Trivializations of Orthogonal Bundles

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1. Introduction. It is known that for any vector bundle E over a compact manifold M , there always exists a complementary bundle E' such that $E \oplus E'$ is isomorphic to a trivial bundle $M \times \mathbf{R}^q$ for some positive integer q . The restriction of this isomorphism to E gives rise to an imbedding of E into $M \times \mathbf{R}^q$. In general, there is no formula for such imbedding. However, if E is an orthogonal vector bundle, then a rather nice formula can be constructed. The imbedding depends on the open covering of M and its associated partition of unity. The imbedding and the exterior derivative on $M \times \mathbf{R}^q$ induce a Riemannian connection with respect to the canonically fibre metric of E . Throughout this paper the partition of unity plays an important role. Also some applications to harmonic sections of E have been made.

2. Imbedding of orthogonal vector bundles. Suppose that M is a separable paracompact manifold of dim n , and suppose that E is an orthogonal vector bundle over M of dim m . Let N denote the set of all natural numbers. Let $\mathfrak{U} = \{U_k\}_{k \in N}$ be a countable locally finite open covering of M such that (i) the intersection of any two members of $\mathfrak{U}$ is connected, (ii) E is trivial over each U_k of $\mathfrak{U}$. Let $\{P_k^2\}_{k \in N}$ be a partition of unity, and let $\{G_{k1}\}$ be the family of transition functions of E associated to the open covering $\mathfrak{U}$. On each $U_k \in \mathfrak{U}$, we define a bundle map $\varphi_k : E_k = E|U_k \rightarrow M \times \mathbf{R}^\infty$ as follows: Suppose $U_{k_1}, U_{k_2}, \dots, U_{k_r}$ are the only members of $\mathfrak{U}$ intersecting U_k , and suppose that $k_1 < k_2 < \dots < k_{i-1} < k < k_i < \dots < k_r$. Let Φ_k be a $\infty \times m$ matrix whose transpose Φ'_k is defined by

$$\Phi'_k = (0, \dots, P_{k_1} G'_{k_1 k}, \dots, P_{k_{i-1}} G'_{k_{i-1} k}, \dots, P_k G'_{kk}, \dots, P_{k_i} G'_{k_i k}, \dots, P_{k_r} G'_{k_r k}, 0, 0, \dots).$$

We note that Φ'_k is an $m \times \infty$ matrix whose entries are almost all zero and that $P_{k_1} G_{k_1}$ is in the k_1 -th place, and so on. Suppose $\tau_k : U_k \times \mathbf{R}^m \rightarrow E_k$ is a trivialization. Let $\varphi_k : E_k \rightarrow U_k \times \mathbf{R}^\infty$ be a bundle map such that the linear map $\varphi_k \circ \tau_k : U_k \times \mathbf{R}^m \rightarrow U_k \times \mathbf{R}^\infty$ has the matrix representation Φ_k with respect to the

standard bases of $\mathbf{R}^m$ and $\mathbf{R}^n$. It is clearly that φ_k is an imbedding of E_k into $U_k \times \mathbf{R}^n$. Since the relation $\varphi_k G_{k1} = \varphi_1$ holds on $U_k \cap U_1$, the collection $\{\varphi_k\}_{k \in N}$ defines a bundle map $\varphi: E \rightarrow M \times \mathbf{R}^n$, which is an imbedding. From the above construction, we may think φ_k as an imbedding of E_k into a subbundle $U_k \times \mathbf{R}^p$ where $p = (r+1)m$.

From now on, we will consider φ_k as an imbedding of E_k into $U_k \times \mathbf{R}^p$. The corresponding matrix Φ_k will then become an $p \times m$ matrix. For each $x \in U_k$, if $A_1(x), \dots, A_m(x)$ are column vectors of Φ_k , then the column vectors are mutually orthogonal. In fact,

$$\begin{aligned}\Phi'_k \Phi_k &= P_{k1}^2 G'_{k1k} G_{k1k} + \dots + P_{kr}^2 G'_{krk} G_{krk} + \dots + P_{kr}^2 G'_{krk} G_{krk} \\ &= (P_{k1}^2 + \dots + P_{kr}^2 + \dots + P_{kr}^2) I_m = I_m\end{aligned}$$

where I_m is the $m \times m$ identity matrix. Let $B_{m+1}(x), B_{m+2}(x), \dots, B_p(x)$ be vectors of $\mathbf{R}^p$ such that $\{A_1(x), \dots, A_m(x), B_{m+1}(x), \dots, B_p(x)\}$ are mutually orthogonal. If we set $\Psi_k = (B_{m+1} \dots B_p)$, then Ψ_k is an $p \times (p-m)$ matrix and $\Phi'_k \Psi_k = 0$. If we consider $A_1, \dots, A_m, B_{m+1}, \dots, B_p$ as functions from U_k to $\mathbf{R}^p$, we have the following

$$dA_i = \sum_{j=1}^m A_j \theta_{ji} + \sum_{t=m+1}^p B_t \Theta_{ti} \quad \text{for } i = 1, \dots, m.$$

The above equation may be written as follows:

$$d\Phi_k = \Phi_k \theta_k + \Psi_k \Theta_k$$

where $\theta_k = (\theta_{ji})$ is an $m \times m$ matrix of 1-forms on U_k and $\Theta_k = (\Theta_{ti})$ is a $(p-m) \times m$ matrix of 1-forms on U_k . Multiplying the second equation by Φ'_k , we get $\theta_k = \Phi'_k d\Phi_k$. Since the relation $\theta_k = G_{1k}^{-1} \theta_1 G_{1k} + G_{1k}^{-1} dG_{1k}$ holds on $U_k \cap U_1$, the collection $\theta = \{\theta_k\}_{k \in N}$ defines a connection of E . The matrix θ_k is also skew-symmetric. In fact, $\theta'_k = d\Phi'_k \Phi_k = -\Phi'_k d\Phi_k = -\theta_k$.

If F is the canonically fibre metric of E , then locally F is expressed as an $m \times m$ identity matrix F_k relative to the standard base of $\mathbf{R}^m$. It follows that φ is an isometry. Since F_k is an identity matrix and θ_k is skew-symmetric, the equation $dF_k = \theta_k F_k + F'_k \theta'_k$ is satisfied. Hence θ is a Riemannian connection. Summarizing the above results, we obtain the following

Theorem 1. *Suppose M is a paracompact separable manifold, and suppose that E is an orthogonal vector bundle over M . Then there exists an isometric imbedding $\varphi: E \rightarrow M \times \mathbf{R}^n$. Furthermore, the imbedding φ and the exterior derivative d on $M \times \mathbf{R}^n$ induce a Riemannian connection θ on E .*

Corollary. *Suppose that M is a compact manifold, and suppose that E is an orthogonal vector bundle. Then there exists an isometric imbedding of E into $M \times \mathbf{R}^q$ for some positive integer q .*

Let $T^p(M)$ denote the bundle of p -covectors of M . Let $\Gamma(E)$ denote the set of all smooth sections of E . It follows from the above construction that $\varphi(E)$ is

a direct summand of $M \times \mathbf{R}^\infty$. Thus $T^p(M) \otimes \varphi(E)$ is a direct summand of $T^p(M) \otimes (M \times \mathbf{R}^\infty)$. Let ρ denote the projection of $T^p(M) \otimes (M \times \mathbf{R}^\infty)$ into $T^p(M) \otimes \varphi(E)$. The imbedding φ and projection ρ induce linear maps

$$\varphi_* : \Gamma(T^p(M) \otimes E) \rightarrow \Gamma(T^p(M) \otimes \varphi(E))$$

and

$$\rho_* : \Gamma(T^p(M) \otimes (M \times \mathbf{R}^\infty)) \rightarrow \Gamma(T^p(M) \otimes \varphi(E)).$$

If $\tilde{d}$ denotes the exterior differential operator on E defined by the connection θ , then we have the following commutative diagram:

$$\begin{array}{ccc} \Gamma(T^p(M) \otimes E) & \xrightarrow{\varphi_*} & \Gamma(T^p(M) \otimes \varphi(E)) \\ \downarrow \tilde{d} & & \downarrow \rho_* d \\ \Gamma(T^{p+1}(M) \otimes E) & \xrightarrow{\varphi_*} & \Gamma(T^{p+1}(M) \otimes \varphi(E)) \end{array}$$

which exhibits a close relation between operators $\tilde{d}$ and d .

3. Harmonic sections. Suppose now that M is a compact Riemannian manifold. We recall that a section s of E is harmonic if the equation $\tilde{\Delta}s = 0$ is satisfied, which is equivalent to $\tilde{d}s = 0$. Let $\mathfrak{U} = \{U_k\}_{k \in A}$ be an open covering of M , where A is a subset of the set N of integers. When E is an orthogonal vector bundle over M of dim 2, the transition function G_{k1} can be represented as

$$G_{k1} = \begin{pmatrix} \cos g_{k1} & \sin g_{k1} \\ -\sin g_{k1} & \cos g_{k1} \end{pmatrix}$$

where g_{k1} is a smooth function on $U_k \cap U_1$. For each U_k of $\mathfrak{U}$, let $U_{k_1}, \dots, U_{k_r}$ be all the members of $\mathfrak{U}$ intersecting U_k . On U_k the connection is given by

$$\theta_k = \sum_{i=1}^r P_{k_i}^2 G'_{k_i k} dG_{k_i k} = \begin{pmatrix} 0 & \sum_{i=1}^r P_{k_i}^2 dg_{k_i k} \\ -\sum_{i=1}^r P_{k_i}^2 dg_{k_i k} & 0 \end{pmatrix}.$$

Since the exterior differential operator $\tilde{d}$ depends only on the connection θ and there exists a Riemannian metric for any compact manifold, we may define a section s of E to be harmonic if $\tilde{d}s = 0$.

Theorem 2. Suppose that M is a compact manifold, and suppose that E is an orthogonal vector bundle over M of dim 2. (i) If there exists a harmonic section of E , then the 1-form $\sum_{i=1}^r P_{k_i}^2 dg_{k_i k}$ is exact on U_k . (ii) There exist at most two linearly independent harmonic sections of E .

Proof. For simplicity we set $\Theta = \sum_{i=1}^r P_{k_i}^2 dg_{k_i k}$. On U_k a section s is represented by a 2-tuple (s_1, s_2) of functions on U_k . The condition $\tilde{d}s = 0$ gives

$$(1) \quad ds_1 = \Theta s_2$$

$$(2) \quad ds_2 = -\Theta s_1.$$

The operation $(1) \times s_1 + (2) \times s_2$ yields $d(s_1^2 + s_2^2) = 0$. Hence $s_1^2 + s_2^2 = c^2$ and $s_2 = (c^2 - s_1^2)^{1/2}$ where c is a constant. Substituting into equation (1), we get $(c^2 - s_1^2)^{-1/2} ds_1 = \Theta$. Similarly, we have $(c^2 - s_2^2)^{-1/2} ds_2 = -\Theta$. Thus Θ is an exact 1-form on U_k if the harmonic section $s = (s_1, s_2)$ exists. It follows from an elementary calculation that the solution is given by $s_1 = c \sin(\int \Theta + c')$ and $s_2 = c \cos(\int \Theta + c')$ where c' is also a constant. By setting $c = 1$ and $c' = 0$ or $\pi/2$ we get two harmonic sections $(\sin \int \Theta, \cos \int \Theta)$ and $(\cos \int \Theta, \sin \int \Theta)$, which are linearly independent. We conclude that there exist at most two linearly independent harmonic sections of E , and this completes the proof.

Theorem 3. Suppose that M is a manifold diffeomorphic to the one sphere S^1 , and suppose that E is an orthogonal vector bundle over M of $\dim 2$. Let $\mathfrak{U} = \{U_k\}_{k=0}^r$ be a finite open covering of M such that (i) $U_r \cap U_0 \neq \emptyset$ and $U_k \cap U_{k+1} \neq \emptyset$ for $k = 0, \dots, r-1$, (ii) the intersection of any three members of $\mathfrak{U}$ is empty, (iii) E is trivial over each U_k of $\mathfrak{U}$. Let x_k and y_k be boundary points of U_k such that $x_k \in U_{k-1}$ and $y_k \in U_{k+1}$ for $k = 1, \dots, r-1$, and $x_0 \in U_r, y_r \in U_0$. Then there are two linearly independent harmonic sections of E if and only if the real number

$$\sum_{i=1}^r g_{i(i+1)}(x_{i+1}) + \sum_{i=1}^r \int_{x_{i+1}}^{y_i} P_i^2 dg_{i(i+1)}$$

is an integral multiple of 2π , where we set $g_{r(r+1)} = g_{r0}$ and $x_{r+1} = x_0$.

Proof. Suppose that a harmonic section s of E exists. On each U_i , the harmonic section s is expressed as

$$\left(c_i \sin \left(\int_{x_i}^x \Theta_i + c'_i \right), \quad c_i \cos \left(\int_{x_i}^x \Theta_i + c'_i \right) \right)$$

where $\Theta_i = P_{i-1}^2 dg_{(i-1)i} + P_{i+1}^2 dg_{(i+1)i}$, c_i and c'_i are constants. By direct calculation, it can be shown that the constants c'_i and c'_{i+1} satisfy the equation

$$c'_i + g_{i(i+1)}(x_{i+1}) + \int_{x_{i+1}}^{y_i} P_i^2 dg_{i(i+1)} = c'_{i+1} \pmod{2\pi}$$

for $i = 0, \dots, r$. Adding all these $r+1$ equations together, we obtain

$$\sum_{i=1}^r g_{i(i+1)}(x_{i+1}) + \sum_{i=1}^r \int_{x_{i+1}}^{y_i} P_i^2 dg_{i(i+1)} = 0 \pmod{2\pi}.$$

This proves the first part of our theorem. The converse part now is obvious.

It follows easily from the proof of the above theorem that

Theorem 4. Suppose that M is a manifold diffeomorphic to a finite interval of the real line $\mathbf{R}^1$. If E is an orthogonal vector bundle over M of $\dim 2$, then there exist two linearly independent harmonic sections of E .

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