu = 0 \quad \text{for } x' \in G, \quad t \geq 0,$$

$$(0.2) \quad \mu_n = 0 \quad \text{for } x' \in \Gamma', \quad t \geq 0,$$

$$(0.3) \quad \mu_\nu + \mu_{tt} = 0 \quad \text{for } x' \in \Gamma, \quad t \geq 0,$$

$$(0.4) \quad \mu = \mu^0, \quad \mu_t = \mu_t^0 \quad \text{for } x' \in \Gamma, \quad t = 0.$$

Here μ^0, μ_t^0 are given functions, *i.e.*, the initial conditions of the problem. We think of t as time, which makes its sole appearance as a parameter in the boundary conditions. We call (0.1–0.4) the linearized equations of water waves.

The purpose of this work is to solve (0.1–0.4). In this connection the first general result seems to be that of Finkelstein (see [8]). He considered (0.1–0.4) for $n = 3$, $\Gamma = \mathbf{R}^2$ and G the half-space with a finite number of compact manifolds removed. He explicitly constructed a Green's function for the half-space and used it to reduce the problem of solving (0.1–0.4) to an integral equation. Whereas Finkelstein's construction extends to the case where G is the strip $\{(x, y); x \in \mathbf{R}^{n-1}, -h_0 < y < 0\}$ with a finite number of compact manifolds removed, it certainly fails when G is any more general.

The next result was that of Friedman and Shinbrot (see [3]). They have shown that in the case of so-called normal domains, (0.1–0.4) have a "strong" solution. This means that (0.1), (0.2) and (0.4) are satisfied classically but (0.3) is satisfied

in some weak sense. With a bit of smoothness on the initial data, they have shown (0.3) is satisfied a.e. on Γ . Thus, (0.3) is satisfied classically if μ, μ_y, μ_t are smooth up to the boundary.

The difficulty with (0.3) occurs in the following way: if we assume that $\mu(x', t) = w(x')e^{i\lambda t}$, then $w(x')$ satisfies the following equations:

$$(0.5) \quad \Delta w = 0 \quad \text{for } x' \in G,$$

$$(0.6) \quad w_n = 0 \quad \text{for } x' \in \Gamma',$$

$$(0.7) \quad w_y = \lambda^2 w \quad \text{for } x' \in \Gamma.$$

This is an eigenvalue problem. In general, the solutions to this problem have singularities at $\partial\Gamma$. In particular, if $n = 2$ and $G = \{(x, y); x > 0, 0 > y > -ax\}$ for some $a > 0$, i.e., G is a wedge, then the solution to (0.5–0.7) has a logarithmic singularity at (0,0) (see [8]). Thus, in general, we cannot expect to prove μ is continuous in $\bar{G}$ for all $t \geq 0$. The best we can try for is smoothness in $\bar{G} \setminus \partial\Gamma$.

Physically this means that if we allow singularities at $\partial\Gamma$ when $t = 0$, do they propagate inward? In this connection Garipov [5] has proved the next result. If $n = 3$, $\Gamma = \mathbf{R}^2$ and Γ' is bounded away from Γ , then the solution is as smooth as the initial data. He even allowed Γ' to vary with time. Physically Garipov has proved that if the initial data have no singularities, then none are propagated in from infinity.

Shinbrot and Friedman then proved a result similar to Garipov's (see [4]). They proved that if $\Gamma = \mathbf{R}^{n-1}$ and Γ' is bounded away from Γ , then the "strong" solution (0.1–0.4) of [3] is almost as smooth as the initial data. (The "almost" is explained in Chapter 2.) When $n = 3$, Friedman's and Shinbrot's results are contained in those of Garipov.

In this paper we extend the previous results in two different directions. In the first place the results of [4] and [5] assume that Γ' is bounded away from Γ . We allow $\bar{\Gamma}'$ to intersect $\bar{\Gamma}$ vertically. This result is valid for G bounded or unbounded. Thus, if $\bar{\Gamma}'$ intersects $\bar{\Gamma}$ vertically, then smooth initial data which satisfy certain "wave conditions" at $\partial\Gamma$ give rise to solutions which are as smooth as the data will allow.

In the second case $n = 2$, Γ' is given by straight line segments in a neighborhood of Γ and the angles formed by $\bar{\Gamma}'$ and $\bar{\Gamma}$ are less than $\pi/2$. With these hypotheses the solution to (0.1–0.4) is as smooth as the initial data, up to a certain point. We then weaken the hypotheses on Γ' slightly and prove the existence and uniqueness of a classical solution. These results are valid for G bounded or unbounded.

The essential technique used is separation of variables. In Chapter 1 we separate the time variable to reduce (0.1–0.4) to a stationary problem. In Chapter 2 we separate the y variable. This allows us to find a representation for solutions to the stationary problem. In Chapter 3 we separate variables in polar coordinates. We are then able to represent solutions to the stationary problem in two dimensions.

Chapter 1. In this chapter we define a transformation T as follows: Given $f \in L^2(\Gamma)$, let U be a solution of

$$\begin{aligned}\Delta U &= 0 & \text{for } x' \in G, \\ U_n &= 0 & \text{for } x' \in \Gamma', \\ U_\nu + U &= f & \text{for } x' \in \Gamma.\end{aligned}$$

Then set $(Tf)(x) = U(x, 0)$. T is a bounded self-adjoint positive operator; hence, it has an inverse. Define $A = T^{-1}$. Using the spectral family of A , we construct a weak solution of (0.1–0.4). We then prove the weak solution is smooth if the initial data are in $D(A^s)$ for large s . A new uniqueness theorem is proved for unbounded domains.

Several of the results of this chapter are similar to known results (see [3]). We present the new proofs for the light they shed on the problem for normal domains and for the fine points which are cleared up.

Chapter 2. Let

$$\begin{aligned}\Delta' &= \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2}, \\ \Delta' &= \left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}} \right].\end{aligned}$$

Assume $\bar{\Gamma}'$ intersects $\bar{\Gamma}$ vertically. Consider the following problem: if $f \in L^2(\Gamma)$, then

$$\begin{aligned}\Delta' v &= f & \text{for } x \in \Gamma, \\ v_n &= 0 & \text{for } x \in \partial\Gamma,\end{aligned}$$

where n denotes the outward normal to $\partial\Gamma$. $f \rightarrow v$ is a bounded self-adjoint positive operator. Let B be its inverse. We use the spectral family of B to prove $D(B^s) = D(A^{2s})$. This result is then used to prove the weak solution is smooth for appropriate initial conditions. These results are also valid for bounded or unbounded domains.

Chapter 3. In this chapter $n = 2$ but G is allowed to be a little more general than in Chapter 2. The existence and uniqueness of a classical solution is proved if the initial data are smooth.

The methods of Chapter 1 and Chapter 2 can be extended to apply to any problem where Δ is replaced by any self-adjoint partial differential operator with constant coefficients with (0.3) and (0.4) replaced by

$$\begin{aligned}P(\partial/\partial t)\mu + \mu_\nu &= 0 & \text{for } x' \in \Gamma, \quad t \geq 0, \\ (\partial^i/\partial t^i)\mu &= \mu_i^0 & \text{for } x' \in \Gamma, \quad t = 0 \quad (i = 0, 1, \dots, s),\end{aligned}$$

where $P(\lambda)$ is any polynomial of degree $s + 1$ all of whose zeros have non-

negative real parts (see [3]). We will not concern ourselves with this problem, but it is easily handled using the techniques of Chapters 1 and 2.

Chapter 1. Weak solutions. In this chapter we prove the existence and uniqueness of a weak solution to (0.1–0.4). We then derive some general smoothness conditions. In Section 1 we define the basic type of domain and prove some lemmas concerning function spaces. In Section 2 we define an operator A and prove the existence and uniqueness of a weak solution. In Section 3 we prove the result which relates the condition $\mu^0 \in D(A^s)$, $\mu_i^0 \in D(A^{s-1/2})$ and smoothness of the weak solution.

1. Normal domains.

Definition. Let G be a domain in $\mathbb{R}^n$. Suppose $G = \Omega \cap \{(x, y); y < 0\}$. We say G is a *normal domain* if Ω is a domain satisfying the following conditions: Ω is a bounded C^{2+p} ($0 < p < 1$) domain, whose boundary consists of two parts. The boundary of Ω consists of an exterior part $\mathfrak{M}$ and a finite number of compact manifolds which bound domains d_i ($i = 1, 2, \dots, i_0$) in the interior of Ω . Assume the intersection of each $\bar{d}_i$ with the hyperplane $y = 0$ is a compact manifold (with boundary) of dimension less than n . Assume the intersection of $\mathfrak{M}$ with the plane $y = 0$ has a finite number of connected components, each of which is a manifold of dimension less than n .

Let Γ be the intersection of Ω with $y = 0$. We assume Γ is non-empty and forms an $(n - 1)$ dimensional manifold.

This is the definition of a normal domain. Let $\Gamma' = \partial G \setminus \bar{\Gamma}$. We call Γ' the bottom and Γ the top. At different times we will need to differentiate in the direction of an outward normal. $\partial/\partial n$ or a subscript n will mean differentiation with respect to the outward normal of Γ' or $\partial\Gamma$. It will be made clear which meaning is intended. The letter c will be used throughout for a generic constant.

Definition. We say G is an *unbounded normal domain* if the following conditions are satisfied: G is an unbounded domain. There exists an increasing sequence of normal domains $\{G_i\}$ such that $G_i \cap \{(x, y); |x| < a_i\} = G \cap \{(x, y); |x| < a_i\}$ for an increasing unbounded sequence of positive numbers $\{a_i\}$. Assume there exists an $R > 0$ and $d > 0$ such that the boundary of $G \cap \{(x, y); |x| > R\}$ consists of three manifolds; the first one lies on $|x| = R$, the second, called Γ^* , lies on $y = 0$, and the third is given by $y = -h(x)$, where $h(x)$ is defined on Γ^* . We assume $0 < h(x) \leq d$ on the interior of Γ^* and $h \in C^{2+p}(\Gamma^*)$. Assume the first and second derivatives of h are uniformly bounded in Γ^* . Γ is defined to be the union of the Γ_i 's associated with the G_i 's. This is the definition of an unbounded normal domain.

This definition of unbounded normal domain is more restrictive than that given in [3]. Friedman and Shinbrot do not assume the derivatives of h are uniformly bounded. This assumption is needed for some of the later estimates, so I assume it now for convenience.

In the rest of the paper we will be consistently concerned with functions defined on G or Γ and the restrictions of such functions. Functions defined on Γ will be denoted by small letters such as f, g , etc. Functions defined on G will be denoted by capitals such as V, W , etc. The only exception is the function $\mu(x', t)$ which appears in (0.1–0.4). If V is defined in G , its restriction to Γ (in a sense which will be made precise in each case) is denoted by v^Γ . In the case of μ appearing in (0.1–0.4) its restriction to Γ is denoted by μ^Γ . If V is defined on G and $b > 0$, the restriction of V to $\{(x, y); y = -b\} \cap G$ is denoted by v^b . A similar notation for the restriction of $\mu(x', t)$ to the plane $y = -b$ will be used, i.e., $\mu^b(x, t)$.

Let M be an open set contained in $\mathbf{R}^n$ or $\mathbf{R}^{n-1}$ and k a non-negative integer. The space $H^k(M)$ is the set of all complex-valued functions defined on M with k distributional derivatives in $L^2(M)$. The norm for $H^k(M)$ is given by

$${}_k||\varphi||_M^2 = \sum_{|\alpha| \leq k} \int_M |D^\alpha \varphi|^2 dz.$$

I will use the notation ${}_k||\cdot||_M$ to represent the norm of $H^k(M)$. Notice that ${}_0||\cdot||_M = ||\cdot||_{L^2(M)}$. $(\cdot, \cdot)_M$ will denote the inner product of $H^0(M) = L^2(M)$ and $(\nabla F, \nabla G)_M = \sum_{|\alpha|=1} (D^\alpha F, D^\alpha G)_M$.

In [1] Lions develops a method for interpolating between Hilbert spaces. We will use this method to define $H^k(M)$ for all real $k \geq 0$ by interpolating between every pair of consecutive non-negative integers. In [1] Lions proves

Theorem 1.1. *Let $W_1 \subset W_0$ and $V_1 \subset V_0$ be Hilbert spaces with continuous injections and W_1 dense in W_0 , V_1 dense in V_0 . Let W_p and V_p , for p between zero and one, be the interpolation spaces defined in [1]. If T maps V_0 into W_0 and V_1 into W_1 continuously, then T maps V_p into W_p continuously.*

Let $H_0^k(\Gamma)$ denote the closure of $C_0^\infty(\Gamma)$ in $H^k(\Gamma)$. If $\Gamma = \mathbf{R}^{n-1}$, then $H_0^k(\Gamma) = H^k(\Gamma)$ for all $k \geq 0$. If $f \in H_0^k(\Gamma)$ for some $k \geq 0$, then $f \in L^2(\Gamma)$. Let $\hat{f}$ denote its Fourier transform. An equivalent norm for $H_0^k(\Gamma)$ is given by

$$\int_{\mathbf{R}^{n-1}} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 d\xi.$$

Lemma 1.2. *Let G be a bounded or unbounded normal domain. If $F \in H'(G)$, an equivalent norm for $H'(G)$ is given by:*

$$(|f^\Gamma|_\Gamma^2 + \|\nabla F\|_G^2)^{1/2}.$$

Friedman and Shinbrot have proved this for G bounded [3]. A slight change in their proof is all that is needed to prove Lemma 1.2. The change needed is in Lemma 1.3 of [3].

In the following sections we will be consistently concerned about functions defined on G and the restrictions of such functions to Γ . The following lemma, which is the main point of this section, sheds some light on the converse problem, i.e., can we extend a function defined on Γ to G ?

Lemma 1.3. *Let $k \geq \frac{1}{2}$ and $f \in H_0^k(\Gamma)$. Then there exists a constant c depending only on G and a function $F(x, y)$ such that:*

- i) $F \in H^{k+1/2}(G)$,
- ii) ${}_{k+1/2}||F||_G \leq c {}_k||f||_\Gamma$,
- iii) $f^\Gamma = f$,
- iv) $\Delta F = 0$ in G .

Proof. We can consider the extension procedure we are about to give as a linear map from $H_0^k(\Gamma)$ to $H^{k+1/2}(G)$. By Theorem 1.1 it is sufficient to prove i)–iii) for k such that $k + \frac{1}{2}$ is a positive integer.

Suppose $k + \frac{1}{2}$ is an integer. If $f \in H_0^k(\Gamma)$, then there exists a sequence $\{f_n\} \subset C_0^\infty(\Gamma)$ converging to f in $H_0^k(\Gamma)$. Define $F_n(x, y)$ by

$$\hat{F}_n(\xi, y) = \hat{f}_n(\xi) e^{y|\xi|},$$

where $\hat{f}_n$ denotes the Fourier transform of f_n . If l is a non-negative integer, then we have

$$(1.1) \quad |D_y^l \hat{F}_n(\xi, y)|^2 \leq |\xi|^{2l} |\hat{f}_n(\xi)|^2 e^{2y|\xi|}.$$

Choose $a > 0$ such that $G \subset \{(x, y); -a < y < 0\}$. Integrating (1.1), we have

$$(1.2) \quad \int_{-a}^0 |D_y^l \hat{F}_n(\xi, y)|^2 dy \leq c |\xi|^{2l-1} |\hat{f}_n(\xi)|^2 \quad \text{for } |\xi| \geq 1, \\ \leq c |\hat{f}_n(\xi)|^2 \quad \text{for } |\xi| \leq 1.$$

Let $W = \{(x, y); -a < y < 0\}$. If m is a non-negative integer, then multiply both sides of (1.2) by $|\xi|^{2m}$ and integrate over $\mathbf{R}^{n-1}$. If α is an $n-1$ multi-index with $|\alpha| = m$, we have by Parseval's theorem,

$$(1.3) \quad {}_0||D_x^\alpha D_y^l F_n||_W^2 \leq c \int_{\mathbf{R}^{n-1}} \int_{-a}^0 |\xi|^{2m} |D_y^l \hat{F}_n(\xi, y)|^2 dy d\xi \\ \leq c \left[\int_{\mathbf{R}^{n-1}} |\xi|^{2m+2l-1} |\hat{f}_n(\xi)|^2 d\xi + \int_{\mathbf{R}^{n-1}} |\hat{f}_n(\xi)|^2 d\xi \right].$$

If $2m + 2l \leq 2k + 1$, then

$$(1.4) \quad {}_0||D_x^\alpha D_y^l F_n||_W^2 \leq c \int_{\mathbf{R}^{n-1}} (1 + |\xi|^2)^k |\hat{f}_n(\xi)|^2 d\xi \\ \leq c {}_k||f_n||_\Gamma^2.$$

If we sum (1.4) over all α and l such that $|\alpha| + l \leq k + \frac{1}{2}$, then

$$(1.5) \quad {}_{k+1/2}||F_n||_G^2 \leq {}_{k+1/2}||F_n||_W^2 \leq c {}_k||f_n||_\Gamma^2.$$

Thus, $F_n \in H^{k+1/2}(G)$.

A similar calculation shows $\{F_n\}$ is Cauchy in $H^{k+1/2}(G)$ since $\{f_n\}$ is Cauchy in $H_0^k(\Gamma)$. Define $F = \lim_{n \rightarrow \infty} F_n$. Hence, i), ii), and iii) are satisfied.

If $k \geq \frac{1}{2}$, then $\{F_n\}$ is Cauchy in $H'(G)$. If $\Phi \in C_0^\infty(G)$, then $(\nabla \Phi, \nabla F_n)_G = 0$ since $\Delta F_n = 0$ in G . Letting $n \rightarrow \infty$, we have $(\nabla \Phi, \nabla F)_G = 0$ for all $\Phi \in C_0^\infty(G)$. Hence, iv) is satisfied (see [7]). This completes the proof.

2. Definition of A . Let $g \in L^2(\Gamma)$. By Lemma 1.2, the functional $\nu(F) = (f^\Gamma, g)_\Gamma$ is defined and continuous on $H'(G)$. Therefore, there exists a unique $U \in H'(G)$ such that

$$(2.1) \quad (f^\Gamma, u^\Gamma)_\Gamma + (\nabla F, \nabla U)_G = (f^\Gamma, g)_\Gamma$$

for all $F \in H'(G)$.

Define an operator by the formula $Tg = u^\Gamma$.

Lemma 2.1. *If G is a normal domain (bounded or unbounded), then T is a bounded self-adjoint positive operator. Furthermore, the norm of T is less than or equal to one.*

Proof. Set $F = U$ in (2.1). Hence,

$$(2.2) \quad {}_0\|u^\Gamma\|_\Gamma^2 + {}_0\|\nabla U\|_G^2 = (Tg, g)_\Gamma.$$

Thus, T is self-adjoint positive. By Schwartz's inequality, and $u^\Gamma = Tg$ we find

$${}_0\|Tg\|_\Gamma^2 \leq {}_0\|Tg\|_\Gamma {}_0\|g\|_\Gamma.$$

Thus, T is bounded. This completes the proof of the lemma.

Friedman and Shinbrot have proved Lemma 2.1 (see [3]). They defined T in a different manner. Later we will show the two definitions are equivalent.

There are several equations involving T which are consequences of (2.1) and (2.2). By Lemma 2.1, T has a positive square root. Let $T^{1/2}$ be the positive square root of T . If $u^\Gamma = Tg$, then we have

$$(2.3) \quad {}_0\|Tg\|_\Gamma^2 + {}_0\|\nabla U\|_G^2 = {}_0\|T^{1/2}g\|_\Gamma^2.$$

Define an operator A by $A = T^{-1}$. If $Au^\Gamma = g$, then from (2.1) we have

$$(2.4) \quad (f^\Gamma, u^\Gamma)_\Gamma + (\nabla F, \nabla U)_G = (f^\Gamma, Au^\Gamma)_\Gamma$$

for all $F \in H'(G)$. We have the following corollary to Lemma 2.1:

Corollary 2.2. *Let G be a normal domain; then A is a positive self-adjoint operator (unbounded).*

If $A^{1/2}$ is the positive square root of A , then $A^{1/2}u^\Gamma = T^{1/2}g$. By (2.3), we have

$$(2.5) \quad {}_0\|u^\Gamma\|_\Gamma^2 + {}_0\|\nabla U\|_G^2 = {}_0\|A^{1/2}u^\Gamma\|_\Gamma^2.$$

The operator A will be the main concern throughout this paper. In this section we will prove some results relating A to the original problem. First we'll need to consider a particular subspace of $H'(G)$.

From Lemma 1.2 we know that

$$(2.6) \quad (f^\Gamma, u^\Gamma)_\Gamma + (\nabla F, \nabla U)_G$$

is an equivalent inner product for $H'(G)$. Let $H'_\Delta(G)$ be the subspace of $H'(G)$ orthogonal (with respect to (2.6)) to all functions in $H'(G)$ whose restriction to Γ is zero. I use the notation $H'_\Delta(G)$ because of the following lemma:

Lemma 2.3. *If $F \in H'_\Delta(G)$, then F is harmonic in G . Furthermore, $F \in C'(G \cup \Gamma')$ and*

$$\partial F / \partial n = 0 \text{ on } \Gamma',$$

where n is the outward normal to Γ' .

Proof. Let $U \in C_0^\infty(G)$. From (2.6) we have

$$(\nabla F, \nabla U)_G = 0,$$

since $u^\Gamma = 0$. Hence, F is harmonic (see [7]).

Let z be a point of Γ' . Let D be the intersection of a closed ball with $\bar{G}$. Take the center to be z and choose the radius so small that the intersection of the ball with ∂G is a connected $n - 1$ dimensional neighborhood of z . Let D' be the intersection of $\bar{G}$ with a ball with center z and half the radius of D . Let S and S' be the spherical portion of the boundaries of D and D' respectively. Let N and N' be the rest of the respective boundaries. Construct a family of sets D_s with the following properties:

$$D_s \subset D \text{ for } 0 \leq s \leq s' \text{ and some } s' > 0, D_0 = D.$$

Each D_s ($s \neq 0$) has a smooth boundary and they decrease monotonically and smoothly with s . The distance between D_s and ∂G is greater than $s/2$. All points of ∂D_s whose distance from ∂G is greater than s , lie on S . Call this portion of the boundary of D_s , S_s . Let $N_s = \partial D_s \setminus S_s$.

Let $p(x, y) \in C^\infty(\bar{G})$ be such that $p(x, y) = 1$ in D' and $p(x, y) = 0$ outside of D . Let R be a fundamental solution of Laplace's equation defined in a neighborhood of D satisfying $\partial R / \partial n = 0$ on N , where n is the outward normal to N . If (x, y) is in the interior of D' , then (x, y) is in the interior of D_s for s sufficiently small. If we apply Green's Theorem to (pR) in D_s , then we have the following representation for F :

$$(2.7) \quad F(x, y) = - \int_{D_s \setminus D'} F \Delta(pR) dx' + \int_{N_s} F(pR)_n dA - \int_{N_s} (F)_n(pR) dA,$$

where the subscript n means differentiation with respect to the outward normal of N_s . Let $s \searrow 0$. The first term becomes an integral over $D \setminus D'$. Since the boundary of G is C^{2+p} in a $\bar{G}$ neighborhood of z , we can choose R so that the first integral is C' up to the boundary in a $\bar{G}$ neighborhood of z . A proof similar to that of Lemma 1.2 will show that F has an L^2 restriction to N . On the other hand $(pR)_n = 0$ for $x' \in N'$ since $p = 1$ for $x' \in D'$ and $R_n = 0$ for $x' \in N$. Thus as $s \searrow 0$ the second term of (2.7) becomes

$$\int_{N \setminus N'} F(pR)_n \, dA,$$

which is C' up to the boundary in a $\tilde{G}$ neighborhood of z .

If I can prove the third term goes to zero, then this will prove F is C' in a $\tilde{G}$ neighborhood of z .

If $\chi \in H'(G)$ and $\chi^\Gamma = 0$, then $(\nabla \chi, \nabla F)_G = 0$. Assume χ has support contained in D . By Green's theorem,

$$(2.8) \quad \int_{D_s} \nabla \chi \cdot \nabla F = \int_{N_s} \chi F_n.$$

Let $s \searrow 0$ in (2.8), then we have

$$(2.9) \quad \lim_{s \searrow 0} \int_{N_s} \chi F_n = 0$$

for all $\chi \in H'(G)$ and support in D . Restrict (pR) to $D \setminus D_{s''}$, where s'' is such that $(x, y) \in D_{s''}$. Extend $(pR)|_{D \setminus D_{s''}}$ to G in such a way that the extension is in $H'(G)$ and has support in D . Set χ equal to this function and we see that the third term goes to zero. Hence, F is C' in a neighborhood of z . Since (2.9) is true for all $\chi \in H'(G)$ with support in D , $\partial F / \partial n = 0$ in a neighborhood of z . This completes the proof of the lemma.

If H is a Hilbert space, then $C^j([0, \infty]; H)$ is the set of functions defined on $[0, \infty)$ with values in H and having j continuous strong derivatives.

Definition. Let $\mu(x', t)$ be a function lying in $H'(G)$ for each fixed $t \geq 0$. Let μ^Γ denote the restriction of μ to Γ . Assume $\mu^\Gamma \in C'([0, \infty); L^2(\Gamma))$. We say μ is a *weak solution* of (0.1–0.4), if for every $F \in H'(G)$,

$$(2.10) \quad (\nabla F, \nabla \mu)_G + \frac{d}{dt} (f^\Gamma, \mu_t^\Gamma)_\Gamma = 0 \quad \text{for } t \geq 0,$$

$$(2.11) \quad \mu^\Gamma(0) = \mu^0,$$

$$(2.12) \quad \mu_t^\Gamma(0) = \mu_t^0.$$

If $\mu^\Gamma \in C^2([0, \infty); L^2(\Gamma))$ and μ satisfies (2.10–2.12), then we say μ is a *strong solution*.

Friedman and Shinbrot [3] have proved the existence of a strong solution if $\mu^0 \in D(A)$ and $\mu_t^0 \in D(A^{1/2})$. If G is bounded, they proved the existence of a weak solution when $\mu^0 \in D(A^{1/2})$ and $\mu_t^0 \in L^2(\Gamma)$. They also proved that if G is bounded and the weak solution satisfies a certain energy equality, then the solution is unique. I will construct a weak solution if G is bounded or unbounded and show it is unique. This closes the gap in [3] between existence and uniqueness. Friedman and Shinbrot partially closed the gap in [4] but I will settle the question completely.

Lemma 2.4. *A function f is an element of $D(A^{1/2})$ if and only if there exists a function $F \in H'(G)$ such that $f^\Gamma = f$. If f has such an extension, then $\|A^{1/2}f\|_\Gamma \leq c_1 \|F\|_G$.*

Proof. If $f \in D(A^{1/2})$, then there exists a sequence $\{f_n\} \subset D(A)$ such that $A^{1/2}f_n \rightarrow A^{1/2}f$. Since $f_n \in D(A)$, there exist $F_n \in H'(G)$ such that $f_n^\Gamma = f_n$ and

$$(2.13) \quad (w^\Gamma, f_n^\Gamma)_\Gamma + (\nabla W, \nabla F_n)_G = (w^\Gamma, Af_n^\Gamma)_\Gamma$$

for all $W \in H'(G)$. Using (2.13) for F_n, F_m , subtracting and setting $W = F_n - F_m$, we then obtain

$$(2.14) \quad \|f_n^\Gamma - f_m^\Gamma\|_\Gamma^2 + \| \nabla(F_n - F_m) \|_G^2 = \|A^{1/2}(f_n - f_m)\|_\Gamma^2.$$

Hence, $\{F_n\}$ is Cauchy in $H'(G)$ since $\{A^{1/2}f_n\}$ is Cauchy in $L^2(\Gamma)$. Hence, there exists F in $H'(G)$ such that $F_n \rightarrow F$ in $H'(G)$. By Lemma 1.2, $f_n^\Gamma \rightarrow f^\Gamma$ in $L^2(\Gamma)$. Furthermore, $f_n \rightarrow f$ in $L^2(\Gamma)$, since $T^{1/2}$ is continuous and $A^{1/2}f_n \rightarrow A^{1/2}f$. Hence, since $f_n^\Gamma = f_n$, we have $f^\Gamma = f$. Thus we have proved the first part of the lemma.

Suppose there exists $F \in H'(G)$ such that $f^\Gamma = f$. Let $\tilde{H}'$ denote the set of functions in $H'(G)$ whose restriction to Γ is zero. $(\nabla W, \nabla F)$ is a continuous linear functional for $W \in \tilde{H}'$. Since $\tilde{H}'$ is a Hilbert space, there exists $\tilde{F} \in \tilde{H}'$ such that $(\nabla W, \nabla F)_G = (\nabla W, \nabla \tilde{F})_G$ for all $W \in \tilde{H}'$. Hence, $F - \tilde{F}$ is an extension of f in $H'_\Delta(G)$ and $\|F - \tilde{F}\|_G \leq 2\|F\|_G$. Call this new extension F' . Thus, f is the restriction of a function in $H'_\Delta(G)$. This extension is unique. Let F' and F'' be two extensions in $H'_\Delta(G)$, then $(\nabla W, \nabla(F' - F''))_G = 0$ for all W in $\tilde{H}'$. Since $F' - F'' \in \tilde{H}'$, set $W = F' - F''$. Thus, $F' - F'' = 0$.

If $u \in D(A)$, then there exists $U \in H'(G)$ such that $u^\Gamma = u$ and

$$(w^\Gamma, u^\Gamma)_\Gamma + (\nabla W, \nabla U)_G = (w^\Gamma, Au)_\Gamma$$

for all $W \in H'(G)$. If $w^\Gamma = 0$, then

$$(\nabla W, \nabla U)_G = 0.$$

Hence, $U \in H'_\Delta(G)$. Therefore, each element of $D(A)$ has a unique extension to $H'_\Delta(G)$. Call $\widetilde{D(A)}$ the set of these extension. I claim that $\widetilde{D(A)}$ is dense in $H'_\Delta(G)$.

Let V be orthogonal to $\widetilde{D(A)}$ in $H'_\Delta(G)$ with respect to the inner product $(\cdot, \cdot)_\Gamma + (\nabla, \nabla)_G$. Hence,

$$0 = (v^\Gamma, w^\Gamma)_\Gamma + (\nabla V, \nabla W)_G = (v^\Gamma, Aw^\Gamma)_\Gamma$$

for all $W \in \widetilde{D(A)}$. Since A^{-1} is bounded, there exists $w \in D(A)$ such that $Aw = v^\Gamma$.

Therefore, $v^\Gamma = 0$. Hence, $\widetilde{D(A)}$ is dense in $H'_\Delta(G)$. Thus, there exists a sequence

$\{F_n\} \subset D(A)$ such that $F_n \rightarrow F'$ in $H'_\Delta(G)$. (2.14) is still valid with f_n, f_m replaced by f_n^Γ, f_m^Γ respectively. Hence, $\{A^{1/2}f_n^\Gamma\}$ is Cauchy in $L^2(\Gamma)$. By Lemma 1.2, $\{f_n^\Gamma\}$ converges to f in $L^2(\Gamma)$. Since A is closed, $f \in D(A^{1/2})$ and $\lim_{n \rightarrow \infty} A^{1/2}f_n^\Gamma = A^{1/2}f$. Using (2.5) for F_n and taking limits we find

$${}_0\|A^{1/2}f\|_\Gamma^2 = {}_0\|f\|_\Gamma^2 + {}_0\|\nabla F'\|_G^2 \leq c_1\|F\|_G^2,$$

where the second inequality follows from Lemma 1.2 and the estimate for ${}_1\|F'\|_G$. This completes the proof of the lemma.

The proof of the lemma shows that

Lemma 2.5. *Every function f in $D(A^{1/2})$ has a unique extension F' to $H'_\Delta(G)$ and ${}_1\|F'\|_G \leq c_0\|A^{1/2}f\|_\Gamma$.*

Lemma 2.6. $H_0^{1/2}(\Gamma) \subset D(A^{1/2})$.

Proof. This follows from Lemmas 1.3 and 2.4.

Remark. Friedman and Shinbrot [3] have shown that if G is bounded, then a function is in $H'(G)$ if and only if its restriction to Γ is an element of $H^{1/2}(\Gamma)$. Hence, $D(A^{1/2}) = H^{1/2}(\Gamma)$.

Theorem 2.7. *Let G be a normal domain (bounded or unbounded). If $\mu^0 \in D(A^{1/2})$ and $\mu_t^0 \in L^2(\Gamma)$, then there exists a unique weak solution of (0.1–0.4). Furthermore,*

$$(2.15) \quad {}_0\|\nabla \mu(t)\|_G^2 + {}_0\|\mu_t^\Gamma(t)\|_\Gamma^2 = {}_0\|\nabla \mu(0)\|_G^2 + {}_0\|\mu_t^0\|_\Gamma^2,$$

and

$$\mu(t) \in C^0([0, \infty); H'_\Delta(G)) \text{ for } t.$$

Proof. Let $\{E_s\}$ be the spectral family of A . $f \in D(A^{1/2})$ if and only if $\int_1^\infty s {}_0\|dE_s f\|_\Gamma^2 < \infty$. Define $\mu^\Gamma(t)$ by the formula,

$$(2.16) \quad \mu^\Gamma(t) = \int_1^\infty \cos(t(s-1)^{1/2}) dE_s \mu^0 + \int_1^\infty \frac{\sin(t(s-1)^{1/2})}{(s-1)^{1/2}} dE_s \mu_t^0.$$

(2.11) and (2.12) are easily seen to be satisfied. We'll now prove that $\mu^\Gamma(t)$ is an element of $C^0([0, \infty); D(A^{1/2}))$. We'll need to estimate

$${}_0\|A^{1/2}\mu^\Gamma(t) - A^{1/2}\mu^\Gamma(t')\|_\Gamma \text{ as } t' \rightarrow t.$$

Thus,

$$(2.17) \quad \begin{aligned} & \left\| \int_1^\infty s^{1/2} [\cos(t(s-1)^{1/2}) - \cos(t'(s-1)^{1/2})] dE_s \mu^0 \right\|_\Gamma^2 \\ & \leq 4 \left[\int_1^N s [\cos(t(s-1)^{1/2}) - \cos(t'(s-1)^{1/2})]^2 {}_0\|dE_s \mu^0\|_\Gamma^2 \right. \\ & \quad \left. + \int_N^\infty s {}_0\|dE_s \mu^0\|_\Gamma^2 \right]. \end{aligned}$$

Let $\epsilon > 0$ and choose N so large that the second integral of (2.17) is less than $(\epsilon/8)^2$. Cosine is uniformly continuous for bounded argument. Choose $\delta > 0$ such that the first integral of (2.17) is less than $(\epsilon/8)^2$ when $|t - t'| < \delta$. The second integral of (2.16) is estimated similarly. Hence, ${}_0\|A^{1/2}\mu^\Gamma(t) - A^{1/2}\mu^\Gamma(t')\|_\Gamma \leq \epsilon$ when

$|t - t'| < \delta$. Thus, $\mu^\Gamma \in C^0([0, \infty); D(A^{1/2}))$. Since ${}_0\|\mu^\Gamma\|_\Gamma \leq {}_0\|A^{1/2}\mu^\Gamma\|_\Gamma$, $\mu^\Gamma \in C^0([0, \infty); L^2(\Gamma))$. In the same way as above, we can show $\mu_t^\Gamma \in C^0([0, \infty); L^2(\Gamma))$. Hence, $\mu^\Gamma \in C^1([0, \infty); L^2(\Gamma))$. For each fixed t , $\mu^\Gamma(t) \in D(A^{1/2})$, so by Lemma 2.5, $\mu^\Gamma(t)$ has a unique extension to $H'_\Delta(G)$. Call this $\mu(t)$. Equation (2.14) is still valid with $\mu(t) = F_n$ and $\mu(t') = F_m$. Thus,

$${}_0\|\mu^\Gamma(t) - \mu^\Gamma(t')\|_\Gamma + {}_0\|\nabla(\mu(t) - \mu(t'))\|_\mathcal{G}^2 = {}_0\|A^{1/2}(\mu^\Gamma(t) - \mu^\Gamma(t'))\|_\Gamma^2.$$

Hence, as $t \rightarrow t'$, $\mu(t) \rightarrow \mu(t')$ in $H'(G)$. Since $\mu(t) \in H'_\Delta(G)$ for each fixed t , we find $\mu(t)$ is an element of $C^0([0, \infty); H'_\Delta(G))$.

If $W \in H'(G)$ and $f \in D(A)$, then $(w^\Gamma, Af)_\Gamma = (A^{1/2}w^\Gamma, A^{1/2}f)_\Gamma$ since $A^{1/2}$ is self-adjoint. If F is the unique extension of f to $H'_\Delta(G)$, then from 2.4 we have

$$(2.18) \quad (w^\Gamma, f)_\Gamma + (\nabla w, \nabla F)_\mathcal{G} = (A^{1/2}w^\Gamma, A^{1/2}f)_\Gamma$$

for all $W \in H'(G)$. Since $D(A)$ is dense in $D(A^{1/2})$, (2.18) is valid for all $f \in D(A^{1/2})$. Thus we have

$$(2.19) \quad (\nabla w, \nabla \mu)_\mathcal{G} = (A^{1/2}w^\Gamma, A^{1/2}\mu^\Gamma)_\Gamma - (w^\Gamma, \mu^\Gamma)_\Gamma$$

for all $W \in H'(G)$. Hence, μ satisfies (2.10) if μ^Γ satisfies

$$(2.20) \quad (A^{1/2}w^\Gamma, A^{1/2}\mu^\Gamma)_\Gamma + \frac{d}{dt}(w^\Gamma, \mu_t^\Gamma)_\Gamma = (w^\Gamma, \mu^\Gamma)_\Gamma$$

for all $W \in H'(G)$. An easy computation shows that μ^Γ satisfies (2.20). Therefore, μ is a weak solution. Set $W = \mu(t)$ in (2.19) and we obtain

$$(2.21) \quad {}_0\|\nabla \mu(t)\|_\mathcal{G}^2 = {}_0\|A^{1/2}\mu^\Gamma(t)\|_\Gamma^2 - {}_0\|\mu^\Gamma(t)\|_\Gamma^2.$$

Thus, (2.15) is satisfied if

$$(2.22) \quad {}_0\|A^{1/2}\mu^\Gamma(t)\|_\Gamma^2 - {}_0\|\mu^\Gamma(t)\|_\Gamma^2 + {}_0\|\mu_t^\Gamma(t)\|_\Gamma^2 \\ = {}_0\|A^{1/2}\mu^\Gamma(0)\|_\Gamma^2 - {}_0\|\mu^\Gamma(0)\|_\Gamma^2 + {}_0\|\mu_t^\Gamma(0)\|_\Gamma^2$$

is satisfied. A simple computation using (2.16) shows that (2.22) is satisfied.

All that remains to be proven is uniqueness. Let $\mu(t)$ be any weak solution such that $\mu^\Gamma(0) = \mu_t^\Gamma(0) = 0$. μ^Γ satisfies (2.20). Thus,

$$(Aw^\Gamma, \mu^\Gamma)_\Gamma + \frac{d}{dt}(w^\Gamma, \mu_t^\Gamma)_\Gamma = (w^\Gamma, \mu^\Gamma)_\Gamma$$

for all $W \in H'(G)$ such that $w^\Gamma \in D(A)$. Integrating this equation from 0 to t we find that

$$(Aw^\Gamma, \bar{\mu}^\Gamma)_\Gamma + (w^\Gamma, \bar{\mu}_t)_\Gamma = (w^\Gamma, \bar{\mu}^\Gamma)_\Gamma,$$

where

$$\bar{\mu}^\Gamma = \int_0^t \mu^\Gamma(s) ds.$$

Since $\bar{\mu}^\Gamma$ is an element of $C^2([0, \infty); L^2(\Gamma))$ and every element of $D(A)$ has a unique extension to $H'_\Delta(G)$, we find that $\bar{\mu}^\Gamma$ is an element of $D(A)$ for each $t \geq 0$. Thus,

$$\bar{\mu}_{it}^\Gamma + (A - I)\bar{\mu}^\Gamma = 0.$$

Taking the inner product of this equation with $\bar{\mu}_t^\Gamma$ and integrating from 0 to t , we find

$$o||\bar{\mu}_t^\Gamma||_\Gamma^2 + o||(A - I)^{1/2}\bar{\mu}^\Gamma||_\Gamma^2 = 0.$$

Thus, $\bar{\mu}_t^\Gamma = 0$. Since $\bar{\mu}^\Gamma(0) = 0$, $\bar{\mu}^\Gamma(t) = 0$. Thus $\mu^\Gamma(t) = 0$. From (2.10) we see that $\mu \in H'_\Delta(G)$ for each $t \geq 0$. Thus by Lemma 2.5, $\mu(t) = 0$ for $t \geq 0$. This completes the proof of the theorem.

3. Smoothness conditions. To find sufficient conditions for smoothness we need a more useful characterization of $D(A)$. Let W be in $L^2(G)$, such that W restricted to Γ_b ($\Gamma_b = G \cap \{(x, y); y = -b\}$) is in $L^2(\Gamma_b)$ for every b such that $-b_0 < b < 0$ for some $b_0 > 0$. The restriction to Γ_b is denoted by w^b . We say W has a weak L^2 restriction to Γ if there exists a w^Γ in $L^2(\Gamma)$ such that

$$\lim_{b \searrow 0} \int_{\Gamma_b} w^b f^b dx = \int_\Gamma w^\Gamma f^\Gamma dx$$

for all $f \in H'(G)$. If $f \in L^2(\Gamma)$, consider the problem

$$(3.1) \quad \Delta U = 0 \quad \text{for } x' \in G,$$

$$(3.2) \quad \mu^\Gamma = f,$$

$$(3.3) \quad U_n = 0, \quad x' \in \Gamma',$$

$$(3.4) \quad U \in H'(G).$$

The following lemma is similar to a result proved in [4]. The difference is in the definition of weak L^2 restriction to Γ , not in the statement of the lemma.

Lemma 3.1. *Let G be a normal domain (bounded or unbounded). If (3.1–3.4) has a solution and U_y has a weak L^2 restriction to Γ , then $f \in D(A)$, and*

$$(3.5) \quad Af = u_y^\Gamma + u^\Gamma.$$

Conversely, if $f \in D(A)$, then (3.1–3.4) has a solution U , the weak L^2 restriction to Γ of U_y exists and (3.5) is satisfied.

Proof. Let $f \in L^2(\Gamma)$ be such that (3.1–3.4) has a solution and U_y has a weak L^2 restriction to Γ . Define g by the formula $g = u_y^\Gamma + u^\Gamma$. Let $G_b = G \cap \{(x, y); y < -b\}$. Let $W \in H'(G)$. Apply Green's Theorem to W and U to find that

$$(3.6) \quad \int_{G_b} \nabla W \cdot \nabla U dx' = \int_{\Gamma_b} w^b u_y^b dx.$$

Let $b \searrow 0$ in (3.6). We find that

$$(\nabla W, \nabla U)_G = (w^\Gamma, u_y^\Gamma)_\Gamma = (w^\Gamma, g - u^\Gamma)_\Gamma$$

for all W in $H'(G)$. By (3.2) and the definition of T , we find $Tg = f$. Thus (3.5) is satisfied.

Let $f \in D(A)$. Since $D(A^{1/2}) \supset D(A)$, by Lemmas 2.3 and 2.5, there exists a U satisfying (3.1–3.4). Now I will show U_y has a weak L^2 restriction to Γ . If $W \in H'(G)$, then (3.6) is still satisfied. Using (3.6) and (2.4), we find

$$\int_{G_b} \nabla W \cdot \nabla U \, dx' - (\nabla W, \nabla U)_G = \int_{\Gamma_b} w^b u_y^b \, dx - (w^\Gamma, Au^\Gamma - u^\Gamma)_\Gamma.$$

Let $b \searrow 0$. The limit on the left exists, hence the limit on the right exists. Therefore, U_y has a weak L^2 restriction to Γ and $u_y^\Gamma = Au^\Gamma - u^\Gamma$. This completes the proof of the lemma.

Lemma 3.2. *Let $K \subset M \subset \Gamma$ be open sets such that K has a closure which is a positive distance from ∂M and M has a closure which is a positive distance from $\partial \Gamma$. Then there exists a constant depending on G, k, K and M , such that*

$$(3.7) \quad {}_{k+1} \|Tf\|_K \leq c_k \|f\|_M + o \|f\|_\Gamma$$

for all $f \in L^2(\Gamma)$, such that f restricted to M is in $H^k(M)$.

Proof. By Theorem 1.2, it is sufficient to prove the lemma for k a non-negative integer.

Let $f \in L^2(\Gamma)$, such that f restricted to M is in $H^k(M)$. Multiply f by a fixed C^∞ function equal to 1 in a neighborhood of $\bar{K}$ and with support contained in M . Call this new function g . $g \in H_0^k(\Gamma)$ and ${}_k \|g\|_\Gamma \leq c_k \|f\|_M$. Let $\hat{g}$ be the Fourier transform of g . Define W by the formula

$$(3.8) \quad \hat{W}(\xi, y) = \frac{\hat{g}(\xi) e^{|\xi|y}}{1 + |\xi|}.$$

Hence, $\Delta W = 0$ in G , $w^\Gamma \in H_0^{k+1}(\Gamma)$, $w_y^\Gamma + w^\Gamma = g$, and

$$(3.9) \quad \begin{aligned} {}_{k+1} \|w^\Gamma\|_\Gamma^2 &\leq c \int_{\mathbb{R}^{n-1}} (1 + |\xi|^2)^{k+1} |\hat{w}^\Gamma(\xi)|^2 \, d\xi \leq c_k \|g\|_\Gamma^2 \\ &\leq c_k \|f\|_M^2. \end{aligned}$$

Let $v^\Gamma = Tf - w^\Gamma$. w^Γ has an extension, (3.8), to G and by Lemma 2.5, Tf has an extension to G . Hence, v^Γ has an extension, V , to G such that $\Delta V = 0$ in G and $v_y^\Gamma + v^\Gamma = 0$ in a neighborhood of $\bar{K}$. Since V is harmonic and satisfies a homogeneous boundary condition in a neighborhood of $\bar{K}$, v^Γ is real analytic in a neighborhood of $\bar{K}$. Next I need to derive estimates on v^Γ and its derivatives.

Let $x_0 \in K$. Let B_p be the intersection of G with the ball of radius p and center x_0 . For p small enough this is half of a ball. Let S_p be the spherical portion of the boundary of B_p . Choose $p'' > p' > 0$ in such a way that $B_{p''}$ is half of a ball

and $\bar{B}_{p''} \cap \Gamma \subset M$. Let B be the $n - 1$ dimensional ball lying in Γ with center x_0 and radius $p'/2$. Let R be a fundamental solution of Laplace's equation defined in a neighborhood of $B_{p''}$ satisfying $R_\nu + R = 0$ in an $n - 1$ dimensional neighborhood of $\bar{B}_{p''} \cap \Gamma$. If $p'' > p > p'$ and $x \in B$, then we have the representation

$$(3.10) \quad v^\Gamma(x) = V(x, 0) = \int_{S_p} [V(\eta)R_n(x, 0; \eta) - V_n(\eta)R(x, 0; \eta)] dS_\eta,$$

where n denotes the outward normal to S_p . If α is an $n - 1$ dimensional multi-index, then there exists a constant $c > 0$ such that

$$(3.11) \quad \begin{aligned} |D_x^\alpha R| &\leq c \quad \text{for } x \in B, \quad \eta \in S_p, \\ |D_x^\alpha R_n| &\leq c \quad \text{for } x \in B, \quad \eta \in S_p, \end{aligned}$$

for p such that $p'' > p > p'$. If we differentiate both sides of (3.10) and use (3.11) we obtain

$$|D_x^\alpha v^\Gamma| \leq c \int_{S_p} (|V| + |V_n|) dS_\eta.$$

Square both sides of the above equation, integrate x over B , integrate p between p'' and p' and we obtain

$$(3.12) \quad \int_B |D_x^\alpha v^\Gamma|^2 dx \leq c_1 \|V\|_{B_{p''}}^2.$$

If $\bar{K}$ is bounded, a finite number of balls like B cover K . Summing over the B 's we obtain

$$(3.13) \quad \int_K |D_x^\alpha v^\Gamma|^2 dx \leq c_1 \|V\|_\sigma^2.$$

If $\bar{K}$ is unbounded, we can choose a sequence of lattice points so all the associated B 's cover K . Since K is a positive distance from $\partial\Gamma$ we can choose the lattice points such that if we sum both sides of (3.12) over all lattice points, we still obtain (3.13).

Call the extension of Tf to $H'_\Delta(G) \cup U$. By definition of V we have

$$_1 \|V\|_\sigma \leq c[_1 \|U\|_\sigma + _1 \|W\|_\sigma].$$

Use (2.3) and Lemma 1.2 to estimate the first term on the right. Use the argument of Lemma 1.3 to estimate the second term on the right. Thus, we obtain

$$(3.14) \quad \begin{aligned} _1 \|V\|_\sigma &\leq c[_0 \|T^{1/2}f\|_\Gamma + _0 \|g\|_\Gamma], \\ _1 \|V\|_\sigma &\leq c_0 \|f\|_\Gamma. \end{aligned}$$

The second inequality follows since $T^{1/2}$ is bounded and by definition of g .

Sum (3.13) over all α such that $|\alpha| \leq k + 1$. From this and (3.14) we obtain

$$_{k+1} \|v^\Gamma\|_\sigma^2 \leq c_1 \|V\|_\sigma^2 \leq c_0 \|f\|_\Gamma^2.$$

Applying this inequality and (3.9) to ${}_k T f|_K \leq [{}_0 \|v^r\|_K + {}_0 \|w^r\|_K]$ we obtain (3.7). This completes the proof of the lemma.

Lemma 3.3. *Let $K \subset \Gamma$ be an open set lying a positive distance from $\partial\Gamma$. If $f \in D(A^k)$ for some $k \geq 0$, then there exists a constant $c > 0$ independent of f such that*

$$(3.15) \quad {}_k \|f\|_K \leq c {}_0 \|A^k f\|_\Gamma.$$

Proof. By Theorem 1.2 it is sufficient to prove (3.15) for k a non-negative integer. I will use induction on k . When $k = 0$ (3.15) is obvious for all K . Suppose (3.15) is true for $0, 1, 2, \dots, k-1, k$. Choose M as in Lemma 3.2 and use f in place of Tf and Af in place of f in (3.7). Hence,

$${}_{k+1} \|f\|_K \leq c [{}_k \|Af\|_M + {}_0 \|Af\|_\Gamma].$$

We may use the induction hypothesis to estimate the first term and the fact that A satisfies ${}_0 \|g\|_\Gamma \leq {}_0 \|A^k g\|_\Gamma$ to estimate the second. This completes the proof of the lemma.

I need to prove one more crucial lemma before proceeding to the main theorems of this section. This lemma will be the main tool used in proving the smoothness theorem.

Lemma 3.4. *Let $K \subset \bar{G} \setminus \partial\Gamma$ be an open set lying a positive distance from $\partial\Gamma$. If $f \in D(A^k)$ for some $k \geq \frac{1}{2}$, then f has a unique extension U in $H'_\Delta(G)$ and there exists a constant c , independent of f , such that*

$$(3.16) \quad {}_{k+1/2} \|U\|_K \leq c {}_0 \|A^k f\|_\Gamma.$$

Proof. I have already proven the existence and uniqueness of the extension. The main point of the lemma is (3.16). As before, it is sufficient to prove the lemma when $k + \frac{1}{2}$ is a positive integer. There are three cases to consider: 1) $K \subset G$, 2) K intersects Γ' but not Γ , 3) K intersects Γ but not Γ' . All other cases can be treated by partitioning K into appropriate subsets.

Case 1. $K \subset G$. Let $x_0 \in K$. Let B_p , for p such that $0 < p' \leq p \leq p''$, be a family of balls centered at x_0 , with radius p , and contained in G . Let S_p be the boundary of each ball. Let R be a fundamental solution of Laplace's equation defined in a neighborhood of $B_{p''}$. Let B be the ball centered at x_0 with radius $p'/2$. If $x' \in B$, then we have the following representation for U ,

$$U(x') = \int_{S_p} [U(\eta)R_n(x'; \eta) - U_n(\eta)R(x'; \eta)] dS_\eta.$$

If α is an n dimensional multi-index, we can use the argument used to prove (3.12) to obtain the estimate

$$\int_B |D_x^\alpha U|^2 dx' \leq c_1 \|U\|_{B_{p''}}^2.$$

The argument used to prove (3.13) can be used to prove

$$(3.17) \quad \int_K |D_x^\alpha U|^2 dx' \leq c_1 \|U\|_G^2.$$

Sum (3.17) over all α such that $|\alpha| \leq k + \frac{1}{2}$ and we obtain

$$(3.18) \quad {}_{k+1/2} \|U\|_K \leq c_1 \|U\|_G.$$

Hence, by Lemma 2.5 and (3.18) we obtain

$${}_{k+1/2} \|U\|_K \leq c_0 \|A^{1/2} f\|_r.$$

(3.16) follows from ${}_0 \|f\|_r \leq {}_0 \|A^s f\|_r$ for all $s \geq 0$ and the above inequality.

Case 2. The proof is similar to Case 1 but R is taken to have normal derivative zero on $\bar{K} \cap \Gamma'$.

Case 3. The proof in this case is similar to the proof of Lemma 3.2 and Lemma 3.3. The representation (3.10) can be used for $V(x, y)$ in a $\bar{G}$ neighborhood of $\bar{K} \cap \Gamma$. The estimate (3.16) then follows from the arguments of Lemma 3.2 and Lemma 3.3. This completes the proof of the lemma.

The set $D(A^k)$ can be made into a Hilbert space in the following way: if $f \in D(A^k)$, then its norm is given by the formula ${}_0 \|A^k f\|_r$. We use this in the following theorem:

Theorem 3.5. *Let G be a normal domain (bounded or unbounded). If $\mu^0 \in D(A^k)$ and $\mu_t^0 \in D(A^{k-1/2})$ for some $k \geq \frac{1}{2}$, then equations (0.1–0.4) have a weak solution. If K is an open set contained in Γ , lying a positive distance from $\partial\Gamma$, then the restriction of the weak solution to Γ is in $C^j([0, \infty); H^{k-i/2}(K))$, $j = 0, 1, 2, \dots, [2k]$.*

Proof. I constructed a weak solution in Theorem 2.7. The point of this theorem is to show the restriction to Γ of the weak solution has the required smoothness properties. $\mu^\Gamma(t)$ is given by formula (2.16). The argument used in the first part of the proof of Theorem 2.7 can be used to show

$$\mu^\Gamma(t) \in C^j([0, \infty); D(A^{k-i/2})), \quad j = 0, 1, 2, \dots, [2k].$$

We need only to use the fact that $f \in D(A^k)$ if and only if $\int_1^\infty s^{2k} {}_0 \|dE_s f\|_r^2 < \infty$. Let j be an integer between 0 and $[2k]$, and let $m = k - j/2$. By Lemma 3.3, we have

$${}_m \left\| \frac{d^j}{dt^j} \mu^\Gamma(t) - \frac{d^j}{dt^j} \mu^\Gamma(t') \right\|_K \leq c {}_0 \left\| A^m \frac{d^j}{dt^j} \mu^\Gamma(t) - A^m \frac{d^j}{dt^j} \mu^\Gamma(t') \right\|_r.$$

The right side of the above inequality goes to zero as t' approaches t , hence the left side goes to zero. This completes the proof of the theorem. (2.16), Lemma 3.3 and the proof of Theorem 3.5 can be used to show that

Corollary 3.6. *Assume the hypotheses of Theorem 3.5 are satisfied, then if t is bounded $\mu^\Gamma(t)$ satisfies*

$$\left\| \frac{d^j}{dt^j} \mu^r(t) \right\|_K \leq c [{}_0 \|A^k \mu^0\|_r + {}_0 \|A^{k-1/2} \mu_t^0\|_r],$$

$j = 0, 1, 2, \dots, [2k], m = k - j/2.$

Theorem 3.7. *Let G be a normal domain (bounded or unbounded). Let M be an open set contained in $\bar{G} \setminus \partial\Gamma$ and lying a positive distance from $\partial\Gamma$. If $\mu^0 \in D(A^k)$ and $\mu_t^0 \in D(A^{k-1/2})$ for some $k \geq \frac{1}{2}$, then the weak solution of Theorem 3.5 is an element of $C^j([0, \infty); H^{k-i/2+1/2}(M))$, $j = 0, 1, \dots, [2k] - 1$.*

Proof. I will use induction on j . If $j = 0$, from Lemma 3.4 we have

$${}_{k+1/2} \|\mu(t) - \mu(t')\|_M \leq c {}_0 \|A^k \mu^r(t) - A^k \mu^r(t')\|_r.$$

Let $t \rightarrow t'$. By Theorem 3.5 the right side goes to zero, hence the left side goes to zero. This proves the case $j = 0$.

Assume the theorem is true for $0, 1, 2, \dots, j < [2k] - 1$. Let $\mu(t)$ be the solution of Theorem 3.5. Define $\mu_j = d^j \mu / dt^j$. By inductive hypothesis μ_j is an element of $C^0([0, \infty); H^{k-i/2-1/2}(M))$. By Theorem 3.5, $d^{j+1} \mu^r / dt^{j+1}$ exists. Call this μ_{j+1}^r . We also know $\mu_{j+1}^r \in C^0([0, \infty); D(A^{k-i/2-1/2}))$ (see the proof of Theorem 3.5). Since $j < [2k] - 1$, this implies $\mu_{j+1}^r \in C^0([0, \infty); D(A^{1/2}))$. Let $\tilde{\mu}_{j+1}(t)$ be the extension of μ_{j+1}^r given by Lemma 2.5 for each fixed t . If we use the argument of the case $j = 0$, we find $\tilde{\mu}_{j+1} \in C^0([0, \infty); H^{k-i/2}(M))$. If $r = k - \frac{1}{2}j$, then by (3.16) we have

$$\begin{aligned} & \left\| \frac{\mu_j(t) - \mu_j(t')}{t - t'} - \tilde{\mu}_{j+1}(t) \right\|_M \leq c \\ & \left\| A^{r-1/2} \left[\frac{\mu_j^r(t) - \mu_j^r(t')}{t - t'} - \mu_{j+1}^r(t) \right] \right\|_r. \end{aligned}$$

Let $t' \rightarrow t$. Since the limit on the right is zero, the limit on the left is zero. Hence, $\mu_j \in C^1([0, \infty); H^{k-i/2}(M))$. This completes the proof of the theorem.

We can prove any amount of smoothness by assuming the initial data lie in $D(A^k)$ for large k . There is one case of particular interest.

If

$$(3.19) \quad \mu, \Delta\mu, \mu_t, \mu_{tt}$$

are continuous in $\bar{G} \setminus \partial\Gamma \times [0, \infty)$, then we say μ is a classical solution.

By the Sobolev imbedding theorem [7] we have the following corollary:

Corollary 3.8. *Let G be a normal domain (bounded or unbounded). If $\mu^0 \in D(A^k)$ and $\mu_t^0 \in D(A^{k-1/2})$ for some $k > (n+1)/2$, then (0.1–0.4) have a unique classical solution.*

The uniqueness follows from Theorem 2.7.

Chapter 2. Standard domains. In this chapter we study the domain of A when G is a standard domain. In Section 1 we study an auxiliary operator B

and use it to derive a representation for functions harmonic in G . In Section 2 we relate the operators B and A and prove conditions sufficient for the initial data to be in $D(A^k)$ for large k .

1. Definition of B .

Definition. Let G be a normal domain (bounded or unbounded). Suppose there exists $h_0 > 0$ such that $\partial\Gamma \times (-h_0, 0)$ is contained in ∂G and $\Gamma \times (-h_0, 0) \subset G$. Assume further that $\partial\Gamma$ is C^{2+p} and the first and second derivatives of some set of functions which locally represent $\partial\Gamma$ are uniformly bounded. In such a case we call G a *standard domain*. The boundary of $\Gamma \times (-h_0, 0)$ consists of three parts: Γ , the set $\{(x, y); y = -h_0, x \in \Gamma\}$ and $V = \{(x, y); x \in \partial\Gamma, 0 > y > -h_0\}$. This definition generalizes that of a standard domain introduced by Friedman and Shinbrot in [3], where it was assumed that $\Gamma = \mathbb{R}^{n-1}$.

I seek a representation for the function U which satisfies the following problem: if $f, g \in L^2(\Gamma)$, then U satisfies

$$(1.1) \quad \Delta U = 0 \quad \text{in} \quad \Gamma \times (-h_0, 0),$$

$$(1.2) \quad u^\Gamma = f,$$

$$(1.3) \quad u^{h_0} = g,$$

$$(1.4) \quad U_n = 0 \quad \text{on} \quad V.$$

If $f \in L^2(\Gamma)$, then $\alpha(f) = (w, f)_\Gamma$ is a continuous linear functional for all w in $H'(\Gamma)$. Hence, there exists $v \in H'(\Gamma)$ such that

$$(w, v)_\Gamma + (\nabla' w, \nabla' v)_\Gamma = (w, f)_\Gamma$$

for all $w \in H'(\Gamma)$, where $\nabla' = [\partial/\partial x_1, \dots, \partial/\partial x_{n-1}]$. As in Lemma 2.1, the map $f \rightarrow v$ is a bounded self-adjoint positive operator on $L^2(\Gamma)$. Let B denote the inverse of this operator. B also is a self-adjoint positive (unbounded) operator which satisfies

$$(1.5) \quad (w, v)_\Gamma + (\nabla' w, \nabla' v)_\Gamma = (w, Bv)$$

for all $w \in H'(\Gamma)$. Let $\{F_s\}$ be the spectral family of B . Define $U(x, y)$ by the formula

$$(1.6) \quad U(\cdot, y) = \int_1^\infty \frac{\sinh(s-1)^{1/2}(y+h_0)}{\sinh((s-1)^{1/2}h_0)} dF_s f - \int_1^\infty \frac{\sinh(s-1)^{1/2}y}{\sinh((s-1)^{1/2}h_0)} dF_s g.$$

Clearly (1.2) and (1.3) are satisfied. Since $U(\cdot, y) \in D(B)$ for $-h_0 < y < 0$, (1.4) is satisfied in some weak sense. If we can prove (1.1), a proof similar to that of Lemma 2.3 of Chapter 1 will show (1.4) is satisfied in the classical sense. By Weyl's Lemma [7], it is sufficient to prove $(\nabla W, \nabla U)_{\Gamma \times (-h_0, 0)} = 0$ for all $W \in C_0^\infty(\Gamma \times (-h_0, 0))$. From (1.6) we find

$$(1.7) \quad \left[(I - B) + \frac{\partial^2}{\partial y^2} \right] U = 0$$

for each fixed y with $-h_0 < y < 0$. Let $W \in C_0^\infty(\Gamma \times (-h_0, 0))$. Multiply (1.7) by W and integrate over Γ . The result is

$$(1.8) \quad (w'', u'')_\Gamma - (w'', Bu'')_\Gamma + (w'', u''_{yy})_\Gamma = 0.$$

Use (1.5) in (1.8) to find

$$(1.9) \quad -(\nabla' w'', \nabla' u'')_\Gamma + (w'', u''_{yy})_\Gamma = 0.$$

Now, integrate (1.9) with respect to y between the limits $-h_0$ and 0 , and use integration by parts on the second term. (1.1) then follows.

Remark. If Γ is bounded, then B^{-1} is compact. The integral in (1.6) is then a series.

2. The main theorems. In this section we will use the representation derived in the last section to prove the following theorem:

Theorem 2.1. *If G is a standard domain, then $D(A^{2k}) = D(B^k)$ for $k \geq 0$.*

The proof requires several lemmas.

Lemma 2.2. *If $k \geq \frac{1}{2}$, then A maps $D(B^k)$ into $D(B^{k-1/2})$ continuously.*

Proof. Let $f \in D(B^k)$. Define W by the formula

$$(2.1) \quad W(\cdot, y) = \int_1^\infty \exp[y(s-1)^{1/2}] dF_s f.$$

As in Section 1 we can see that W satisfies (1.1) and (1.2). By (2.1), we have $w'' \in D(B)$ for $y < 0$. Set $v = w''$, $w = w''$ and $f = Bw''$ in (1.5). Since B has a self-adjoint square root we find

$$(2.2) \quad {}_0\|w''\|_\Gamma^2 + {}_0\|\nabla' w''\|_\Gamma^2 = {}_0\|B^{1/2} w''\|_\Gamma^2$$

for $y < 0$. From (2.1) we obtain ${}_0\|\partial w''/\partial y\|_\Gamma^2 \leq c {}_0\|B^{1/2} w''\|_\Gamma^2$. We also find that ${}_0\|B^{1/2} w''\|_\Gamma^2 \leq c {}_0\|B^{1/2} f\|_\Gamma^2$. Hence, from (2.2) and the above inequalities we obtain

$$(2.3) \quad {}_0\|\partial w''/\partial y\|_\Gamma^2 + {}_0\|\nabla' w''\|_\Gamma^2 \leq c {}_0\|B^{1/2} f\|_\Gamma^2$$

for $y < 0$. Integrate (2.3) with respect to y between the limits $-h_0$ and 0 . Thus, f has an extension to $H'(\Gamma \times (-h_0, 0))$ whose norm can be estimated in terms of ${}_0\|B^{1/2} f\|_\Gamma$. This extension can be extended to $H'(G)$ and its norm can be estimated in terms of ${}_0\|B^{1/2} f\|_\Gamma$. Hence, by Lemma 2.4 and Lemma 2.5 in Chapter 1, f has a unique extension to $H'_\Delta(G)$ and its $H'(G)$ norm can be estimated in terms of ${}_0\|B^{1/2} f\|_\Gamma$. Call this extension U . Define $g(x) = U(x, -h_0)$. Thus, U satisfies (1.1–1.4). From (1.6) we find that U_y has a weak L^2 restriction to Γ and

(2.4)

$$u_v^\Gamma = \int_1^\infty ((s-1)^{1/2}) \cotanh(h_0(s-1)^{1/2}) dF_s f - \int_1^\infty \frac{(s-1)^{1/2}}{\sinh(h_0(s-1)^{1/2})} dF_s g.$$

Therefore, $u_v^\Gamma \in D(B^{k-1/2})$ and

$$o||B^{k-1/2}u_v^\Gamma||_\Gamma^2 \leq c[o||B^k f||_\Gamma^2 + o||g||_\Gamma^2].$$

Since $g(x) = U(x, -h_0)$, the L^2 norm of g can be estimated in terms of the $H'(G)$ norm of U . This can be estimated in terms of the L^2 norm of $B^{1/2}f$. Hence, the L^2 norm of g can be estimated in terms of the L^2 norm of $B^{1/2}f$. Since B has a bounded inverse and $k \geq \frac{1}{2}$ we have $o||B^{1/2}f||_\Gamma \leq c o||B^k f||_\Gamma$. Thus, the norm of $B^{k-1/2}u_v^\Gamma$ is bounded by a constant times the L^2 norm of $B^k f$. This completes the proof of the lemma.

Lemma 2.3. *If $k \geq 0$, then T maps $D(B^k)$ into $D(B^{k+1/2})$ continuously.*

Proof. Let $f \in D(B^k)$. Let $U \in H'_\Delta(G)$ such that $Tf = u^\Gamma$. Let $U = \Phi + \chi$ in $\Gamma \times (-h_0, 0)$, where

$$(2.5) \quad \Phi(\cdot, y)$$

$$= \int_1^\infty \frac{\sinh[(s-1)^{1/2}(y+h_0)]}{[(s-1)^{1/2} \cotanh(h_0(s-1)^{1/2}) + 1] \sinh(h_0(s-1)^{1/2})} dF_s f.$$

From (2.5) we obtain $(\partial/\partial y)\varphi^\Gamma + \varphi^\Gamma = f$ on Γ . Therefore, by Lemma 3.1, $\chi_v + \chi = 0$ on Γ , in the sense that χ_v has a weak L^2 restriction to Γ . A proof similar to that of Lemma 2.3 of Chapter 1 will show that $\chi_v + \chi = 0$ on Γ in the classical sense; choose the fundamental solution R to satisfy $R_v + R = 0$ on Γ . Thus we have

$$(2.6) \quad \chi(\cdot, y) = \int_1^\infty \frac{(s-1)^{1/2} \cosh(y(s-1)^{1/2}) - \sinh(y(s-1)^{1/2})}{(s-1)^{1/2} \cosh(h_0(s-1)^{1/2}) + \sinh(h_0(s-1)^{1/2})} dF_s g$$

where $g(x) = U(x, -h_0)$. From (2.5) and (2.6) we see that $\varphi^\Gamma \in D(B^{k+1/2})$ and $\chi^\Gamma \in \bigcap_{k \geq 0} D(B^k)$. A simple computation with (2.5) and (2.6) will show

$$o||B^{k+1/2}(Tf)||_\Gamma \leq c o||B^k f||_\Gamma.$$

This completes the proof of the lemma.

We can now proceed to the proof of Theorem 2.1.

Proof. By Theorem 1.1 of Chapter 1, it is sufficient to prove Theorem 2.1 for k a non-negative integer. I will use induction on k . If $k = 0$, then the theorem is obvious. Assume Theorem 2.1 is true for $0, 1, 2, \dots, k$. Hence, $D(B^k) = D(A^{2k})$. By Lemma 2.3, $T : D(B^k) \rightarrow D(B^{k+1/2})$. Hence by inductive hypothesis $D(A^{2k+1}) \subset D(B^{k+1/2})$. By Lemma 2.3, $T : D(B^{k+1/2}) \rightarrow D(B^{k+1})$. Hence, $D(A^{k^2+2}) \subset D(B^{k+1})$. By Lemma 2.2,

$$A : D(B^{k+1}) \rightarrow D(B^{k+1/2}) \quad \text{and} \quad A : D(B^{k+1/2}) \rightarrow D(B^k).$$

Hence by inductive hypothesis, $D(B^{k+1}) \subset D(A^{2k+2})$. Therefore, $D(B^{k+1}) = D(A^{2k+2})$. This completes the proof of the theorem.

Let

$$\mathcal{H}_n = \left\{ f \in \bigcap_{k \geq 0} H^k(\Gamma); \left(\frac{\partial}{\partial n} \right) (\Delta')^l f = 0 \text{ on } \partial\Gamma \text{ for } l = 0, 1, 2, \dots \right\}.$$

Define $H_n^k(\Gamma)$, $k \geq 0$, to be the closure of $\mathcal{H}_n$ in $H^k(\Gamma)$.

Theorem 2.4. *If G is a standard domain, then $H_n^k(\Gamma) \subset D(A^k)$, $k \geq 0$.*

Proof. It is sufficient to prove the theorem for $k = 0, 1, 2, \dots$. I will use induction on k . The case $k = 0$ is obvious. An argument similar to the proof of Lemma 2.4 of Chapter 1 can be used to prove $H'(\Gamma) = D(B^{1/2})$. Therefore, by Theorem 2.1, $D(A) = H'(\Gamma) \supset H_n^1(\Gamma)$. Thus the theorem is true for $k = 1$.

Assume Theorem 2.4 is true for $0, 1, 2, 3, \dots, k$ where $k \geq 1$. Let $v \in H_n^{k+1}(\Gamma)$. Let $(1 - \Delta')v = f$. By Green's Theorem we have

$$(w, v)_\Gamma + (\nabla' w, \nabla' v)_\Gamma = (w, f)_\Gamma$$

for all $w \in H'(\Gamma)$. Hence, by definition of B , $Bv = f$. By definition of f , we have $f \in H_n^{k-1}(\Gamma)$. By inductive hypothesis and Theorem 2.1, $H_n^{k-1}(\Gamma) \subset D(B^{(k-1)/2})$. Thus, $v \in D(B^{(k+1)/2})$. Therefore, by Theorem 2.1, $v \in D(A^{k+1})$. This completes the proof of the theorem.

Theorem 2.5. *Let G be a standard domain and K be an open set contained in $\bar{G} \setminus \partial\Gamma$, which lies at a positive distance from $\partial\Gamma$. If $\mu^0 \in H_n^k(\Gamma)$ and $\mu_t^0 \in H_n^{k-1/2}(\Gamma)$ for some $k \geq \frac{1}{2}$, then there exists a unique weak solution of (0.1–0.4). Furthermore, it is an element of*

$$C^j([0, \infty); H^{k+1/2-i/2}(K)), \quad j = 0, 1, 2, \dots, [2k - 1].$$

This is a direct consequence of Theorem 3.7 of Chapter 1 and Theorem 2.4 of Chapter 2.

As a consequence of the Sobolev Imbedding Theorem we have:

Corollary 2.6. *If $\mu^0 \in H_n^k(\Gamma)$ and $\mu_t^0 \in H_n^{k-1/2}(\Gamma)$ for some $k > (n + 1)/2$ then (0.1–0.4) have a unique classical solution.*

Remark. If I could show $H_n^{2k}(\Gamma) = D(B^k)$, then I could reprove Theorem 3.7 of Chapter 1 showing that the solution is an element of $C^j([0, \infty); H^{k+1/2-i/2}(G))$, $j = 0, 1, 2, \dots, [2k - 1]$ when $\mu^0 \in H_n^k(\Gamma)$ and $\mu_t^0 \in H_n^{k-1/2}(\Gamma)$ for some $k \geq \frac{1}{2}$.

If $\Gamma = \mathbf{R}^{n-1}$, then $H_n^k(\Gamma) = H^k(\Gamma)$. In this case we can use Fourier Transforms to prove $H^{2k}(\Gamma) = D(B^k)$. For $n = 3$ Theorem 3.5 is precisely Garipov's result (see [5]) when Γ' is stationary. Friedman and Shinbrot have proved Theorem 2.5 with $j = 0, 1, \dots, [2k - 2]$ and $\Gamma = \mathbf{R}^{n-1}$. The reason for the difference lies in their proof of Lemma 3.4 of Chapter 1. If one uses Lemma 1.3 of Chapter 1, instead of the lemma they prove, Friedman's and Shinbrot's method can be

used to prove Theorem 2.5 of Chapter 2. Thus, their result can be brought into agreement with Garipov's.

Chapter 3. Semi-standard domains. In this chapter we will restrict ourselves to two dimensions. In this case a standard domain is a normal domain with Γ' formed by vertical straight line segments in a neighborhood of Γ . In the first half of this chapter we still assume Γ' is made up of straight line segments near Γ but we allow Γ' to intersect Γ in angles of less than $\pi/2$. In the second half of the chapter we weaken the hypotheses on Γ' near Γ . In Section 1 we give the basic definitions and prove some lemmas. In Section 2 we prove the main result of this chapter. Section 3 generalizes the main result. In Section 4 we prove the properties of a certain conformal representation needed in Section 3.

1. Definitions.

Definitions. We say G is a *semi-standard domain* if the following conditions are satisfied:

G is a normal domain (bounded or unbounded). There exists $h_0 > 0$ such that $\Gamma' \cap \Gamma \times (-h_0, 0)$ is made up of a finite number of straight line segments. Each line segment forms an angle of less than $\pi/2$ with Γ when measured through G .

If $f \in L^2(\Gamma)$, then $\tilde{f}$ denotes its Fourier transform. If $f(x)/x^{1/2} \in L^2([0, \infty))$, then $\tilde{f}$ denotes its Mellin transform. If $f \in C_0^\infty((0, \infty))$, the Mellin transform is given by $\tilde{f}(s) = \int_0^\infty r^{s-1} f(r) dr$.

Lemma 1.1. If $k \geq 0$, then $H_0^k([0, \infty))$ has an equivalent norm defined by

$$\int_{-\infty}^{\infty} (1 + |s|^2)^k |\tilde{f}(\frac{1}{2} - k + is)|^2 ds.$$

Proof. $H_0^k([0, \infty))$ has an equivalent norm defined by

$$\int_{-\infty}^{\infty} (1 + |\xi|^2)^k |\tilde{f}(\xi)|^2 d\xi$$

where $\tilde{f}$ is the Fourier transform of f . Using this norm and Theorem 2 of [6], we see that it is sufficient to prove Lemma 1.1 for $k = 0, 1, 2, 3, \dots$. Moreover, I can prove the lemma using the original norm for $H_0^k([0, \infty))$. I will use induction on k to prove the theorem.

Let $k = 0$. If $f \in C_0^\infty((0, \infty))$, let $g(r) = e^{r/2} f(e^r)$. We then obtain

$$(1.1) \quad \|f\|_{[0, \infty)}^2 = \int_0^\infty |f(r)|^2 dr = \int_{-\infty}^\infty |g(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^\infty |\vartheta(\xi)|^2 d\xi.$$

From the definition of $\tilde{f}$ we have

$$(1.2) \quad \tilde{f}(\frac{1}{2} + is) = \int_0^\infty r^{is} r^{-1/2} f(r) dr = \int_{-\infty}^\infty e^{irs} g(r) dr = \vartheta(s).$$

From (1.1) and (1.2) we have

$$(1.3) \quad {}_0||f||_{[0,\infty)}^2 = \frac{1}{2\pi} \int_0^\infty |\tilde{f}(\tfrac{1}{2} + is)|^2 ds.$$

This is valid for all $f \in C_0^\infty((0, \infty))$. If $f \in L^2([0, \infty))$, then there exists $\{f_n\} \subset C_0^\infty((0, \infty))$ converging to f in $L^2([0, \infty))$. Applying (1.3) to $\{f_n - f_m\}$ shows $\tilde{f}_n$ is Cauchy in the norm of the right hand side of (1.3). Apply (1.3) to $\{f_m\}$ and take limits. Thus (1.3) is satisfied for all $f \in L^2([0, \infty))$.

Suppose the lemma is proved for $0, 1, 2, \dots, k$. Let $f \in C_0^\infty((0, \infty))$. By definition we have

$$(1.4) \quad {}_{k+1}||f||_{[0,\infty)}^2 = {}_k||f||_{[0,\infty)}^2 + {}_0||f^{(k+1)}||_{[0,\infty)}^2.$$

Using (1.3), we find

$$(1.5) \quad {}_0||f^{(k+1)}||_{[0,\infty)}^2 = \frac{1}{2\pi} \int_{-\infty}^\infty |\tilde{f}^{(k+1)}(\tfrac{1}{2} + is)|^2 ds.$$

Using the definition of Mellin transform and integrating by parts, we find $\tilde{f}^{(k+1)}(\tfrac{1}{2} + is) = (-1)^{k+1} [(-\tfrac{1}{2} + is)(-\tfrac{3}{2} + is) \cdots (-k - \tfrac{1}{2} + is)] \tilde{f}(-k + \tfrac{1}{2} + is)$. Hence, there exists a constant $c > 0$ such that

$$(1.6) \quad \left(\frac{1}{c}\right)(1 + |s|^2)^{k+1} |\tilde{f}(\tfrac{1}{2} - k - 1 + is)|^2 \leq |\tilde{f}^{(k+1)}(\tfrac{1}{2} + is)|^2 \\ \leq c(1 + |s|^2)^{k+1} |\tilde{f}(\tfrac{1}{2} - k - 1 + is)|^2.$$

Hence, by (1.4), (1.5), (1.6) and the inductive hypotheses, the two norms are equivalent for functions in $C_0^\infty((0, \infty))$. The argument in the case of $k = 0$ can be used to prove the lemma for $H_0^k((0, \infty))$.

Lemma 1.2. *If $0 < a < \pi/2$ and $0 \leq s < \pi/2a + 1/2$, then $\cos(s-1)a \neq 0$. If $0 \leq k < \pi/2a$, then $\cos ia \neq 0$, $i = 0, 1, \dots, k$.*

Proof. By hypothesis $-1 \leq s-1 < \pi/2a - 1/2$. Multiply both sides by a and we find $-\pi/2 < -a \leq (s-1)a < \pi/2 - a/2 < \pi/2$. Hence, $\cos(s-1)a \neq 0$. If i is such that $0 \leq i \leq k < \pi/2a$, then $0 \leq ia < \pi/2$. Hence, $\cos ia \neq 0$. This completes the proof of the lemma.

2. The main theorem. The main theorem is stated globally, but the proof is local. Let γ be the largest angle formed by Γ' and Γ .

Theorem 2.1. *Let G be a semi-standard domain. If $1 \leq k < \pi/2\gamma + 1/2$, then A maps $H^k(\Gamma)$ into $H^{k-1}(\Gamma)$ continuously.*

Proof. $f \in H^k(\Gamma)$. Let $f = f_1 + f_2$, where $f_1 \in H_0^k(\Gamma)$, $f_2 \in C^\infty(\bar{\Gamma})$, ${}_k||f_1||_\Gamma \leq c {}_k||f||_\Gamma$, ${}_k||f_2||_\Gamma \leq c {}_k||f||_\Gamma$ and each point of $\partial\Gamma$ has a neighborhood where f_2 equals a polynomial of degree less than $k - \frac{1}{2}$. We may take f_2 to have bounded support. By Lemma 2.6 of Chapter 1, $f_1 \in D(A^{1/2})$. f_2 certainly has an extension to $H'(G)$. Hence, by Lemmas 2.4 and 2.5 of Chapter 1, $f \in D(A^{1/2})$ and has a

unique extension in $H'_\Delta(G)$. Call this extension U . By Lemma 3.1 of Chapter 1, it is sufficient to prove U_v has a weak L^2 restriction to Γ and this restriction is an element of $H^{k-1}(\Gamma)$. I will show this in two parts: first I will show that if K is an open set with closure contained in Γ , then u_v^Γ restricted to K is an element of $H^{k-1}(K)$; next we will prove that each point of $\partial\Gamma$ is the end-point of a closed interval N such that u_v^Γ restricted to N is an element of $H^{k-1}(N)$. We then decompose $\bar{\Gamma}$ into an open set K with closure in Γ and a finite number of intervals $N_1, \dots, N_l$ with the properties given above. Thus, we have

$${}_{k-1}||u_v^\Gamma||_\Gamma \leq {}_{k-1}||u_v^\Gamma||_K + {}_{k-1}||u_v^\Gamma||_{N_1} + \dots + {}_{k-1}||u_v^\Gamma||_{N_l}.$$

Thus, the proof of the theorem depends on estimates for ${}_{k-1}||u_v^\Gamma||_K$ and ${}_{k-1}||u_v^\Gamma||_N$.

Let K be an open set with closure contained in Γ . Multiply f by a cutoff function. Call this new function g . Choose the cutoff function in such a way that $g \in H_0^k(\Gamma)$, ${}_k||g||_\Gamma \leq c {}_k||f||_\Gamma$ and $f = g$ in a neighborhood of $\bar{K}$. Let $\hat{W}(\xi, y) = \hat{g}(\xi)e^{y|\xi|}$, where $\hat{}$ represents Fourier transform. By Lemma 1.3 of Chapter 1, W is harmonic in G and $w^\Gamma = g$. An easy calculation shows $w_v^\Gamma \in H_0^{k-0}(\Gamma)$ and ${}_{k-1}||w_v^\Gamma||_\Gamma \leq c {}_k||g||_\Gamma$. Let $V = U - W$. V is harmonic in G and $v^\Gamma = 0$ in a neighborhood of $\bar{K}$. Hence, V is real analytic in a neighborhood of $\bar{K}$. Let R be a fundamental solution which vanishes in a one-dimensional neighborhood of $\bar{K}$ contained in Γ . Using the argument of Lemma 3.2 in Chapter 1, we find ${}_{k-1}||v_v^\Gamma||_K \leq c {}_{1/2}||f||_\Gamma$. Hence,

$${}_{k-1}||u_v^\Gamma||_K \leq {}_{k-1}||v_v^\Gamma||_K + {}_{k-1}||w_v^\Gamma||_K \leq c {}_k||f||_\Gamma.$$

Consider a point of $\partial\Gamma$. Without loss of generality, we may assume it has coordinates $(0, 0)$. Suppose $(0, \delta) \subset \Gamma$ for $\delta > 0$ and δ sufficiently small. I will show u_v^Γ restricted to $(0, \delta)$ is an element of $H^{k-1}([0, \delta])$ for δ sufficiently small. The proof in the case $(-\delta, 0) \subset \Gamma$ for $\delta > 0$ and δ sufficiently small is very similar, hence we omit this case.

We will consider polar coordinates with center at $(0, 0)$. By definition of a semi-standard domain, there exists a $\delta > 0$ and an $a > 0$, such that $M = \{(r, \theta); 0 < r < \delta, -a < \theta < 0\} \subset G$, $\{(r, -a); 0 < r < \delta\} \subset \Gamma'$ and $\{(r, 0); 0 < r < \delta\} \subset \Gamma$. Restrict f_1 to $\Gamma \cap [0, \infty)$ and extend this restriction to $[0, \infty)$ by defining it to be zero outside of Γ . Call this new function f_1 . $f_1 \in H^k([0, \infty))$ and ${}_k||f_1||_{[0, \infty)} \leq {}_k||f_1||_\Gamma$. Let $U = \Phi + \Psi + \chi$, where

$$(2.1) \quad \tilde{\Phi}(s, \theta) = \tilde{f}_1(s) \left[\frac{\cos s(\theta + a)}{\cos sa} \right],$$

$$(2.2) \quad \Psi(r, \theta) = \sum_{i=0}^{\infty} f_2^{(i)}(0) \frac{r^i}{i!} \left[\frac{\cos i(\theta + a)}{\cos ia} \right],$$

" $\sim$ " indicates the Mellin transform. Since $f_2(r)$ is equal to a polynomial for r near zero, the sum in (2.2) is finite and $\Psi(r, 0) = f_2(r)$ for $0 < r < \delta$ and δ sufficiently small. Hence, χ satisfies

$$(2.3) \quad \Delta\chi = 0 \quad \text{for } (r, \theta) \in M,$$

$$(2.4) \quad \chi^\Gamma = 0 \quad \text{for } r \in (0, \delta),$$

$$(2.5) \quad \chi_\theta(r, -a) = 0 \quad \text{for } r \in (0, \delta),$$

$$(2.6) \quad \chi(\delta, \theta) = L(\theta) \quad \text{for } \theta \in (-a, 0),$$

where $L(\theta) = U(\delta, \theta) - \Phi(\delta, \theta) - \Psi(\delta, \theta)$ for $-a < \theta < 0$. Since χ is harmonic and (2.4) and (2.5) are homogeneous boundary conditions, χ is real analytic for $\theta = 0$ or $-a$ and $r \in (0, \delta)$. Thus,

$$(2.7) \quad \chi(r, \theta) = \sum_{i=0}^{\infty} b_i \left(\frac{r}{\delta}\right)^{\lambda_i} \sin \lambda_i \theta \quad \text{for } (r, \theta) \in M,$$

with $\lambda_i = (\frac{1}{2} + i)\pi/a$, $i = 0, 1, 2, \dots$ and $b_i = (2/a) \int_{-a}^0 \sin \lambda_i \theta L(\theta) d\theta$.

From (2.1) and the definition of Mellin transform we obtain

$$\tilde{\varphi}_y^\Gamma(s) = (s-1)\tilde{f}_1(s-1) \tan a(s-1).$$

Hence, by Lemmas 1.1 and 1.2 of this chapter, we have $\varphi_y^\Gamma \in H^{k-1}([0, \infty))$ and ${}_{k-1}||\varphi_y^\Gamma||_{[0, \infty)} \leq c {}_k||f_1||_\Gamma$. Thus, the restriction of φ_y^Γ to $[0, \delta)$ is an element of $H^{k-1}([0, \delta))$ and

$$(2.8) \quad {}_{k-1}||\varphi_y^\Gamma||_{[0, \delta)} \leq c {}_k||f_1||_\Gamma.$$

From (2.2) we obtain

$$\Psi_y(r, 0) = - \sum_{i=1}^{\infty} \frac{f_2^{(i)}(0)}{(i-1)!} r^{i-1} \tan ia.$$

Hence, $\Psi_y^\Gamma \in H^{k-1}([0, \delta))$ and ${}_k||\Psi_y^\Gamma||_{[0, \delta)} \leq c \max_{0 \leq i < k-1/2} \{|f_2^{(i)}(0)|\}$. By the Sobolev inequality (see [7]), $\max_{0 \leq i < k-1/2} \{|f_2^{(i)}(0)|\} \leq c {}_k||f_2||_{[0, \delta)}$. Thus,

$$(2.9) \quad {}_{k-1}||\Psi_y^\Gamma||_{[0, \delta)} \leq c {}_k||f_2||_{[0, \delta)} \leq c {}_k||f||_\Gamma.$$

From (2.7) we obtain

$$\chi_y = \frac{1}{\delta} \sum_{i=0}^{\infty} \lambda_i b_i \left(\frac{r}{\delta}\right)^{\lambda_i-1}.$$

Let $\nu_1(r) = (1/\delta)\lambda_0 b_0(r/\delta)^{\lambda_0-1}$ and $\nu_2(r) = \chi_y(r) - \nu_1(r)$. Let $m = [\lambda_0 - 1]$ and $\epsilon = \lambda_0 - [\lambda_0 - 1]$. Thus $\nu_1(r) \in C^m[0, \delta]$. Let $\bar{\nu}_1(r) = (d^m/dr^m)\nu_1(r)$, $\bar{\nu}_1(r) = cb_0r^\epsilon$. A simple computation shows $\bar{\nu}_1(r) \in H^\eta[0, \delta]$ for $0 \leq \eta < \epsilon + 1/2$. Thus $\nu_1(r) \in H^\eta[0, \delta]$ for $0 \leq \eta < m + \epsilon + 1/2 = \lambda_0 - 1/2 = \pi/2a - 1/2$.

Thus, since $k-1 < \pi/2a - 1/2$ we have $\nu_1(r) \in H^{k-1}[0, \delta]$ and

$${}_{k-1}||\nu_1||_{[0, \delta]}^2 \leq c |b_0|^2.$$

On the other hand $\lambda_i \geq 3\pi/2a$ for $i = 1, 2, 3, \dots$. Thus $[k] < \pi/2a + 1/2 \leq 3\pi/2a \leq \lambda_1$ for $i = 1, 2, 3, \dots$. Hence, $r^{\lambda_i-1} \in C^{[k]}[0, \delta]$ for $i = 1, 2, 3, \dots$. If $0 \leq j \leq [k]$, then

$$\max_{[0, \delta/2]} \left| \frac{d^j}{dr^j} \nu_2(r) \right|^2 \leq \left(\sum_{i=1}^{\infty} |\lambda_i| |\lambda_i - 1| \cdots |\lambda_i - j + 1| |b_i| \frac{(\delta/2)^{\lambda_i-j}}{\delta^{\lambda_i}} \right)^2$$

$$\begin{aligned} &\leq c \left(\sum_{j=1}^{\infty} |b_j|^2 \right) \left(\sum_{i=1}^{\infty} |\lambda_i|^{2j+2} \left(\frac{1}{2}\right)^{2\lambda_i} \right) \\ &\leq c \left(\sum_{i=1}^{\infty} |b_i|^2 \right). \end{aligned}$$

Thus

$$_{[k]} ||\nu_2||_{[0, \delta/2]}^2 \leq c \left(\sum_{i=1}^{\infty} |b_i|^2 \right).$$

Hence

$$(2.10) \quad {}_{k-1} ||\chi_y^r||_{[0, \delta/2]} \leq c \left(\sum_{i=1}^{\infty} |b_i|^2 \right).$$

By Parseval's inequality we have

$$(2.11) \quad \sum_{i=0}^{\infty} |b_i|^2 \leq \int_{-a}^0 |\chi(\delta, \theta)|^2 d\theta.$$

A proof similar to that of Lemma 1.2 in Chapter 1 will show

$$(2.12) \quad \int_{-a}^0 |\chi(\delta, \theta)|^2 d\theta \leq c {}_1 ||\chi||_M^2.$$

By definition we have

$$(2.13) \quad {}_1 ||\chi||_M \leq {}_1 ||U||_M + {}_1 ||\Phi||_M + {}_1 ||\Psi||_M.$$

By Lemmas 2.4 and 2.5 of Chapter 1, ${}_1 ||U||_G$ can be estimated in terms of any $H'(G)$ extensions of f . Use the extension W for f_1 given in Lemma 1.3 of Chapter 1. Define $V(x, y) = f_2(x)$. V is clearly an $H'(G)$ extension of f_2 and ${}_1 ||U||_G \leq c_1 ||f_2||_r$. Thus, ${}_1 ||U||_G$ can be estimated in terms of ${}_1 ||W + V||_G$, which in turn can be estimated in terms of ${}_1 ||f||_r$ because of the hypotheses of f_1 and f_2 . Hence,

$$(2.14) \quad {}_1 ||U||_M \leq c {}_k ||f||_r.$$

On the other hand we have

$$\begin{aligned} {}_0 ||\nabla \Phi||_M &= \iint_M (|\Phi_x|^2 + |\Phi_y|^2) dx dy = \int_{-a}^0 \int_0^{\delta} \left(\frac{1}{r} |\Phi_{\theta}|^2 + r |\Phi_r|^2 \right) dr d\theta \\ &\leq \int_{-a}^0 \int_0^{\infty} \left(\frac{1}{r} |\Phi_{\theta}|^2 + r |\Phi_r|^2 \right) dr d\theta. \end{aligned}$$

By definition of the Mellin transform and Lemma 1.1 of this chapter we obtain

$${}_0 ||\nabla \Phi||_M \leq \int_{-a}^0 \int_{-\infty}^{\infty} (|\tilde{\Phi}_{\theta}(is, \theta)|^2 + |s|^2 |\tilde{\Phi}(is, \theta)|^2) ds d\theta.$$

If we use (2.1) in the above inequality, then we find

$$\begin{aligned}
{}_0||\nabla\Phi||_{\mathcal{M}} &\leq \int_{-a}^0 \int_{-\infty}^{\infty} |s|^2 |\tilde{f}_1(is)|^2 \left[\frac{\sinh^2 s(\theta + a) + \cosh^2 s(\theta + a)}{\cosh^2 sa} \right] ds d\theta \\
&\leq \int_{-\infty}^{\infty} |s|^2 |\tilde{f}_1(is)|^2 \left(\int_{-a}^0 \frac{\sinh^2 s(\theta + a) + \cosh^2 s(\theta + a)}{\cosh^2 sa} d\theta \right) ds \\
&\leq \int_{-\infty}^{\infty} |s|^2 |\tilde{f}_1(is)|^2 \left(\frac{\tanh sa}{s} \right) ds \\
&\leq c \int_{-\infty}^{\infty} |s| |\tilde{f}_1(is)|^2 ds \leq c_{1/2} \|f_1\|_{[0, \infty)} \leq c_k \|f\|_{\Gamma}.
\end{aligned}$$

Thus, by Lemma 1.2 of Chapter 1, we obtain

$$(2.15) \quad {}_1||\Phi||_{\mathcal{M}} \leq c_k \|f\|_{\Gamma}.$$

An argument similar to the proof of (2.9) of this chapter can be used to prove that

$$(2.16) \quad {}_1||\Psi||_{\mathcal{M}} \leq c_k \|f\|_{\Gamma}.$$

If we combine (2.10), (2.11), (2.12), (2.13), (2.14), (2.15) and (2.16), then we obtain

$${}_{k-1}||\chi_y^{\Gamma}||_{[0, \delta/2]} \leq c_k \|f\|_{\Gamma}.$$

Hence, the above inequality, (2.8) and (2.9) imply that

$${}_{k-1}||u_y^{\Gamma}||_{[0, \delta/2]} \leq c_k \|f\|_{\Gamma}.$$

This completes the proof of the theorem.

Corollary 2.2. *If G is a semi-standard domain, then $D(A^k)$ contains $H^k(\Gamma)$ for k such that $0 \leq k < \pi/2\gamma + 1/2$.*

Proof. By Theorem 2.1, A maps $H'(\Gamma)$ continuously into $L^2(\Gamma)$. Hence, by Theorem 2 of [6], $D(A^k) \subset H^k(\Gamma)$ for $0 \leq k \leq 1$. If k is such that $1 < k < \pi/2\gamma + 1/2$, let $l = [k]$. By Theorem 2.1, A^l maps $H^k(\Gamma)$ into $H^{k-l}(\Gamma)$. A^{k-l} maps $H^{k-l}(\Gamma)$ into $L^2(\Gamma)$. Hence, A^k maps $H^k(\Gamma)$ into $L^2(\Gamma)$. This completes the proof of the corollary.

Theorem 2.3. *Let G be a semi-standard domain. If $\mu^0 \in H^k(\Gamma)$ and $\mu_i^0 \in H^{k-1/2}(\Gamma)$ for some k with $1/2 \leq k < \pi/2\gamma + 1/2$, then there exists a unique weak solution of (0.1–0.4) satisfying (2.15) of Chapter 1. If K is an open set with closure contained in $\bar{G} \setminus \partial\Gamma$, then the solution is an element of $C^j([0, \infty); H^{k+1/2-j/2}(K))$, $j = 0, 1, \dots, [2k - 1]$.*

Proof. Use Corollary 2.3 of Chapter 3 and Theorem 3.7 of Chapter 1. Sobolev's imbedding theorem and Corollary 3.8 of Chapter 1 imply:

Corollary 2.4. *If G is a semi-standard domain and $\mu^0 \in H^k(\Gamma)$ and $\mu_i^0 \in H^{k-1/2}(\Gamma)$ for some k with $k > 3/2$, then (0.1–0.4) have a unique classical solution.*

3. Pseudo-standard domains. In this section we generalize the results of the last section by allowing Γ' to deviate "slightly" from a straight line segment near Γ . In this case we map G locally onto a semi-standard domain. We can then estimate A on this new domain. Unfortunately this attack does not succeed very well. The conditions on Γ' are very stringent. In order to obtain better results, methods similar to those used in Lemma 2.4 of Chapter 1 will probably have to be used.

Definition. Let G be a normal domain (bounded or unbounded) in $\mathbf{R}^2$. Assume each point x_0 of $\partial\Gamma$ is the endpoint of a closed interval I contained in $\bar{\Gamma}$ such that the set $\{(x, y); x \in I\} \cap \Gamma'$ is given by $y = -h(x)$ where h satisfies the following properties: i) $h'(x_0) > 0$ if x_0 is the left-hand endpoint and $h'(x_0) < 0$ if x_0 is the right-hand endpoint of I , ii) if $\alpha = \tan^{-1}(|h'(x_0)|)$ and k is the smallest integer larger than $\pi/2\alpha$, then $h \in C^k(I)$ and $h^{(j)}(x_0) = 0$, $j = 2, 3, \dots, k$. If G satisfies the above conditions, then we say G is a *pseudo-standard domain*.

Define $\gamma = \max \{\tan^{-1}(|h'(x_0)|); x_0 \in \partial\Gamma\}$. Note that $0 < \gamma < \pi/2$. Before proving the main theorem of this section we shall need the following lemma:

Lemma 3.1. Let I and I' be two closed bounded intervals. Let σ map I onto I' . If $\sigma \in C^2(I)$ and $\sigma'(x) > 0$, $x \in I$, then $f \in H^k(I')$ if and only if $f \circ \sigma \in H^k(I)$ for $0 \leq k \leq 1$. If $1 \leq k \leq 2$, then $f \in H^k(I')$ implies $f \circ \sigma \in H^k(I)$.

Proof. Let $f \in L^2(I')$. By the inverse function theorem $\sigma^{-1'}(x) > 0$ for $x \in I'$. Hence, by the change of variables formula for the Lebesgue integral, we have

$$(3.1) \quad \int_I |f \circ \sigma(x)|^2 dx = \int_{I'} |f(x)|^2 \sigma^{-1'}(x) dx.$$

Since I' is compact, there exists a constant $c > 0$ such that $c \leq \sigma^{-1'}(x) \leq 1/c > 0$ for $x \in I'$. Hence,

$$(3.2) \quad \frac{1}{c} \|f\|_{I'}^2 \leq \|f \circ \sigma\|_I^2 \leq c \|f\|_{I'}^2.$$

Thus, $f \in L^2(I')$ if and only if $f \circ \sigma \in L^2(I)$. Let $f \in H^1(I')$. From the chain rule we obtain $(f \circ \sigma)' = (f' \circ \sigma)\sigma'$. By substituting this in (3.1) and using the inverse function theorem we obtain

$$\int_I |(f \circ \sigma)'(x)|^2 dx = \int_{I'} \frac{|f'(x)|^2}{\sigma^{-1'}(x)} dx.$$

Since $\sigma^{-1'}(x)$ is bounded above and below on I' , we have

$$(3.3) \quad \frac{1}{c} \|f'\|_{I'}^2 \leq \|(f \circ \sigma)'\|_I^2 \leq c \|f'\|_{I'}^2.$$

Hence, by (3.2) and (3.3), $f \in H^1(I')$ if and only if $f \circ \sigma \in H^1(I)$. By Theorem 1.1 of Chapter 1, the first part of the lemma follows.

Let $f \in H^2(I')$. By the chain rule we have $(f \circ \sigma)'' = \sigma''(f' \circ \sigma) + (\sigma')^2(f'' \circ \sigma)$. If we use $(f \circ \sigma)''$ in (3.1) and the bounds on $\sigma^{-1'}$ and σ'' we find

$$\int_I |(f \circ \sigma)''(x)|^2 dx \leq c[|f'|_{I'}^2 + |f''|_{I'}^2].$$

Hence, if $f \in H^2(I')$, then $f \circ \sigma \in H^2(I')$. By Theorem 1.1 of Chapter 1 the proof is complete.

Theorem 3.2. *If G is a pseudo-standard domain, then A maps $H^k(\Gamma)$ continuously into $H^{k-1}(\Gamma)$ for k with $1 \leq k < \min(2, \pi/2\gamma + 1/2)$.*

Proof. As in the proof of Theorem 2.1 of the last section, I will decompose $\bar{\Gamma}$ into an open set K with closure in Γ and a finite number of closed intervals each of which has an element of $\partial\Gamma$ as one endpoint. Let $f \in H^k(\Gamma)$ for k with $1 \leq k < \min(2, \pi/2 + 1/2)$, f has an extension U which is an element of $H'_\Delta(G)$. As before we need to estimate $_{k-1}||u_\nu^\Gamma||_\Gamma$. The estimate for $_{k-1}||u_\nu^\Gamma||_K$ is exactly the same as in Theorem 2.1 of the last section. As before it is sufficient to estimate u_ν^Γ restricted to $[0, \delta]$ where $(0, 0) \in \partial\Gamma$ and δ is sufficiently small. Choose δ so small that $\Gamma' \cap \{(x, y); x \in [0, \delta]\}$ is given by $y = -h(x)$. Map the set $M = \{(x, y); x \in [0, \delta]\} \cap G$ conformally onto the set $M' = \{(x, y); x \in [0, \delta], 0 > y > -(h'(0))x\}$. Call the map Σ . In Section 4 we prove the following facts about Σ : i) Σ has a continuous extension to $\bar{M}$, call this extension Σ again, ii) if Σ restricted to $[0, \delta]$ is σ , then $\sigma \in C^2[0, \delta]$ and $\sigma'(x) > 0$ for $x \in [0, \delta]$.

We will consider f as being restricted to $[0, \delta]$ and U as being restricted to M . Define $\bar{U}$ on M' by the formula $\bar{U} = U \circ \Sigma^{-1}$. Therefore U satisfies

$$\begin{aligned} \Delta \bar{U} &= 0 \quad \text{in } M', \\ \bar{u}^\Gamma &= f \circ \sigma^{-1}, \\ \bar{U}_n &= 0 \quad \text{for } y = -xh'(0), \quad x \in (0, \delta), \\ \bar{U}(\delta, y) &= (U \circ \Sigma^{-1})(\delta, y), \quad 0 > y > -\delta h'(0). \end{aligned}$$

By Lemma 3.1 of this chapter we have $f \circ \sigma^{-1} \in H^k([0, \delta])$. Multiply $f \circ \sigma^{-1}$ by a C^∞ cutoff function which is one in $[0, 3\delta/4]$ and zero to the right of δ . Call this modified function g . Let $\bar{U} = \Phi + \Psi + \chi$ in M' where Φ and Ψ are defined as in Theorem 2.1 with g in place of f . Thus from (2.8) and (2.9) we have

$$(3.4) \quad _{k-1}||\varphi_\nu^\Gamma||_{[0, \delta/2]} + _{k-1}||\Psi_\nu^\Gamma||_{[0, \delta/2]} \leq c_k ||g||_{[0, \delta]} \leq c_k ||f \circ \sigma^{-1}||_{[0, \delta]}.$$

The arguments which lead to (2.11) show that

$$\begin{aligned} _{k-1}||\chi_\nu^\Gamma||_{[0, \delta/2]} &\leq c \int_{-a}^0 |\chi(\delta, \theta)|^2 d\theta \\ &\leq c \left(\int_{-a}^0 (|\bar{U}(\delta, \theta)|^2 + |\Phi(\delta, \theta)|^2 + |\Psi(\delta, \theta)|^2) d\theta \right). \end{aligned}$$

In the proof of Lemma 1.2 of Chapter 1, (2.15) and (2.16) show that

$$\begin{aligned} \int_{-a}^0 (|\Phi(\delta, \theta)|^2 + |\Psi(\delta, \theta)|^2) d\theta &\leq c_1 \|\Phi\|_{M'}^2 + \|\Psi\|_{M'}^2 \\ &\leq c_k \|g\|_{[0, \infty)}^2 \leq c_k \|f \circ \sigma^{-1}\|_{[0, \delta]}^2. \end{aligned}$$

On the other hand, by the change of variables formula, we have

$$\int_{-a}^0 |\bar{U}(\delta, \theta)|^2 d\theta = \int_{\beta} |U|^2 |\Sigma'| d\theta$$

where β is a simple Jordan arc in M . By the theorem of Section 152 of [2], $|\Sigma'|$ is bounded on β . This fact combined with the argument of Lemma 1.2 implies

$$\int_{-a}^0 |\bar{U}(\delta, \theta)|^2 d\theta \leq c \int_{\beta} |U|^2 d\theta \leq c_1 \|U\|_M^2.$$

Thus, by the above and (2.14) of this chapter, we obtain

$$\begin{aligned} {}_{k-1} \|\chi_{\mathbf{y}}^{\Gamma}\|_{[0, \delta/2]}^2 &\leq c \left(\int_{-a}^0 |\bar{U}(\delta, \theta)|^2 d\theta + {}_k \|f \circ \sigma^{-1}\|_{[0, \delta]}^2 \right) \\ &\leq c_1 \|U\|_M^2 + {}_k \|f \circ \sigma^{-1}\|_{[0, \delta]}^2 \\ &\leq c({}_k \|f\|_{\Gamma}^2 + {}_k \|f \circ \sigma^{-1}\|_{[0, \delta]}^2). \end{aligned}$$

Thus, by Lemma 3.1 of this chapter,

$$(3.5) \quad {}_{k-1} \|\chi_{\mathbf{y}}^{\Gamma}\|_{[0, \delta/2]} \leq c_k \|f\|_{\Gamma}.$$

Combining (3.4) and (3.5) and the definition of Φ and Ψ , we obtain

$${}_{k-1} \|\bar{u}_{\mathbf{y}}^{\Gamma}\|_{[0, \delta/2]} \leq c_k \|f\|_{\Gamma}.$$

A straight-forward calculation will show that if $U_{\mathbf{y}}$ is continuous in $\bar{G} \setminus \partial\Gamma$, then $u_{\mathbf{y}}^{\Gamma} = (\bar{u}_{\mathbf{y}}^{\Gamma} \circ \sigma)\sigma'$ for $x \in (0, \delta)$. If $f \in C^{\infty} \cap H^k(\Gamma)$, then $U_{\mathbf{y}}$ will have the required properties. Thus,

$$\begin{aligned} (3.6) \quad {}_{k-1} \|u_{\mathbf{y}}^{\Gamma}\|_{[0, \delta/2]} &= {}_{k-1} \|(\bar{u}_{\mathbf{y}}^{\Gamma} \circ \sigma)\sigma'\|_{[0, \delta/2]} \leq c_{k-1} \|\bar{u}_{\mathbf{y}}^{\Gamma} \circ \sigma\|_{[0, \delta/2]} \\ &\leq c_{k-1} \|\bar{u}_{\mathbf{y}}^{\Gamma}\|_{[0, \delta/2]} \leq c_k \|f\|_{\Gamma}, \end{aligned}$$

where σ' is bounded was used to derive the first inequality and Lemma 1.3 was used for the second inequality. (3.6) is true for a dense set in $H^k(\Gamma)$. Therefore take a sequence of smooth functions converging to f . (3.6) shows the $u_{\mathbf{y}}^{\Gamma}$'s form a Cauchy sequence. Then take limits in inequality (3.6), and the theorem is proved.

Corollary 3.3. *If G is a pseudo-standard domain, then $H^k(\Gamma) \subset D(A^k)$ for k with $0 \leq k < \min(2, \pi/2\gamma + 1/2)$.*

The proof is similar to the proof of Corollary 2.2.

Theorem 3.4. *If G is a pseudo-standard domain and $\mu^0 \in H^k(\Gamma)$ and $\mu_i^0 \in H^{k-1/2}(\Gamma)$ for some k with $1/2 \leq k < \min(2, \pi/2\gamma + 1/2)$, then there exists a*

unique weak solution to (0.1–0.4) satisfying (2.15) of Chapter 1. If K is an open set with closure contained in $\bar{G} \setminus \partial\Gamma$, then the solution is an element of $C^j([0, \infty); H^{k+1/2-j/2}(K))$, $j = 0, 1, \dots, [2k - 1]$.

Corollary 3.5. *If G is a pseudo-standard domain, $\mu^0 \in H^k(\Gamma)$ and $\mu_t^0 \in H^{k-1/2}(\Gamma)$ for some $k > 3/2$, then (0.1–0.4) have a unique classical solution.*

The proofs of the above theorem and corollaries are exactly the same as in Corollary 2.2, Theorem 2.3 and Corollary 2.4 of the last section.

4. The conformal representation. In this section I will prove σ has the properties needed in proving Theorem 3.2. These properties are the following:

- i) Σ is a conformal representation of M onto M' ,
- ii) Σ has a continuous extension to $\bar{M}$ which is 1–1 and onto, call this extension Σ again,
- iii) if σ is the restriction of Σ to $[0, \delta]$, then $\sigma \in C^2([0, \delta])$ and σ maps $[0, \delta]$ onto $[0, \delta]$,
- iv) $\sigma'(x) > 0$ for $x \in [0, \delta]$.

We can construct Σ such that i) is satisfied, by using the Riemann Mapping Theorem. ii) follows from the theorem of Section 137 of [2]. Let rM be the union of M and its reflection across the real axis. Define rM' similarly. By the Schwarz reflection principle (see Section 141 of [2]) we can continuously extend Σ to rM such that Σ maps rM onto rM' and is conformal on the interior of rM' . Thus, $\sigma \in C^2((0, \delta))$ and $\sigma'(x) \neq 0$ for $x \in (0, \delta)$. By the Schwarz reflection principle, Σ can be extended conformally across $\{(\delta, y); -h(\delta) < y < h(\delta)\}$. Hence, $\sigma \in C^2((0, \delta])$ and $\sigma'(\delta) \neq 0$. Thus it is sufficient to show $\sigma''(x)$ exists and is continuous at 0 and $\sigma'(0) \neq 0$.

Let $\alpha = \tan^{-1}(h'(0))$. Let mM and mM' be the images of rM and rM' under the map $z^{\pi/2\alpha}$. Σ induces a conformal map of mM onto mM' . Call this map ζ . I will show $\zeta''(x)$ is continuous at 0 and $\zeta'(0) \neq 0$. Let $\Psi = \operatorname{Re} \zeta$. Ψ is a harmonic function satisfying Dirichlet boundary conditions. Since the boundary of mM' lies on the imaginary axis near the point 0, $\Psi = 0$ on the boundary of mM near the point 0. If I show the boundary at mM is C^{2+p} near 0, then Ψ has two continuous derivatives up to the boundary near 0. Hence, by the Cauchy–Riemann equations, $\operatorname{Im} \zeta$ has two continuous derivatives up to the boundary near 0. Therefore, $\zeta''(x)$ is continuous at 0. Now I will show mM is C^{2+p} near 0. The upper part of the boundary of mM is given in parametric form by $x(\xi) + iy(\xi) = (\xi + ih(\xi))^\beta$ where $\beta = \pi/2\alpha$. If k is the smallest positive integer larger than β then by hypothesis and Taylor's Formula we find $y(\xi) = c_1 \xi^\beta + O(\xi^{\beta+1})$; $x(\xi) = O(\xi^{\beta+k})$ where $c_1 \neq 0$. These equations can be differentiated. Thus $(dy/d\xi)(\xi) \neq 0$ for $\xi \in (0, \delta)$ and δ sufficiently small. Thus $y(\xi)$ is 1–1 on $[0, \delta]$. Hence ξ can be considered as a function of y . Let $[0, \delta']$ be the image of $[0, \delta]$ under the function $y(\xi)$. Now $(dy/d\xi)(\xi) \neq 0$ for all $\xi \in (0, \delta)$. If ξ_0 is a fixed point of $(0, \delta)$ then by Inverse Function Theorem we know $\xi(y)$ is as smooth as $y(\xi)$ in a neighborhood

of $y(\xi_0)$. On the other hand $y(\xi) \in C^\infty(0, \delta)$. Thus $\xi(y)$ is an element of $C^{2+p}(0, \delta')$ for every $p > 0$. Hence $x(y) \in C^{2+p}(0, \delta')$ for every $p > 0$. Thus to prove $x(y) \in C^{2+p}[0, \delta')$ for some $p > 0$ it is sufficient to show $d^2 x/dy^2 = O(y^p)$ for some $p > 0$. Using the chain rule and the formulas for $x(\xi)$ and $y(\xi)$ we obtain $d^2 x/dy^2 = O(\xi^{k-\beta})$. On the other hand

$$\xi = \left(\frac{y}{c_1}\right)^{1/\beta} + O(\xi^2).$$

Hence, since $y \geq (c_1/2)\xi^\beta$ we find $\xi = (y/c_1)^{1/\beta} + O(y^{2/\beta})$. Thus if $p = (k - \beta)/\beta$, then $d^2 x/dy^2 = O(y^p)$. Thus, $x(y) \in C^{2+p}[0, \delta')$. Furthermore, since $x(y) = x(y^{1+k/\beta})$ and $dx/dy = O(y^{k/\beta})$ we have $x(0) = 0$, $(dx/dy)(0) = 0$ and $(d^2 x/dy^2)(0) = 0$. A similar argument for the lower half of the boundary, which is given by $(\xi - ih(\xi))^\beta$, will prove that the boundary of mM is C^{2+p} in a neighborhood of 0. Thus, $\zeta''(0)$ is continuous at 0.

We now want to show $\zeta'(0) \neq 0$. Let B be the intersection of mM' and a ball with center 0 and radius so small that the intersection forms a half ball. Let χ map the unit circle conformally onto B with $\chi(1) = 0$. By the theorem of Section 152 of [2] we find $\chi'(1) > 0$. On the other hand mM is C^{2+p} near 0, hence mM has the strong inside and outside sphere property. Hence, by the theorem of Section 152 of [2], $(\zeta^{-1} \circ \chi)'(1) > 0$. On the other hand we have

$$(\zeta^{-1} \circ \chi)'(0) = (\zeta^{-1} \circ \chi)(1)\chi'(1) = \zeta^{-1'}(0)\chi'(1) = \left(\frac{1}{\zeta'(0)}\right)\chi'(1).$$

Hence, $\zeta'(0) > 0$.

By definition we have $\sigma(x) = [\zeta(x^\beta)]^{1/\beta}$. Thus we obtain

$$\sigma'(x) = \zeta_0^{1/\beta} + O(x^\beta), \quad x > 0,$$

and

$$\sigma''(x) = O(x^{\beta-1}), \quad x > 0.$$

Hence, $\sigma'(0)$ exists and $\sigma'(0) > 0$. Since $\beta > 1$, $\sigma''(0)$ exists and is equal to 0.

The proof of iv) is easy. $\sigma'(x)$ has one sign and $\sigma'(0) > 0$. This completes Section 4.

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