A Nonlinear Sturm-Liouville Problem

J. W. MACKI & P. WALTMAN

Communicated by JÜRGEN MOSER

1. Introduction. In this paper we consider the nonlinear eigenvalue problem

$$(1)_p \quad -(p(x)u'(x))' + q(x)u(x) = \lambda[a(x) - f(x, u(x), u'(x))]u(x),$$

$$(2) \quad a_1u(0) + a_2u'(0) = 0$$

$$b_1u(1) + b_2u'(1) = 0$$

where when $f = 0$ the resulting linear problem has eigenvalues $0 < \lambda_0 < \lambda_1 < \dots$. This problem was studied by Crandall and Rabinowitz [4] who used the theory of Leray-Schauder degree to study the existence of eigenfunctions $u(x, \lambda)$. The type of questions considered are motivated by the work of Wolkowisky [10] (and references therein) where the existence of eigenfunctions corresponds to the existence of buckled states in elasticity problems [11].

Since we would assume $p(x) > 0$ on $[0, 1]$, we simply take $p(x) \equiv 1$ and treat the equation

$$(1) \quad -u'' + q(x)u = \lambda[a(x) - f(x, u(x), u'(x))]u(x).$$

The methods used are such that $(1)_p$ and (1) are equivalent via a simple change of variable.

In an earlier paper [7], the authors used a simple polar coordinate technique to study eigenvalue problems for two dimensional systems. That work was motivated by the papers of Bailey [1], Estabrooks and Macki [5], and Brauer [2]. We shall apply this same basic approach to the problem $(1)(2)$, to obtain the "local" theorem of [4] without the machinery of functional analysis, degree theory, *etc.* Since these problems are physically important, this puts the mathematical results within the grasp of a wider audience.

Given the "local" theorem (or the "small" solutions), in [4] an additional hypothesis of an *a priori* bound was added to obtain a global theorem; an *a priori* bound was also postulated in [10] (in both cases sufficient conditions were given for its existence). Here we replace the assumption of an *a priori* bound with a geometrical hypothesis. In the presence of the continuability of solutions of

certain initial value problems over $[0, 1]$, our geometrical assumption is weaker than an *a priori* bound; in fact it is necessary and sufficient for the conclusion of the global theorem. Finally we note that Rabinowitz [9] has achieved more general results on this problem, using degree theory.

We set up the polar coordinate system in Section 2. In Section 3 we establish some basic lemmas and prove the existence of small solutions of (1)(2) with k zeros for λ in some interval to the right of λ_k . In Section 4 we establish the existence of solutions of (1)(2) with k zeros for all $\lambda > \lambda_k$. We also show that the *a priori* bound hypothesis of [4] implies our geometrical hypothesis. In Section 5 we comment on one part of one theorem of Wolkowisky [10]. In Section 6 we consider two examples.

2. Preliminaries. For the most part we work with two hypotheses, one concerning the associated linear problem (obtained by setting $f \equiv 0$ in (1)) and one concerning the nature of the functions in (1). Specifically we assume

(H-1) *The linear problem obtained by setting $f \equiv 0$ in (1) and keeping (2) has eigenvalues $0 < \lambda_0 < \lambda_1 < \dots$, accumulating at $+\infty$, and the eigenfunction (unique up to a constant factor) corresponding to λ_k has exactly k simple zeros in $(0, 1)$.*

(H-2) *$f \in C([0, 1] \times \mathbb{R}^2)$, $f(x, 0, 0) \equiv 0$, and there exists a constant ρ such that $f(x, \xi, \eta) > 0$ for $x \in [0, 1]$, $0 < |\xi|, |\eta| < \rho$. Furthermore, $a(x)$ and $q(x)$ are continuous on $[0, 1]$ with $a(x) > 0$ ($p(x) > 0$ and continuous in $(1)_x$).*

Some of our conclusions are true under the weaker hypothesis

$$f(x, \xi, \eta) \geq 0 \quad \text{for } x \in [0, 1], \quad 0 < |\xi|, |\eta| < \rho.$$

We refer to (H-2) with this weaker inequality as (H-2)'.

Note that if $y(x, \lambda)$ solves (1) for a given λ , with $y'(x_0, \lambda) = y(x_0, \lambda) = 0$ for some $x_0 \in [0, 1]$, then $y \equiv 0$ on $[0, 1]$, since both y and the null function solve the linear equation

$$y'' + B(x)y = 0, \quad B(x) = -q(x) + \lambda[a(x) - f(x, y(x), y'(x))].$$

(This observation is due to Jay Wolkowisky [12], p. 651.)

(H-1)(H-2) are the hypotheses of [4]. Note in particular that we do not assume the uniqueness of solutions of initial value problems for equation (1). To avoid this assumption, we need to make use of a strengthened form of Kneser's theorem. Specifically let $\mathbf{P}$ be a compact connected subset of the (y, y') -plane. Assume all solutions of (1) with initial conditions in $\mathbf{P}$ exist over $[0, 1]$. Let $\{y(1, p)\}$ denote the set of values at $x = 1$ of all solutions through $p \in \mathbf{P}$. Then $\Gamma = \bigcup_{p \in \mathbf{P}} y(1, p)$ is a compact, connected set. This fact appears in Sell [8, Th. 5]. It also appears in more accessible form in Coppel [3, p. 21] under a boundedness assumption, but the boundedness seems to be needed only for continuability, which we prove in Lemma 1 or assume directly in Section 4.

The lack of uniqueness of solutions of the initial value problem and the possibility of noncontinuable solutions cause some complications in the proofs. Since uniqueness of solutions of initial value problems is not assumed in [4], we also wish not to make that assumption here. The reader who does not care to bother with such details, should assume uniqueness and continuability of solutions of the initial value problem and thus simplify the proofs. The theorems on solution funnels are usually stated for autonomous systems and for initial conditions rather than for parameters. They can be applied in circumstances such as ours by introducing a new independent variable t and new equations $x' = 1$ and $\lambda' = 0$ and treating a vector equation.

In association with (1)(2) we use the polar coordinate functions (r, θ) defined (up to an additive factor of 2π in the case of θ) by $y(x) = r(x) \cos \theta(x)$, $y' = r(x) \sin \theta(x)$, yielding the equivalent system

$$(3) \quad \begin{aligned} \theta' &= -\sin^2 \theta - (\lambda a - q) \cos^2 \theta + \lambda f \cos^2 \theta, \\ r' &= r \cos \theta \sin \theta (1 + \lambda f - \lambda a + q), \end{aligned}$$

$$(4) \quad \theta(0) = \text{Arctan}(-a_1/a_2) \equiv \alpha, \theta(1) = \text{Arctan}(b_1/b_2) - k\pi,$$

where $-\pi/2 < \alpha \leq \pi/2$ and k is any integer.

For convenience we define $\beta_k = \text{Arctan}(-b_1/b_2) - k\pi$. Note that $\theta'(x) < 0$ if $\theta = (2j+1)\pi/2$, so $\theta(x) < \pi/2$ on $(0, 1]$ and we need only consider $k \geq 0$. For the associated linear problem we use corresponding polar coordinates (ρ, φ) , obtaining the problem (3'), (4'), where

$$(3') \quad \begin{aligned} \varphi' &= -\sin^2 \varphi - (\lambda a - q) \cos^2 \varphi, \\ \rho' &= \rho \cos \varphi \sin \varphi (1 + q - \lambda a) \end{aligned}$$

and (4') is (4) with θ replaced by φ . Note that $\varphi(1, \lambda)$ is strictly decreasing in λ ; therefore $\pi/2 > \varphi(1, \lambda) > \beta_0$ for $\lambda < \lambda_0$; note also that φ is completely independent of ρ .

When solutions of initial value problems are not unique, inequalities involving the polar coordinates are to be taken to mean that the inequality is satisfied for every solution in the funnel through the prescribed initial conditions.

3. The local theorem. We first prove two lemmas which are basic to our work. (All of (H-1), (H-2)' are not needed here but will be used later.)

Lemma 1. Assume (H-1) and (H-2)' hold. If λ is restricted to $[0, \lambda^*]$, then there is a $\delta(\lambda^*) > 0$ and $K(\lambda^*) > 0$ such that if $r(0, \lambda) < \delta$, then

$$r(x, \lambda) \leq Kr(0, \lambda)$$

for all $x \in [0, 1]$. (In particular, solutions of (1) are continuable if $r(0, \lambda)$ is sufficiently small.)

Proof. From H-2, if λ is fixed then for $\epsilon > 0$ there is a $\delta_1(\epsilon)$ such that if $0 \leq r(x, \lambda) < \delta_1$, then

$$0 \leq f(x, y, y') < \epsilon, \quad \text{for all } x \in [0, 1].$$

If $0 < r(0, \lambda) < \delta$, then (3) implies the estimate (valid as long as $r(x, \lambda) < \delta_1$)

$$\begin{aligned} r(t, \lambda) &\leq r(0, \lambda) \exp \left[\int_0^t (1 + |q| + \lambda a + \lambda f) ds \right] \\ &\leq r(0, \lambda) K \end{aligned}$$

where

$$K = \exp \left[\int_0^1 (1 + |q| + \lambda^* a + \lambda^* \epsilon) ds \right].$$

Choose δ such that $\delta K < \delta_1$. The conclusion of the lemma follows.

Thus far we have suppressed the dependence of θ on the initial value $r(0)$. From now on we write $\theta(x, \lambda, \mu)$ where $\mu = r(0)$; note that $\theta(0) = \alpha$ is fixed. Since $\varphi(x)$ is independent of $\rho(0)$, we continue to write $\varphi(x, \lambda)$.

Lemma 2. Assume (H-1) and (H-2)' hold. If $\lambda > \lambda_i$, then there exists $r_1(\lambda)$ such that if $0 < \mu < r_1$, then $\theta(1, \lambda, \mu) < \varphi(1, \lambda_i) = \beta_i$, for every solution with $r(0) = \mu$, $\theta(0) = \alpha$.

Proof. Fix λ and choose $\delta > 0$ so that if $0 < u^2 + v^2 < \delta^2$, then $0 \leq f(x, u, v) < a_m(\lambda - \lambda_i)/\lambda$, where $a_m = \min_{[0,1]} a(t) > 0$. Using Lemma 1, choose $r_1(\lambda)$ so small that if $0 < \mu < r_1$, then $0 < r(x, \lambda, \mu) < \delta$. Fix a solution of equation (1) in the funnel with $\theta(0) = \alpha$, $r(0) = \mu < r_1$. Then $f(x, y(x), y'(x)) < a_m(\lambda - \lambda_i)/\lambda$, $x \in [0, 1]$. Thus

$$\begin{aligned} \theta'(x, \lambda, \mu) &= -\sin^2 \theta - (\lambda a - q) \cos^2 \theta + \lambda f(x, y, y') \cos^2 \theta \\ &< -\sin^2 \theta - (\lambda a - q) \cos^2 \theta + a(x)(\lambda - \lambda_i) \cos^2 \theta, \quad \text{a.e.} \\ &= -\sin^2 \theta - (\lambda_i a - q) \cos^2 \theta. \end{aligned}$$

(The set $\{t \mid \theta(t, \lambda, \mu) = \pi/2 + n\pi\}$ is discrete for each solution since $\theta' = -1$ there.) Since

$$\varphi'(x, \lambda_i) = -\sin^2 \varphi - (\lambda_i a - q) \cos^2 \varphi,$$

and $\theta(0, \lambda, \mu) = \varphi(0, \lambda_i)$, a standard comparison technique [6, p. 26] implies that

$$\theta(1, \lambda, \mu) < \varphi(1, \lambda_i) = \beta_i.$$

Theorem 1. (Crandall and Rabinowitz [4], Theorem 3.4). Let (H-1) and (H-2) hold. Then for each λ_i , $j = 0, 1, \dots$, there exists a nondegenerate interval $I_j = (\lambda_j, \lambda_j^*)$ such that for each $\lambda \in I_j$, (1)(2) has at least two solutions with exactly j zeros in $(0, 1)$.

Proof. Let δ_1 be as in the proof of Lemma 1. Then if $y(x)$ is a solution of equation (1) with $0 < r(0) = \mu < \delta_1$, and $\theta(0) = \alpha$, we have $f(x, y(x), y'(x)) > 0$. It follows that

$$\begin{aligned}\theta'(x, \lambda_i, \mu) &= -\sin^2 \theta - (\lambda_i a - q) \cos^2 \theta + \lambda_i f(x, y, y') \cos^2 \theta \\ &> -\sin^2 \theta - (\lambda_i a - q) \cos^2 \theta, \quad \text{a.e.}\end{aligned}$$

Since

$$\varphi'(x, \lambda_i) = -\sin^2 \varphi - (\lambda_i a - q) \cos^2 \varphi,$$

and $\varphi(0) = \theta(0)$, we have $\theta(1, \lambda_i, \mu) > \varphi(1, \lambda_i)$ for every solution in the funnel with the selected initial condition. Since funnel sections are compact there is an open set U containing the section and with polar coordinates satisfying $\theta > \varphi(1, \lambda_i)$. Hence [8, Cor. 3.1] there is an open set containing (μ, α, λ_i) whose funnel section at $t = 1$ lies in U . In particular, fixing α, μ we obtain an interval $J_i(\alpha, \mu) = [\lambda_i, \Lambda_i)$ such that $\theta(1, \lambda, \mu) > \beta_i$ for $\lambda \in J_i$. (This is upper-semicontinuity of the solution map viewed as a point to set mapping.) Since α is always fixed, we will write $J_i(\mu)$ instead of $J_i(\alpha, \mu)$. Let

$$I_i = \bigcup_{0 < \mu < \delta} J_i(\mu) - \{\lambda_i\} = (\lambda_i, \lambda_i^*).$$

If $\lambda \in I$, then $\lambda \in J_i(\mu_1) - \{\lambda_i\}$ for some μ_1 , and $\theta(1, \lambda, \mu_1) > \beta_i$. By Lemma 2, there exists $\mu_2 > 0$ such that $\theta(1, \lambda, \mu_2) < \beta_i$. Since all solutions with $\mu \in [\mu_2, \mu_1]$ are continuable over $[0, 1]$, it follows from the connectedness of the sections of solution funnels that there are initial conditions $\theta(0) = \alpha, r(0) = \mu_3$ such that for some solution of the funnel $\theta(1, \lambda, \mu_3) = \beta_i$, i.e., we have a solution of (1)(2).

The nodal properties of this solution follow from a counting of the crossings of $\theta = \pi/2 - k\pi, k = 1, 2, \dots, j$. That there are (at least) two solutions follows from the fact that only $y^2 + y'^2$ and y'/y are specified at $x = 0$. The same arguments apply replacing (y, y') by $(-y, -y')$.

Remark. We note that the strict inequality in $\theta(1, \lambda_i, \mu) > \varphi(1, \lambda_i)$ was necessary to obtain a nondegenerate interval (λ_i, Λ_i) . We can retain this strict inequality and weaken the hypothesis on the sign of f . Specifically, it is sufficient to assume for every $x \in [0, 1]$, $f(x, \xi, \eta) > 0$ on a fixed nondegenerate cone C in the $y - y'$ plane interior to the cone formed by transversing clockwise between the lines $a_1 y + a_2 y' = 0, b_1 y + b_2 y' = 0$.

4. The global existence theorem. Knowing there is a solution of (1)(2) with k interior zeros for λ sufficiently close to λ_k , we wish to add conditions such that (1)(2) has such a solution for each $\lambda > \lambda_k$. In [4] this was accomplished by the following assumption.

There is a continuous, strictly increasing function $M(\lambda)$ such that if (λ, μ) is a solution of (1)(2), then for all $x \in [0, 1]$, $r(x, \lambda) < M(\lambda)$.

We shall show first that if sections of solution funnels are connected, then the above has the following geometric consequence:

(H-3) For every λ , there is a $\mu(\lambda)$ such that $\theta(1, \lambda, \mu) \geq \beta_0$.

Proof. If there are no eigenfunctions corresponding to the eigenvalue λ with $\mu > M(\lambda)$, then for some fixed integer $p \geq 0$ and all $\lambda \in [0, \infty)$, $\mu > M(\lambda)$ implies

$$\beta_p < \theta(1, \lambda, \mu) < \beta_{p-1}.$$

(Otherwise there would be a solution of (1),(2) with $\mu > M(\lambda)$, since $\theta(1, \lambda, \mu)$ would cut one of the planes $\theta = \beta_i$.) Suppose $p \neq 0$. For $\lambda = 0$ the equations for $\theta(x, \lambda, \mu)$ and $\varphi(x, \lambda)$ coincide. Now $\varphi(1, 0) > \beta_0$, so $\theta(1, 0, \mu) > \beta_0$ for all μ . However, if $\mu > M(0)$, then $\theta(1, 0, \mu) < \beta_{p-1}$, $p \neq 0$, which implies $\beta_{p-1} > \beta_0$, a contradiction.

Remark. If the function $M(\lambda)$ is given only for $\lambda > 0$, a somewhat more complicated argument can be used to show that (H-3) holds.

When working with an *a priori* bound, one need only consider functions with bounded $r(x, \lambda)$ (for fixed λ), and hence the continuability of solutions is no problem. Under the assumption (H-3) and the absence of conditions on the nonlinear term f , we cannot assert continuability of solutions, and therefore cannot assume that solution funnel sections are connected.

We therefore make the additional assumption of continuability of solutions to certain initial value problems for (1).

Theorem 2. Assume (H-1) and (H-2) hold and that all solutions of (1) with $a_1 u(0) + a_2 u'(0) = 0$, are extendable over $[0, 1]$. Then (H-3) is a necessary and sufficient condition for the existence of a solution of (1), (2) with exactly k interior zeros for each $\lambda > \lambda_k$, $k = 0, 1, 2, \dots$.

Proof. Fix $\lambda > \lambda_k$. By (H-3) there exists μ_1 such that $\theta(1, \lambda, \mu_1) \geq \beta_0$. By Lemma 2, there exists μ_2 such that $\theta(1, \lambda, \mu_2) < \beta_k$. The result then follows, since the point to set mapping $\theta(1, \lambda, \cdot)$ maps $[\mu_1, \mu_2]$ into a compact connected set.

If (H-3) fails for a given $\lambda > \lambda_0$, then there is no zero-free eigenfunction $u(x, \lambda)$.

5. Other results. In this section we apply Lemma 2 to obtain, under our extendability assumption, a short proof of a part of a theorem of Wolkowsky (Theorem 2 of ([10]).

Theorem 3. If (H-1), (H-2)', (H-3) hold with $\rho = +\infty$ in (H-2)', and if solutions of (1), $a_1 y(0) + a_2 y'(0) = 0$, exist over $[0, 1]$, then for every $\lambda > \lambda_k$, there exists a solution of (1),(2) with k zeros in $(0, 1)$.

Proof. Let $\lambda > \lambda_k$. By Lemma 2 there is a μ_1 such that $\theta(1, \lambda, \mu_1) < \beta_k$. By (H-3) there is a μ_2 such that $\theta(1, \lambda, \mu_2) \geq \beta_0$. The connectivity of sections of solution funnels implies the existence of μ_k such that for some solution, $\theta(1, \lambda, \mu_k) = \beta_k$.

Our assumption $\theta(1, \lambda, \mu) \geq \beta_0$ is implied by the existence of an *a priori* bound on solutions of (1), (2), as shown in the previous section; in particular an *a priori* bound on translates as postulated in [10], would not be required.

If (H-3) fails for a given $\lambda > \lambda_0$, then there is no zero-free eigenfunction $u(x, \lambda)$.

6. Examples. First we show that (H-3) does not imply the existence of an *a priori* bound. Define

$$f^*(x, u, v) = \begin{cases} \text{To satisfy (H-1), (H-2),} & u^2 + v^2 \leq 1; \\ 1 & , \quad 2 \leq u^2 + v^2 \leq 3; \\ 0 & , \quad u^2 + v^2 \geq 4; \end{cases}$$

and let $f(x, u, v)$ be a bounded continuous extension to $\mathbf{R}^3$ (using the Tietze extension theorem).

Consider

$$(4) \quad y'' + \lambda y = \lambda f(x, y, y')y,$$

$$(5) \quad y'(0) = 0 \quad (\theta(0) = 0),$$

$$(6) \quad y(1) + y'(1) = 0, \quad (\theta(1) = -\pi/4).$$

For any λ , $2 \leq \mu \leq 3$, ($\mu = r(0)$), a solution of (4), (5) is given by $y(x) \equiv 1$, i.e., $\theta(x, \lambda, \mu) \equiv 0 > -\pi/4$, so (H-3) holds. Since (4) is linear for $r(x, \lambda) \geq 4$, there is no bound on the eigenfunctions.

Since both the existence of an *a priori* bound and the condition (H-3) are rather intangible, it is of interest to note particular cases where these conditions are met. We consider a problem treated as Theorem 5.1 of [4]. Specifically, we consider (1) with boundary conditions

$$u(0) - c_0 u'(0) = 0, \quad c_0 > 0,$$

$$u(1) + c_1 u'(1) = 0, \quad c_1 > 0,$$

where $f(x, u, 0) > a(x)$ if $|u| > K_1$, $x \in [0, 1]$ and $f \geq -K_2$ for $x \in [0, 1]$, $(u, v) \in \mathbf{R}^2$, and $q \geq 0$. We show essentially that no trajectory crosses the darkened lines in Figure 1. Let λ be fixed.

For $\theta = n\pi + \pi/2$, $\theta'(x, \lambda, \mu) = -1 < 0$, so trajectories cross these half lines clockwise. Furthermore, if $\theta = n\pi$, and $|u(x)| > K_1$, then

$$\theta'(x, \lambda) = -\lambda(a - f) + q > 0$$

so trajectories cross these half-lines counterclockwise. Let a_M and q_M denote the maxima of a and q over $[0, 1]$. Then from (3)

$$r(x, \lambda, \mu) \geq \mu \exp \left[-1 - q_M - a_M + \lambda \int_0^1 \cos \theta \sin \theta f ds \right].$$

So for θ in the interior of the first and third quadrants ($\sin \theta \cos \theta > 0$) we can choose μ so large that $r(x, \lambda, \mu) > K_1$ for $x \in [0, 1]$. For this choice of μ ,

$$\theta(1, \lambda, \mu) > 0 > -\pi/4$$

so that (H-3) is satisfied.

FIGURE 1

REFERENCES

1. P. BAILEY, *An eigenvalue theorem for nonlinear second order differential equations*, J. Math. Anal. Appl. 20 (1967), 94-102.
2. F. BRAUER, *Nonlinear perturbations of Sturm-Liouville boundary value problems*, J. Math. Anal. Appl. 22 (1968), 591-598.
3. W. A. COPPEL, *Stability and asymptotic behavior of differential equations*, Heath and Co., Boston, 1965.
4. M. G. CRANDALL & P. H. RABINOWITZ, *Nonlinear Sturm-Liouville eigenvalue problems and topological degree*, J. Math. Mech. 19 (1970), 1083-1102.
5. M. F. ESTABROOKS & J. W. MACKI, *A nonlinear Sturm-Liouville problem*, J. Differential Equations 10 (1971), 181-187.
6. P. HARTMAN, *Ordinary differential equations*, J. Wiley & Sons, New York, 1964.
7. J. W. MACKI & P. WALTMAN, *A nonlinear boundary value problem of Sturm-Liouville type for a two dimensional system of ordinary differential equations*, SIAM J. Appl. Math. 21 (1971), 225-231.
8. G. R. SELL, *On the fundamental theory of ordinary differential equations*, J. Differential Equations 1 (1965), 370-392.
9. P. H. RABINOWITZ, *Nonlinear Sturm-Liouville problems for second order ordinary differential equations*, Comm. Pure Appl. Math. 23 (1970), 939-961.
10. J. H. WOLKOWISKY, *Nonlinear Sturm-Liouville problems*, Arch. Rat. Mech. Anal. 35 (1969), 299-320.
11. ———, *Existence of buckled states of circular plates*, Comm. Pure Appl. Math. 20 (1967), 549-560.

12. ———, *Buckling of the circular plate embedded in elastic springs. An application to geophysics*, Comm. Pure Appl. Math. 22 (1969), 639–667.

This research was supported by the Defence Research Board of Canada, Grant No. DRB-9540-11 and by National Science Foundation Grant GP-19071.

University of Alberta

University of Iowa

Date communicated: MARCH 2, 1972