Zeros of Power Series with Large Gaps

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1. Introduction and statement of results. Let $\{n_k\}_0^\infty$ denote a strictly increasing sequence of non-negative integers, and let

$$(1) \quad f(z) = \sum_0^\infty c_k z^{n_k}$$

be the Taylor series expansion about zero for a function f which is analytic in $|z| < 1$. Let $M(r) = \max |f(z)|$, $|z| = r$, and let $n(r, 1/f)$ be the number of zeros of f in $|z| \leq r$. We prove the following theorem:

Theorem 1. *Let f be an analytic function in $|z| < 1$ and have the form (1). If the sequence $\{n_k\}$ satisfies*

$$(2) \quad \sum_{\substack{k=0 \\ k \neq \nu}}^\infty \frac{n_\nu^2}{(n_k - n_\nu)^2} = o\{(\log n_\nu)^2\}, \quad (\nu \rightarrow \infty),$$

and if

$$(3) \quad \limsup_{r \rightarrow 1^-} \frac{\log M(r)}{-\log(1-r)} > 2,$$

then

$$(4) \quad \limsup_{r \rightarrow 1^-} \frac{n(r, 1/f)}{-\log(1-r)} > 0.$$

Theorems of the above type for “faster” growth conditions have been proved by A. Wiman [13] (implicitly), F. Sunyer I. Balaguer [11], and the authoress [10]. The method of [10] also yields the following theorem:

Theorem A. *Let f be an analytic function in $|z| < 1$ and have the form (1). Let $\alpha \geq 0$. If*

$$\prod_{\substack{k=0 \\ k \neq \nu}}^\infty \left| \frac{n_k + n_{k+1} + 2 + 2n_\nu}{n_k + n_{k+1} + 2 - 2n_\nu} \right| = O(n_\nu^\alpha), \quad (\nu \rightarrow \infty),$$

and if

$$(5) \quad \limsup_{r \rightarrow 1^-} \frac{\log M(r)}{-\log(1-r)} > 2(2 + \alpha),$$

then (4) holds.

Recently J. M. Anderson [1] showed

Theorem B. *Let f be an analytic function in $|z| < 1$ and have the form (1). Let β satisfy $\frac{1}{2} \leq \beta < 1$. If for each $\epsilon > 0$*

$$(6) \quad \prod_{\substack{k=0 \\ k \neq \nu}}^{\infty} \left| \frac{n_{\nu}^{\beta} + n_k^{\beta}}{n_{\nu}^{\beta} - n_k^{\beta}} \right| = O(n_{\nu}^{\epsilon}), \quad (\nu \rightarrow \infty),$$

and if (3) holds, then f assumes every value infinitely often.

The series in (1) is said to have Hadamard gaps if

$$(7) \quad \frac{n_{k+1}}{n_k} \geq q > 1, \quad (k \geq 1).$$

We note that the series allowed by the gap condition of Theorem B are “very close” to series with Hadamard gaps (the product in (6) is $O(1)$ for Hadamard gaps). Theorem 1 includes series with Hadamard gaps, but, more generally, may be seen to include series for which the sequence $\{n_n\}$ satisfies $n_k = [\exp(k^{\alpha})]$, $\alpha \geq \frac{1}{2}$. Further the method of proof shows that if

$$n_{k+1} - n_k \geq \frac{n_k}{\log n_k} \quad \text{and} \quad \frac{k}{(\log n_k)^2} = O(1), \quad (k \rightarrow \infty),$$

and if

$$\limsup_{r \rightarrow 1^-} \frac{\log M(r)}{-\log(1-r)} \geq 120(\log 2) + 2,$$

then (4) holds. In particular, the series allowed in Theorem 1 are also quite close to series with Hadamard gaps, but one does get the stronger conclusion (4) over Theorem B. Related results appear in [4] and [12].

The product in the hypothesis of Theorem A allows series which consist of say the sum of α series which each have Hadamard gaps (not included in either Theorems 1 or B), but the method necessitates the stronger assumption (5) in order to obtain (4). (See also [9].)

The method of proving Theorem 1 also yields the following corollaries which are stated without proof:

Corollary 1. *Let f be an analytic function in $|z| < 1$ and have the form (1). Suppose $f(0) \neq 0$ and r_0 is the modulus of the zero of f (or a zero of f) nearest the origin. Let $\delta > 0$. If (2) holds and if*

$$\limsup_{r \rightarrow 1^-} \frac{\log M(r)}{-\log(1-r)} \geq 2 + 2\delta,$$

then

$$\limsup_{r \rightarrow 1^-} \frac{n(r, 1/f)}{-\log(1-r)} \geq r_0 \cdot \delta/2.$$

By a path emanating from the origin to $|z| = 1$ we shall mean a Jordan curve emanating from the origin and extending to $|z| = 1$.

Corollary 2. *Let f be an analytic function in $|z| < 1$ and have the form (1). If (2) and (3) hold, then f is unbounded on any path from the origin to $|z| = 1$. In particular, f cannot approach any finite value on a path from the origin to $|z| = 1$.*

If f is analytic and non-constant in $|z| < 1$, then f is said to be of class $\mathfrak{A}$ (see G. R. MacLane [8]) provided there exists an everywhere dense set D of points on $|z| = 1$ with the property that for each ζ in D , there is a complex number a such that on a path tending to ζ we have $f(z) \rightarrow a$. Corollary 2 together with an argument introduced by Anderson [1] shows

Corollary 3. *Let f be an analytic function in $|z| < 1$ and have the form (1). If (2) and (3) hold, and if $f \in \mathfrak{A}$, then $\mathcal{CR}(f, \zeta) = \infty$ for each ζ on $|z| = 1$. ($\mathcal{R}(f, \zeta)$ is the so-called "range set at ζ "—see [3]).*

In the special case of Hadamard gaps the result below of K. G. Binmore [2] enables us to improve Theorem 1.

Theorem C. *Let f be an analytic function in $|z| < 1$ and have the form (1). Suppose the sequence $\{n_k\}$ satisfies (7). Then there exists a constant K depending on q such that*

$$|c_k| \leq K \sup_{z \notin \mathfrak{C}} |f(z)|$$

for every path $\mathfrak{C}$ from the origin to $|z| = 1$.

We obtain:

Theorem 2. *Let f be an analytic function in $|z| < 1$ and have the form (1), where the sequence $\{n_k\}$ satisfies (7). Let $\beta > 0$. If*

$$\limsup_{r \rightarrow 1^-} \frac{\log M(r)}{-\log(1-r)} = 1 + \beta,$$

then (4) holds.

Clearly one obtains corresponding improvements of Corollaries 1 and 2. Corresponding to Corollary 3 we note that Fuchs [5] recently showed that functions satisfying the hypothesis of Theorem C with $c_k \rightarrow 0$, ($k \rightarrow \infty$), assume every value infinitely often in every sector of positive angular opening formed by two radii of the unit disc.

2. Proofs of the theorems. We use the following two lemmas.

Lemma 1. (A. E. Ingham [6]). *For each integer $N \geq 2$ there exists a function P_N defined for $t \geq -1$ with the following properties:*

- (i)
$$P_N(t) = \sum_{i=1}^{2N} p_i 2^{-it}, \quad p_i = p_i(N),$$
- (ii)
$$\sum_1^{2N} |p_i| < 2^{4N},$$
- (iii)
$$0 \leq P_N(t) \leq P_N(0) = 1, \quad (t \geq -1),$$
- (iv)
$$P_N(t) < \frac{5}{N^2 t^2}, \quad (t \geq -1; t \neq 0).$$

Lemma 2. *Let f be an analytic function in $|z| < 1$. Let $\nu(r)$ denote the central index on $|z| = r$ of the Taylor series expansion for f at zero, and let $\mu(r)$ denote the maximum term of the series for $|z| = r$. Then outside a set E of values of r of zero-density the following hold:*

- (i)
$$\nu(r) \leq \frac{C_1}{1-r} (\log \mu(r))^2,$$
- (ii)
$$\frac{M(r)}{(\log M(r))^3} \leq \frac{C_2}{1-r} \mu(r),$$

where C_1 and C_2 are constants.

Part (ii) in Lemma 2 is a special case of a result in T. Kövari [7]. For (i) in Lemma 2 see Anderson [1].

To prove Theorem 1 we will let $n_\nu = \nu(r)$ denote the central index of the series (1) for $|z| = r$ and $\mu(r)$ the maximum term for $|z| = r$. We may assume $f(0) \neq 0$. For $\epsilon' > 0$ condition (2) implies that there is a ν_0 such that

$$(8) \quad \sum_{\substack{k=0 \\ k \neq \nu}}^{\infty} \frac{n_\nu^2}{(n_k - n_\nu)^2} \leq \frac{(\epsilon')^2}{10} (\log n_\nu)^2, \quad (\nu \geq \nu_0).$$

Choose $N^2 = (\epsilon')^2 (\log n_\nu)^2$ where $\nu \geq \nu_0$. We may assume N is an integer and form

$$S = \sum_{k=0}^{\infty} c_k z^{n_k} P_N \left(\frac{n_k - n_\nu}{n_\nu} \right)$$

where P_N is the function of Lemma 1. Using (i) and (ii) of Lemma 1, we see readily that for $z = re^{i\theta}$

$$(9) \quad \begin{aligned} |S| &= \left| \sum_1^{2N} p_i 2^i f(z 2^{-i/n_\nu}) \right| \\ &\leq \sum_{i=1}^{2N} |p_i| 2^{2N} \sup_{1 \leq i \leq 2N} |f(z 2^{-i/n_\nu})| \\ &\leq 2^{6N} \sup_{0 \leq t \leq 1} |f(r t e^{i\theta})|. \end{aligned}$$

On the other hand, (iii) and (iv) of Lemma 1 together with (8) show

$$(10) \quad |S| \geq \mu(r) - \mu(r) \sum_{\substack{k=0 \\ k \neq \nu}}^{\infty} \frac{5}{N^2} \frac{n_{\nu}^2}{(n_k - n_{\nu})^2} \\ \geq \frac{1}{2} \mu(r).$$

Combining (9) and (10) and taking logarithms, we see

$$\log \frac{1}{2} + \log \mu(r) \leq 12\epsilon'(\log n_{\nu}) + \log \left\{ \sup_{0 \leq t \leq 1} |f(rte^{i\theta})| \right\}.$$

The inequalities of Lemma 2 show that for values of r outside E we have

$$\log M(r) - 3 \log \log M(r) - \log \frac{1}{1-r} + O(1) \\ \leq 12\epsilon' \left(\log \frac{1}{1-r} + 2 \log \log \mu(r) \right) + \log \left\{ \sup_{0 \leq t \leq 1} |f(rte^{i\theta})| \right\}.$$

Since (3) holds, for values of r outside E we thus see

$$(11) \quad (1 + o(1)) \log M(r) \leq (1 + 12\epsilon') \log \frac{1}{1-r} + \log \left\{ \sup_{0 \leq t \leq 1} |f(rte^{i\theta})| \right\}.$$

Condition (3) easily implies that there is an $\epsilon > 0$ and values of r arbitrarily near one for which $r \notin E$ and

$$(12) \quad \log M(r) > (2 + \epsilon) \log \frac{1}{1-r}.$$

Recalling that ϵ' is an arbitrary positive number, we note that (11) and (12) show that there are values of r arbitrarily near one for which $r \notin E$ and uniformly in θ

$$(13) \quad \left(1 - \frac{1}{2+\epsilon} - \frac{\epsilon}{8+4\epsilon} \right) \log M(r) < \log \left\{ \sup_{0 \leq t \leq 1} |f(rte^{i\theta})| \right\}.$$

In fact, (13) implies that there exists a curve $\mathcal{K}$ in $|z| \leq r$ about the origin along which

$$(14) \quad |f(z)| \geq (M(r))^{\gamma}$$

where $\gamma = \epsilon/(8 + 4\epsilon)$. If such a curve did not exist, there would be a curve $\mathcal{C}$ from the origin to $|z| = r$ along which $|f(z)| < M(r)^{\gamma}$. Let z_0 be the first point at which $\mathcal{C}$ intersects $|z| = r$ and assume (without loss of generality) that z_0 is real. In order to follow a reflection trick of G. Pólya, we let $\mathcal{R}$ be the radial segment joining the origin to z_0 , and let $\bar{\mathcal{C}} = \{\bar{z} : z \text{ is in } \mathcal{C}\}$. Let $\mathcal{D}$ denote the domain which has its boundary formed by $\mathcal{C}$ and $\bar{\mathcal{C}}$ and which encompasses $\mathcal{R}$. If we consider the function F defined in $\mathcal{D}$ by $F(z) = f(z)\overline{f(\bar{z})}$, we see that the maximum principle implies $|F(z)| < M(r)^{\gamma} \cdot M(r)$ for each z in $\mathcal{D}$. In particular, for all real numbers x on $\mathcal{R}$ we have

$$(15) \quad |f(x)|^2 < M(r)^\gamma \cdot M(r).$$

However, (15) contradicts (13), and so we obtain the desired curve $\mathcal{K}$ for each real number r for which (13) is true.

For each real number r for which (13) is true form

$$P_r(z) = \prod_1^n \left(1 - \frac{z}{z_k}\right)$$

where $z_1, \dots, z_n$ are the zeros of f in $|z| \leq r$ and $0 < |z_1| \leq \dots \leq |z_n|$. In $|z| \leq r$ let

$$\exp(h_r(z)) = \frac{f(z)}{P_r(z)},$$

and note that $\exp(h_r(0)) = f(0)$. Further note that

$$|P_r(z)| \leq \left(1 + \frac{1}{|z_1|}\right)^n, \quad (|z| \leq 1),$$

and

$$\log |P_r(z)| \leq n(r, 1/f) \log \left(1 + \frac{1}{|z_1|}\right), \quad (|z| \leq 1).$$

Assume now that Theorem 1 is false. Then for each real number r for which (13) is true we see $|\exp(h_r(z))|$ assumes on the curve $\mathcal{K}$ its minimum value for complex numbers z inside $\mathcal{K}$, and hence

$$\log |\exp(h_r(0))| \geq \gamma \log M(r) - o\left(\log \frac{1}{1-r}\right), \quad (r \rightarrow 1).$$

Thus there is a number r_0 such that for values of r greater than r_0 for which (13) holds, we have

$$(16) \quad \log |f(0)| = \log |\exp(h_r(0))| \geq \frac{\gamma}{2} \log M(r).$$

However, $M(r)$ gets arbitrarily large as r approaches one, and therefore (16) implies a contradiction.

Theorem 2 is proved as above by using Theorem C (in place of Lemma 1) together with Lemma 2 to obtain the curve $\mathcal{K}$.

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