Thinness and Wiener Criteria for Non-linear Potentials

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This paper is dedicated to Eberhard Hopf on the occasion of his seventieth birthday.

Introduction. In this work we will discuss the theory of thin sets relative to non-linear potentials of the form $u_\mu = k * (k * \mu)^{1/(p-1)}$. Here k is a kernel on $\mathbf{R}^n$, which, among other things, is radial and non-increasing in $|x|$, μ is a positive Radon measure and $1 < p < \infty$. Such potentials play a prominent role in the capacity theory of linear potentials $k * f$, where f is a function in the Lebesgue class $\mathfrak{L}_p$; see [7] and [8]. The continuity properties of the two kinds of potentials are closely tied, so that the theory of thin sets for the non-linear potential gives important information concerning the continuity of the linear potential. For a discussion of this see [6].

We distinguish two classes of non-linear potentials, $\mathcal{P}_{k,p}$ and $\mathcal{F}_{k,p}$. The first class contains more or less all potentials of interest, while the second is the subclass consisting of bounded potentials. We consider thinness relative to these two classes. If $E \subset \mathbf{R}^n$ and $x_0 \notin E$ we say that E is $\mathcal{P}_{k,p}$ -thin at x_0 , if $x_0 \notin \bar{E}$ or there exists $u = u_\mu \in \mathcal{P}_{k,p}$ such that

$$u(x_0) < \liminf_{\substack{x \rightarrow x_0 \\ x \in E}} u(x).$$

$\mathcal{F}_{k,p}$ -thinness is defined in the same way except that $u \in \mathcal{F}_{k,p}$.

Our principal concern is the Wiener Criterion for thinness. Here we impose the restriction that in the neighborhood of 0, $k(x)$ be comparable to a kernel $|x|^{\alpha-n}$, $0 < \alpha < n$. Further, we need consider only the case $\alpha p \leq n$, since, in the contrary case, the potentials are continuous functions. Pointwise upper and lower bounds for the non-linear potential, discussed at length in Section 3, play a crucial role. We will outline some of these results in rough form. Let $v(x_0, r) = \int_{\{|x-x_0|<r\}} d\mu(x)$. In all cases, $1 < p \leq n/\alpha$, we get the *lower* bound,

$$Q \int_0^{r_0} (r^{\alpha p - n} v(x_0, r))^{1/(p-1)} \frac{dr}{r} \leq u(x_0),$$

where Q is a positive constant and r_0 is a fixed radius depending only on k . The *upper* bound is far more difficult. If $2 - \alpha/n < p \leq n/\alpha$, then

$$u(x_0) \leq Q \int_0^{4r_0} (r^{\alpha p - n} v(x_0, r))^{1/(p-1)} \frac{dr}{r} + Qu_r(x_0).$$

Here ν is the measure μ cut down to the complement of the ball $\{|x - x_0| < r_0\}$. For our purposes, only the first term is important. The remaining cases, $1 < p \leq 2 - \alpha/n$, pose additional difficulties stemming from the fact that $k^{1/(p-1)}$ is no longer locally integrable. Here, however, under the additional assumption that $u \leq M < \infty$, we get

$$u(x_0) \leq Q_2 M^{((2-p)n - \alpha)/(n - \alpha p)} \int_0^{8r_0} (r^{\alpha p - n} v(x_0, r))^{(n - \alpha)/(n - \alpha p)} \frac{dr}{r} + Q_2 u_r(x_0),$$

$$1 < p < 2 - \alpha/n,$$

and

$$u(x_0) \leq Q_2 \int_0^{8r_0} \left(r^{\alpha p - n} v(x_0, r) \cdot \log \frac{QM^{p-1}}{r^{\alpha p - n} v(x_0, r)} \right)^{1/(p-1)} \frac{dr}{r} + Q_2 u_r(x_0),$$

$$p = 2 - \alpha/n.$$

The essential form of the upper bound in these cases is no longer the same as that for the lower bound, owing to the replacement of $1/(p-1)$ by the smaller exponent $(n - \alpha)/(n - \alpha p)$ when $1 < p < 2 - \alpha/n$ and the presence of the logarithm when $p = 2 - \alpha/n$. In this connection, Example 3.1 shows that the exponent $1/(p-1)$ in the lower bound cannot be decreased. Example 5.1 shows that the exponent $(n - \alpha)/(n - \alpha p)$ in the upper bound cannot be increased.

With the help of Section 4 where certain relations between capacities are discussed, and Section 6 where an inequality of the Harnack type is presented for non-linear potentials, the above inequalities easily lead to the Wiener Criteria of Section 5.

In case $2 \leq p \leq n/\alpha$, we find the Wiener Criteria for $\mathcal{P}_{k,p}$ and $\mathfrak{F}_{k,p}$ -thinness, which proves their equivalence. This is a fact already known in the classical or linear case $p = 2$, but not sufficiently emphasized in the literature.

In case $2 - \alpha/n < p < 2$, we find the Wiener Criterion for $\mathfrak{F}_{k,p}$ -thinness. Note that if $\alpha p = n$, we must be in the first case or the second. However, for $\alpha p < n$ we can have the remaining case, $1 < p \leq 2 - \alpha/n$. Here we can give only a necessary condition and a different sufficient condition for $\mathfrak{F}_{k,p}$ -thinness, a reflection of the disparate lower and upper bounds.

We have recently learned from L. I. Hedberg that our work overlaps with his and that of V. P. Havin and V. G. Maz'ya; see [6] and [5].

1. Preliminaries. For the purposes of this work a *capacity* will be a function

$$C: \mathcal{A} \rightarrow [0, +\infty],$$

where $\mathfrak{A}$ is a class of subsets of $\mathbf{R}^n$ containing the compact sets and closed under countable unions, such that

- (i) $C(\emptyset) = 0$, $\emptyset =$ empty set.
- (ii) $A_1, A_2 \in \mathfrak{A}$ and $A_1 \subset A_2$ implies $C(A_1) \leq C(A_2)$.
- (iii) $A_i \in \mathfrak{A}$, $i = 1, 2, \dots$ implies $C(\cup_i A_i) \leq \sum_i C(A_i)$.

If, in addition, C has the property

$$A \in \mathfrak{A} \text{ implies } C(A) = \sup_{K \subset A} C(K), \quad K \text{ compact,}$$

C is called an *inner capacity*. If, in addition to (i)–(iii), C has the property

$$A \in \mathfrak{A} \text{ implies } C(A) = \inf_{G \supset A} C(G), \quad G \text{ open,}$$

C is called an *outer capacity*. An arbitrary set $A \subset \mathbf{R}^n$ is called *C-capacitable* if

$$\sup_{K \subset A} C(K) = \inf_{G \supset A} C(G), \quad K \text{ compact, } G \text{ open.}$$

A property which holds except on a set A with $C(A) = 0$ is said to hold *C*-a.e.

$\mathfrak{M}$ will denote the space of real Radon measures on $\mathbf{R}^n$ with its usual weak dual topology. Weak convergence without any mention of the space refers to this topology. $\mathfrak{M}_0$ will be the subspace of $\mathfrak{M}$ consisting of measures with compact support. We adopt the practice of denoting positive elements in a space by a superscript plus. Thus $\mathfrak{M}^+$ is the cone of positive Radon measures. If m is Lebesgue measure, then we will drop m from the notation and write for example dx for $dm(x)$, a.e. for m -a.e. If f is real valued a.e. and locally Lebesgue integrable, then we identify f with the Radon measure $f \cdot m$.

$\mathfrak{L}_1$ will be the subspace of $\mathfrak{M}$ composed of all measures μ with

$$\|\mu\|_1 = \text{total variation } \mu < \infty.$$

For $1 < p < \infty$, $\mathfrak{L}_p$ will be the space of functions $f = f(x)$ which are real valued a.e. and measurable with

$$\|f\|_p = \left(\int |f|^p dx \right)^{1/p} < \infty.$$

By a *kernel* we will mean a function $k: [0, +\infty) \rightarrow [0, +\infty]$ which is non-increasing and continuous. k will always appear together with implicitly understood values of n and p , $1 < p < \infty$. Relative to these, k will be assumed to satisfy:

- (i) $\int_0^1 k(r) r^{n-1} dr < \infty.$
- (ii) $r^{-n} \int_0^r k(t) t^{n-1} dt \leq \text{const. } k(r) \quad \text{for } 0 < r \leq 1.$
- (iii) $\int_1^\infty k^{p'}(r) r^{n-1} dr < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1.$

(i) and (iii) are assumptions without which our theory becomes trivial. (ii) appears for technical reasons and can perhaps be weakened or eliminated. We set $k(x) = k(|x|)$ for $x \in \mathbb{R}^n$ and also refer to $k = k(x)$ as a *kernel*. Convolutions involving k refer to $k = k(x)$.

Two cases of special interest are Riesz kernels,

$$h_\alpha(r) = \frac{\Gamma[(n - \alpha)/2]}{2^\alpha \pi^{n/2} \Gamma(\alpha/2)} r^{\alpha-n}, \quad 0 < \alpha < n,$$

and Bessel kernels,

$$g_\alpha(r) = \frac{1}{2^n \pi^{n/2} \Gamma(\alpha/2)} \int_0^{+\infty} e^{-t} e^{-r^2/4t} t^{(\alpha-n)/2} \frac{dt}{t}, \quad 0 < \alpha < \infty.$$

g_α is comparable to h_α , for $0 < \alpha < n$, in the neighborhood of $r = 0$. g_n is comparable to $\log(1/r)$ in the neighborhood of $r = 0$, while for $\alpha > n$, $g_\alpha(0) < \infty$. All the Bessel kernels have an exponential order of decrease as $r \rightarrow +\infty$. For further properties see [2], pp. 416, 417. In view of property (iii) of kernels, we will consider h_α only for $1 < p < n/\alpha$. The g_α are admissible for all values $1 < p < \infty$.

If $A \subset \mathbb{R}^n$, then we define

$$C_{k,p}(A) = \inf_f \int f^p(x) dx,$$

where $f \in \mathcal{L}_p^+$ and $k * f(x) \geq 1$ for all $x \in E$. In the special case $k = g_\alpha$ we write $B_{\alpha,p} = C_{k,p}$.

Proposition 1.1. $C_{k,p}$ is an outer capacity on the class of all subsets of $\mathbb{R}^n$; see Theorem 1, [7].

Proposition 1.2. If a sequence $A_i \uparrow A$, then $C_{k,p}(A_i) \uparrow C_{k,p}(A)$; see Theorem 6, [7].

Proposition 1.3. Every analytic set is $C_{k,p}$ -capacitable; see Theorem 8, [7].

If $C_{k,p}(A) < \infty$ and there exists $f \in \mathcal{L}_p^+$ such that $k * f(x) \geq 1$, $C_{k,p}$ -a.e. on A with $\|f\|_p^p = C_{k,p}(A)$, then f is called a $C_{k,p}$ -capacitary distribution and $k * f$ a $C_{k,p}$ -capacitary potential of A .

Proposition 1.4. If $C_{k,p}(A) < \infty$, then a $C_{k,p}$ -capacitary distribution exists and is unique in the sense that any two must agree a.e.; see Theorem 9, [7].

Thus, if $C_{k,p}(A) < \infty$ its capacitary potential exists and we denote it by u^A . If $\mu \in \mathfrak{M}^+$ we write

$$u = u_\mu = k * (k * \mu)^{(1/(p-1))}.$$

Notice that u is lower semi-continuous.

Proposition 1.5. Suppose $C_{k,p}(A) < \infty$. Then:

- (i) $u^A = u_\mu$ where $\|\mu\|_1 = C_{k,p}(A)$ and $\text{supp } \mu \subset \bar{A}$; see pp. 277-8, [7].
- (ii) $u^A \leq \max(Q^{1/(p-1)}, Q) = \bar{Q}$, where Q is a constant which depends only on n ; see Corollary 2.1, [1].
- (iii) For A analytic, there exists an increasing sequence of compact sets $K_i \subset A$ such that $u^{K_i} = u_{\mu_i}$, μ_i satisfies the conditions of (i) relative to K_i and

$$\mu_i \rightarrow \mu \text{ weakly,}$$

where $u^A = u_\mu$ and $\|\mu\|_1 = C_{k,p}(A)$; see pp. 277-8, [7].

- (iv) For A open, $u^A(x) \geq 1$ for all $x \in A$; see Proposition 2.1, [1].

In general there is no reason to expect the representation of u^A , in terms of μ , to be unique. Therefore we adopt the convention that for analytic sets A , the measure μ is to be derivable by the process given in (iii) of Proposition 1.5.

We will use the notation $B(x_0, r) = \{x \mid |x - x_0| < r\}$ and χ_A for the indicator function of A .

2. Thinness.

Definition 2.1. Let k be a kernel and $1 < p < \infty$. $\mathfrak{N}_{k,p}^+$ will be the subset of $\mathfrak{N}^+$ consisting of all measures μ such that

$$(2.1) \quad (k * \mu)^{1/(p-1)} \in \mathfrak{N}^+$$

and as $r \uparrow \infty$,

$$(2.2) \quad k * (k * (1 - \chi_{B(0,r)})\mu)^{1/(p-1)}(x) \downarrow 0$$

uniformly on compact sets. $\mathcal{O}_{k,p}$ will be the class of all potentials

$$u = u_\mu = k * (k * \mu)^{1/(p-1)}, \quad \mu \in \mathfrak{N}_{k,p}^+.$$

$\mathfrak{F}_{k,p}$ will be the subclass of $\mathcal{O}_{k,p}$ consisting of bounded potentials.

If we write $u_\mu \in \mathcal{O}_{k,p}$ or $\mathfrak{F}_{k,p}$, it is to be assumed that μ is chosen to be in $\mathfrak{N}_{k,p}^+$. Because of possible non-uniqueness of representation the necessity of this condition is in doubt. In view of Proposition 2.1 (iii) this is consistent with our choice of μ for capacitary potentials.

Condition (2.2) implies that the mass near infinity matters very little and will permit us to localize the mass distribution.

Proposition 2.1. Let $\mu \in \mathfrak{N}^+$. Then:

- (i) If $u_\mu(x) < \infty$ on a dense set and $k * (1 - \chi_{B(0,r)})\mu \in \mathcal{L}_p$ for sufficiently large r , then $\mu \in \mathfrak{N}_{k,p}^+$.
- (ii) If $k * \mu \in \mathcal{L}_p$, then $\mu \in \mathfrak{N}_{k,p}^+$.
- (iii) If $\mu \in \mathcal{L}_1$ and u_μ is bounded, then $\mu \in \mathfrak{N}_{k,p}^+$ and $u_\mu \in \mathfrak{F}_{k,p}$.

Proof. It is clear that if $k(r_0) > 0$ and $u(x) < \infty$ then $(k * \mu)^{1/(p-1)}$ is integrable over $B(x, r_0)$, so that $u_\mu(x) < \infty$ on a dense set implies (2.1). In fact, if k does not have a compact support, (2.1) will follow if $u_\mu(x) < \infty$ at only

a single point. We must now show that under the conditions of (i), (2.2) also holds.

Let $\Gamma_1, \dots, \Gamma_{2^n}$ be the closed 'quadrants' of $\mathbb{R}^n$ ordered in some manner. If $x_0 \in \mathbb{R}^n$, then set $\Gamma_j(x_0) = \Gamma_j + x_0$. Let $K(x_0, h)$ be the closed cube $\{|x^1 - x_0^1| \leq h, \dots, |x^n - x_0^n| \leq h\}$ and finally let $\mu_r = (1 - \chi_{B(0, r)})\mu$. It is clear that as $r \uparrow \infty$

$$k * \mu_r(x) \downarrow 0 \quad \text{a.u.};$$

thus there exists a set E such that

$$\text{meas}(\mathbb{R}^n \sim E) < \infty$$

and

$$k * \mu_r(x) \downarrow 0 \quad \text{uniformly on } E.$$

If x_j is the vertex of $K(x_0, 2h)$ in $\Gamma_j(x_0)$, then there exists $z_j \in \Gamma_j(x_j) \cap E$. It is not hard to see that one can choose r_0 so large that for each $x \in K(x_0, h)$, $y \in \Gamma_j(x) \sim B(0, r_0)$ and $j = 1, \dots, n$

$$|z_j - y| \leq |x - y|.$$

Hence, for $r \geq r_0$ and $x \in K(x_0, h)$

$$k * \mu_r(x) \leq 2^n \sup_E k * \mu_r.$$

Therefore, as $r \uparrow \infty$

$$k * \mu_r(x) \downarrow 0 \quad \text{uniformly on compact sets.}$$

Statement (2.2) now follows easily.

Statement (ii) is a consequence of (i) since $u_\mu(x) < \infty$ a.e.

To prove (iii) note that

$$\int u_\mu d\mu = \int (k * \mu)^{p'} dx. \quad \text{Q.E.D.}$$

Definition 2.2. If $E \subset \mathbb{R}^n$ and $x_0 \notin E$ we say that E is $\mathfrak{F}_{k,p}$ -thin at x_0 if $x_0 \notin \bar{E}$ or there exists $u \in \mathfrak{F}_{k,p}$ such that

$$u(x_0) < \liminf_{\substack{x \rightarrow x_0 \\ x \in E}} u(x).$$

E $\mathcal{O}_{k,p}$ -thin at x_0 , is defined by substituting $\mathcal{O}_{k,p}$ for $\mathfrak{F}_{k,p}$ in the above definition.

Since $\mathfrak{F}_{k,p} \subset \mathcal{O}_{k,p}$, it is clear that E $\mathfrak{F}_{k,p}$ -thin at x_0 implies E $\mathcal{O}_{k,p}$ -thin at x_0 , though the converse proposition is in doubt. We will show later that, for $2 \leq p < \infty$ and a large class of kernels, the converse is indeed true.

Since the potentials we are considering are non-linear in μ except in case $p = 2$, the classes $\mathfrak{F}_{k,p}$ and $\mathcal{O}_{k,p}$ do not form convex cones and our concept of thinness is not contained in the general axiomatic theories already in the literature. For this reason we must derive some elementary properties of thinness.

If a proposition in the following group is stated for both $\mathcal{F}_{k,p}$ and $\mathcal{O}_{k,p}$ and requires proof, we will present the proof only in the case $\mathcal{F}_{k,p}$.

Proposition 2.2. *If $E_1 \subset E_2$ and E_2 is $\mathcal{F}_{k,p}(\mathcal{O}_{k,p})$ -thin at x_0 , then so is E_1 .*

Proposition 2.3. *If V is a neighborhood of x_0 and $E \cap V$ is $\mathcal{F}_{k,p}(\mathcal{O}_{k,p})$ -thin at x_0 , then so is E .*

Proposition 2.4. *If E is $\mathcal{F}_{k,p}(\mathcal{O}_{k,p})$ -thin at x_0 , then there exists $G(\text{open}) \supset E$ which is also $\mathcal{F}_{k,p}(\mathcal{O}_{k,p})$ -thin at x_0 .*

Proposition 2.4 is a direct consequence of the lower semi-continuity of the potentials.

Lemma 2.1. *Let $x_0 \in \bar{E}$ and let $u_\mu \in \mathcal{O}_{k,p}$ with $u_\mu(x_0) < \infty$. Then, if $\lambda = \chi_{B(x_0,r)}\mu$*

$$\liminf_{\substack{x \rightarrow x_0 \\ x \in \bar{E}}} u_\mu(x) - u_\mu(x_0) \leq \max(2^{(2-p)/(p-1)}, 1) (\liminf_{\substack{x \rightarrow x_0 \\ x \in \bar{E}}} u_\lambda(x) - u_\lambda(x_0)).$$

Proof. Set $Q = \max(2^{(2-p)/(p-1)}, 1)$ and $\nu = \mu - \lambda$; then

$$(k * \mu)^{1/(p-1)}(y) \leq Q((k * \lambda)^{1/(p-1)}(y) + (k * \nu)^{1/(p-1)}(y)).$$

Multiply both sides of this inequality by $k(x - y) - k(x_0 - y)$ and integrate with respect to y over the set $\{|x_0 - y| > |x - y|\}$. This leads to

$$u_\mu(x) - u_\mu(x_0) \leq Q(u_\lambda(x) - u_\lambda(x_0) + I + II)$$

where,

$$I = \int_{\{|x_0 - y| \leq |x - y|\}} (k(x_0 - y) - k(x - y))(k * \lambda)^{1/(p-1)}(y) dy$$

and

$$II = \int_{\{|x_0 - y| > |x - y|\}} (k(x - y) - k(x_0 - y))(k * \nu)^{1/(p-1)}(y) dy.$$

By Lebesgue's Dominated Convergence Theorem

$$\lim_{x \rightarrow x_0} I = 0.$$

On the other hand,

$$\begin{aligned} II &\leq \int_{\{|x_0 - y| < \delta\}} k(x - y)(k * \nu)^{1/(p-1)}(y) dy \\ &\quad + \int_{\{|x_0 - y| \geq \delta\}} |k(x - y) - k(x_0 - y)| (k * \nu)^{1/(p-1)}(y) dy = II_1 + II_2. \end{aligned}$$

If we write $\nu = \chi_{B(0,\rho)}\nu + (1 - \chi_{B(0,\rho)})\nu = (\nu - \nu_\rho) + \nu_\rho$, then $k * (\nu - \nu_\rho)$ is bounded on $B(x_0, r/2)$, while from (2.2) $u_{\nu_\rho}(x)$ can be made as small as we wish in the neighborhood of x_0 by making ρ large. Hence, by making δ small and ρ large II_1 can be made as small as we wish in the neighborhood of x_0 .

If we again split ν into $(\nu - \nu_\rho) + \nu_\rho$ and split the range of integration of II_2 into $\{\delta \leq |x_0 - y| \leq \rho_0\} \cup \{|x_0 - y| > \rho_0\}$, ρ and ρ_0 being chosen appropriately large, it follows without difficulty that

$$\lim_{x \rightarrow x_0} II_2 = 0.$$

Thus

$$\lim_{x \rightarrow x_0} II = 0,$$

and the conclusion is immediate. Q.E.D.

Proposition 2.5. *If $u_\mu \in \mathcal{G}_{k,p}$, $x_0 \in \bar{E}$ and*

$$\liminf_{\substack{x \rightarrow x_0 \\ x \in E}} u_\mu(x) - u_\mu(x_0) = J > 0,$$

then for $\lambda = \chi_{B(x_0, r)} \mu$,

$$\liminf_{\substack{x \rightarrow x_0 \\ x \in E}} u_\lambda(x) - u_\lambda(x_0) \geq \min(2^{(p-2)/(p-1)}, 1)J$$

and

$$u_\lambda(x_0) \rightarrow 0 \quad \text{as } r \rightarrow 0.$$

Furthermore, there will exist $u_\nu \in \mathcal{G}_{k,p}$ such that

$$\lim_{\substack{x \rightarrow x_0 \\ x \in E}} u_\nu(x) = \infty,$$

$$u_\nu(x_0) < \infty.$$

Proof. In the first part, the only statement in doubt is $u_\lambda(x_0) \rightarrow 0$. If x_0 carries positive mass, then from $u_\mu(x_0) < \infty$, $k \in \mathcal{L}_{p'}$. But then u_λ is continuous, which is not possible. Therefore $u_\lambda(x_0) \rightarrow 0$. The second conclusion is proved by setting ν equal to an appropriate infinite linear combination of the λ . Q.E.D.

Proposition 2.6. *If $E_1, \dots, E_N$ are $\mathcal{F}_{k,p}(\mathcal{G}_{k,p})$ -thin x_0 , then so is $E = \bigcup_{i=1}^N E_i$.*

Proof. We may assume $x_0 \in \bar{E}_i$ for each j . Choose $u_{\mu_j} \in \mathcal{F}_{k,p}$ so that

$$\liminf_{\substack{x \rightarrow x_0 \\ x \in E_j}} u_{\mu_j}(x) = 1$$

and

$$u_{\mu_j}(x_0) < \min(N^{-1/(p-1)}, N^{-1}),$$

as is clearly possible in view of Proposition 2.5. If $\mu = \mu_1 + \dots + \mu_N$, then $u_\mu \in \mathcal{F}_{k,p}$ and

$$\liminf_{\substack{x \rightarrow x_0 \\ x \in E}} u_\mu(x) \geq 1,$$

$$u_\mu(x_0) < 1.$$

Q.E.D.

Lemma 2.2. *Let $\mu \in \mathfrak{L}_1^+$ and let $u_\mu(x) \leq M < \infty$ for all x . Let $2d_0 = \text{dist}(x_0, \text{supp } \mu) > 0$. Then*

$$u_\mu(x_0) \leq \int_{\{|y| < d_0\}} k(y) dy (k(d_0) \|\mu\|_1)^{1/(p-1)} + \left(\int_{\{|y| \geq d_0\}} k^{p'}(y) dy \right)^{1/p'} (M \|\mu\|_1)^{1/p}.$$

Proof. We have

$$\begin{aligned} u_\mu(x_0) &= \int_{\{|x_0 - y| < d_0\}} k(x_0 - y) (k * \mu)^{1/(p-1)}(y) dy \\ &\quad + \int_{\{|x_0 - y| \geq d_0\}} k(x_0 - y) (k * \mu)^{1/(p-1)}(y) dy = I + II. \\ I &\leq \sup_{B(x_0, d_0)} (k * \mu)^{1/(p-1)} \int_{\{|x_0 - y| < d_0\}} k(x_0 - y) dy \\ &\leq (k(d_0) \|\mu\|_1)^{1/(p-1)} \cdot \int_{\{|y| < d_0\}} k(y) dy. \\ II &\leq \left(\int_{\{|x_0 - y| \geq d_0\}} k^{p'}(x_0 - y) dy \right)^{1/p'} \left(\int (k * \mu)^{p'} dx \right)^{1/p} \\ &\leq \left(\int_{\{|y| \geq d_0\}} k^{p'}(y) dy \right)^{1/p'} (M \|\mu\|_1)^{1/p}. \end{aligned} \quad \text{Q.E.D.}$$

Proposition 2.7. *If $C_{k,p}(E) = 0$ and $x_0 \notin E$ then given $\epsilon > 0$, there exists $G(\text{open}) \supset E$ such that $u^\epsilon(x_0) < \epsilon$. Therefore E is $\mathfrak{F}_{k,p}$ -thin at every point of its complement.*

Proof. Define S_j to be the shell

$$B(x_0, 2^j) \sim B(x_0, 2^{j-1}) \quad (j = 0, \pm 1, \dots).$$

Let $G_i(\text{open}) \supset E \cap S_i$ and let $G_i \subset S_{i-1} \cup S_{i+1}$. Define

$$G = \bigcup_i G_i \supset E.$$

Since the capacity of E is zero, the G_i can be chosen so that their capacities are less than any preassigned positive numbers ϵ_i and in particular so that G has finite capacity. Then $u^G = u_\mu$ exists. It will follow from later considerations (see Proposition 4.6) that if $\mu_i = \chi_{G_i} \mu$, then

$$\|\mu_i\|_1 \leq \tilde{Q}^{p-1} \epsilon_i.$$

From (ii) of Proposition 1.5 and Lemma 2.2 it follows that by properly choosing the ϵ_i the values of the $u_{\mu_i}(x_0)$ can be made arbitrarily small and thus, by a simple inequality, $u^G(x_0)$ can also be made arbitrarily small.

Finally, we see that according to Proposition 1.5, $u^G(x) \geq 1$ for all x in G , so that E is $\mathfrak{F}_{k,p}$ -thin at x_0 . Q.E.D.

3. Bounds for potentials. Throughout this section we will assume, unless there is explicit mention to the contrary, only that $\mu \in \mathfrak{M}^+$. We adopt the notation

$$v(\mu, x_0, r) = v(x_0, r) = \mu(B(x_0, r)).$$

3.1. Lower bounds.

Proposition 3.1. *We have*

$$u(x_0) \geq \omega_n \int_0^\infty (k(2r)v(x_0, r))^{1/(p-1)} r^n dr - k(r),$$

where ω_n is the volume of $B(0, 1)$.

Proof. We have

$$\begin{aligned} u(x_0) &= \int_0^\infty \int_{\{|x-x_0|<r\}} (k * \mu)^{1/(p-1)} dx dr - k(r) \\ &\geq \int_0^\infty \int_{\{|x-x_0|<r\}} \left(\int_{\{|y-x|<2r\}} k(x-y) d\mu(y) \right)^{1/(p-1)} dx dr - k(r) \\ &\geq \int_0^\infty (k(2r)v(x_0, r))^{1/(p-1)} \int_{\{|x-x_0|<r\}} dx dr - k(r), \end{aligned}$$

from which the result follows. Q.E.D.

The above inequality assumes a particularly simple form in the case of Riesz potentials.

Theorem 3.1. *If $k = h_\alpha$, $\alpha p < n$, then*

$$u(x_0) \geq Q \int_0^\infty (r^{\alpha p - n} v(x_0, r))^{1/(p-1)} \frac{dr}{r};$$

$Q > 0$ and is independent of μ .

We also specialize to kernels which are ‘greater than’ Riesz kernels in a neighborhood of 0.

Theorem 3.2. *Let k satisfy*

$$(3.1) \quad k(r) \geq Q_1 h_\alpha(r) \quad \text{for} \quad 0 < r \leq 2r_0$$

where $Q_1 > 0$. Then

$$u(x_0) \geq Q \int_0^{r_0} (r^{\alpha p - n} v(x_0, r))^{1/(p-1)} \frac{dr}{r} + Q \int_{r_0}^\infty (k(2r)v(x_0, r))^{1/(p-1)} r^n dr - k(r);$$

$Q > 0$ and is independent of μ .

Proof. The inequality $k(r) \geq 2^{-1}Q_1 h_\alpha(r)$ holds in some interval $0 < r \leq ar_0$, $a > 2$. Let $\varphi = \varphi(r)$ be continuously differentiable and non-increasing with $\varphi(r) \equiv 4^{-1}$ for $0 \leq r \leq 2^{-1}(2+a)r_0$ and $\varphi(r) \equiv 0$ for $r \geq ar_0$. Define the new kernel k^* by

$$k^* = Q_1 \varphi h_\alpha + 2^{-1}k.$$

Since $k^* \leq k$ we have $u^* \leq u$ for the respective potentials. The result follows by applying Proposition 3.1 to u^* . Q.E.D.

3.2. Upper bounds.

The case of upper bounds is considerably more difficult and to some extent incomplete. As before, we start with a general inequality and then consider the special cases of greatest interest.

Proposition 3.2. *We have*

$$u(x_0) \leq 2^{(2-p)/(p-1)} n \omega_n \int_0^\infty \int_0^r k^{1/(p-1)}(t) t^{n-1} dt \cdot v^{1/(p-1)}(x_0, 2r) d - k(r) \\ + 2^{(2-p)/(p-1)} \omega_n \int_0^\infty \left(\int_r^\infty v(x_0, 2t) d - k(t) \right)^{1/(p-1)} r^n d - k(r), \quad 1 < p < 2,$$

and

$$u(x_0) \leq n^{1/(p-1)} \omega_n \int_0^\infty (r^{n(p-2)} \int_0^r k(t) t^{n-1} dt v(x_0, 2r))^{1/(p-1)} d - k(r) \\ + \omega_n \int_0^\infty \left(\int_r^\infty v(x_0, 2t) d - k(t) \right)^{1/(p-1)} r^n d - k(r), \quad 2 \leq p < \infty.$$

It is clear that the upper bound in case $1 < p < 2$ will be infinite if $\int_0^\infty k^{1/(p-1)}(t) t^{n-1} dt = \infty$, a possibility not ruled out by our assumptions on k . We will handle this problem separately in Proposition 3.3.

Proof of Proposition 3.2. Consider the case $1 < p < 2$.

$$u(x_0) = \int_0^\infty \int_{\{|x-x_0|<r\}} (k * \mu)^{1/(p-1)} dx d - k(r) \\ \leq 2^{(2-p)/(p-1)} \int_0^\infty \int_{\{|x-x_0|<r\}} \left(\int_{\{|y-x|<r\}} k(x-y) d\mu(y) \right)^{1/(p-1)} dx d - k(r) \\ + 2^{(2-p)/(p-1)} \int_0^\infty \int_{\{|x-x_0|<r\}} \left(\int_{\{|y-x|\geq r\}} k(x-y) d\mu(y) \right)^{1/(p-1)} dx d - k(r) \\ = 2^{(2-p)/(p-1)} (I + II).$$

By Hölder's Inequality,

$$\int_{\{|y-x|<r\}} k(x-y) d\mu(y) \leq \left(\int_{\{|y-x|<r\}} k^{1/(p-1)}(x-y) d\mu(y) \right)^{p-1} v^{2-p}(x, r) \\ \leq \left(\int_{\{|y-x_0|<2r\}} k^{1/(p-1)}(x-y) d\mu(y) \right)^{p-1} v^{2-p}(x_0, 2r).$$

Therefore by interchanging the x and y integrations,

$$I \leq n \omega_n \int_0^\infty \int_0^r k^{1/(p-1)}(t) t^{n-1} dt \cdot v^{1/(p-1)}(x_0, 2r) d - k(r).$$

On the other hand,

$$\begin{aligned} II &= \int_0^\infty \int_{\{|x-x_0|<r\}} \left(\int_r^\infty v(x, t) dt - k(t) \right)^{1/(p-1)} dx dt - k(r) \\ &\leq \omega_n \int_0^\infty \left(\int_r^\infty v(x_0, 2t) dt - k(t) \right)^{1/(p-1)} r^n dt - k(r), \end{aligned}$$

which completes this case. In case $2 \leq p < \infty$ we have

$$u(x_0) \leq I + II.$$

Now

$$\begin{aligned} &\int_{\{|x-x_0|<r\}} \left(\int_{\{|y-x|<r\}} k(x-y) d\mu(y) \right)^{1/(p-1)} dx \\ &\leq (\omega_n r^n)^{(p-2)/(p-1)} \left(\int_{\{|x-x_0|<r\}} \int_{\{|y-x_0|<2r\}} k(x-y) d\mu(y) dx \right)^{1/(p-1)}. \end{aligned}$$

By interchanging the x and y integrations,

$$I \leq n^{1/(p-1)} \omega_n \left(r^{n(p-2)} \int_0^r k(t) t^{n-1} dt v(x_0, 2r) \right)^{1/(p-1)}.$$

II is bounded just as before.

Q.E.D.

We now turn to the case $\int_0^1 k^{1/(p-1)}(t) t^{n-1} dt = \infty$. The estimate involves not only $v(x_0, r)$ but variations in the neighborhood of *all* points x (at least *all* x in a neighborhood of x_0). For this reason we introduce

$$v(\mu, x_0, r, t) = v(x_0, r, t) = \sup_{B(x_0, r)} v(x, t).$$

Proposition 3.3. *If $\int_0^1 k^{1/(p-1)}(t) t^{n-1} dt = \infty$, $1 < p < 2$, and the discrete part of μ equals zero, then*

$$\begin{aligned} u(x_0) &\leq 2^{n+(2-p)/(p-1)} (\log 2)^{-1/(p-1)} \omega_n \\ &\quad \cdot \int_0^\infty v(x_0, 2r) \left(\int_0^r s^{n(p-1)} k(s/2) \cdot v^{2-p}(x_0, 2r, 2s) \frac{ds}{s} \right)^{1/(p-1)} dt - k(r) \\ &\quad + 2^{(2-p)/(p-1)} \omega_n \int_0^\infty \left(\int_r^\infty v(x_0, 2t) dt - k(t) \right)^{1/(p-1)} r^n dt - k(r). \end{aligned}$$

The restriction that the discrete part, ν , be zero is not a serious one. If $\text{supp } \nu$ intersects the interior of $\text{supp } k(x_0 - x)$, then $u(x_0) = \infty$; in particular, this is so if $\text{supp } k = \mathbf{R}^n$ and $\nu \neq 0$. If $\text{supp } \nu$ is sufficiently far from $\text{supp } k(x_0 - x)$, then ν has no effect on $u(x_0)$. In general we can write

$$u(x_0) \leq 2^{(2-p)/(p-1)} (u_{\mu-\nu}(x_0) + u_\nu(x_0)).$$

If $\nu = \sum_i a_i \delta_{y_i}$, then

$$u_\nu(x_0) = \int k(x_0 - x) \left(\sum_i a_i k(x - y_i) \right)^{1/(p-1)} dx,$$

and one can find an upper bound for $u_\nu(x_0)$ by suitably modifying the proof of Proposition 3.2. Finally, notice that if the potential is assumed to be bounded, then $\nu = 0$.

Proof of Proposition 3.3. As in Proposition 3.2

$$u(x_0) \leq 2^{(2-p)/(p-1)}(I + II).$$

II is bounded just as before. We now consider

$$\begin{aligned} I' &= \int_{\{|x-x_0|<r\}} \left(\int_{\{|y-x|<r\}} k(x-y) d\mu(y) \right)^{1/(p-1)} dx \\ &= \int_{\{|x-x_0|<r\}} \left(\sum_{j=0}^{\infty} \int_{\{2^{-j-1}r \leq |y-x| < 2^{-j}r\}} k(x-y) d\mu(y) \right)^{1/(p-1)} dx, \end{aligned}$$

which by Minkowski's Inequality is

$$\begin{aligned} &\leq \left(\sum_{j=0}^{\infty} \left(\int_{\{|x-x_0|<r\}} \left(\int_{\{2^{-j-1}r \leq |y-x| < 2^{-j}r\}} k(x-y) d\mu(y) \right)^{1/(p-1)} dx \right)^{p-1} \right)^{1/(p-1)} \\ &\leq \left(\sum_{j=0}^{\infty} \left(\int_{\{|x-x_0|<r\}} \left(\int_{\{2^{-j-1}r \leq |y-x| < 2^{-j}r\}} k^{1/(p-1)}(x-y) d\mu(y) \right) dx \right. \right. \\ &\quad \left. \cdot v^{(2-p)/(p-1)}(x_0, 2r, 2^{-j}r) \right)^{p-1} \Big)^{1/(p-1)}, \end{aligned}$$

by Hölder's Inequality. If we now interchange x and y integrations,

$$\begin{aligned} I' &\leq n\omega_n v(x_0, 2r) \left(\sum_{j=0}^{\infty} \left(\int_{2^{-j-1}r}^{2^{-j}r} k^{1/(p-1)}(t) t^{n-1} dt \right)^{p-1} \cdot v^{2-p}(x_0, 2r, 2^{-j}r) \right)^{1/(p-1)} \\ &\leq \omega_n v(x_0, 2r) \left(\sum_{j=0}^{\infty} (2^{-j}r)^{n(p-1)} k(2^{-j-1}r) v^{2-p}(x_0, 2r, 2^{-j}r) \right)^{1/(p-1)} \\ &\leq 2^n (\log 2)^{-1/(p-1)} \omega_n \left(\int_0^r s^{n(p-1)} k(s/2) v^{2-p}(x_0, 2r, 2s) \frac{ds}{s} \right)^{1/(p-1)} v(x_0, 2r). \end{aligned}$$

If we now substitute into I , the inequality follows. Q.E.D.

We will now consider the upper bounds for special kernels.

Theorem 3.3. *If $k = h_\alpha$, $2 - \alpha/n < p < n/\alpha$, then*

$$u(x_0) \leq Q \int_0^\infty (r^{\alpha p - n} v(x_0, r))^{1/(p-1)} \frac{dr}{r};$$

Q is independent of μ .

Proof. Throughout the proof Q will stand for constants independent of μ . First consider the case $1 < p < 2$. With p subject to the above restriction, $h_\alpha^{1/(p-1)}(x)$ is locally integrable and from Proposition 3.2,

$$u(x_0) \leq Q \int_0^\infty (r^{\alpha p - n} v(x_0, 2r))^{1/(p-1)} \frac{dr}{r} \\ + Q \int_0^\infty r^\alpha \left(\int_r^\infty t^{-\alpha(p-1)} (t^{\alpha p - n} v(x_0, 2t)) \frac{dt}{t} \right)^{1/(p-1)} \frac{dr}{r} = Q(I + II).$$

By Hardy's Inequality (see p. 245, [4])

$$II \leq Q \int_0^\infty (r^{\alpha p - n} v(x_0, 2r))^{1/(p-1)} \frac{dr}{r}.$$

The change of variables $2r$ to r completes this case.

In case $2 \leq p < \infty$, rewrite

$$II = \int_0^\infty r^\alpha \left(\sum_{i=0}^\infty \int_{2^i r}^{2^{i+1} r} t^{\alpha - n} v(x_0, 2t) \frac{dt}{t} \right)^{1/(p-1)} \frac{dr}{r} \\ \leq \sum_{i=0}^\infty \int_0^\infty r^\alpha \left(\int_{2^i r}^{2^{i+1} r} t^{\alpha - n} v(x_0, 2t) \frac{dt}{t} \right)^{1/(p-1)} \frac{dr}{r} \\ \leq (\log 2)^{1/(p-1)} \sum_{i=0}^\infty 2^{-\alpha i} \int_0^\infty ((2^i r)^{\alpha p - n} v(x_0, 2^{i+2} r))^{1/(p-1)} \frac{dr}{r} \\ = (\log 2)^{1/(p-1)} \left(\sum_{i=0}^\infty 2^{-\alpha i} \right) \int_0^\infty (r^{\alpha p - n} v(x_0, 4r))^{1/(p-1)} \frac{dr}{r}.$$

The inequality now follows by a change of variables.

Q.E.D.

Lemma 3.1. (i) If $k = h_\alpha$ and $u \leq M$, then

$$v(x, r) \leq e^{-1} Q M^{p-1} r^{n-\alpha p}, \quad 0 < \alpha p < n;$$

Q is independent of μ , M and r .

(ii) If

$$k(r) \geq Q_1 h_\alpha(r) \quad \text{for } 0 < r \leq 2r_0,$$

where $Q_1 > 0$, then

$$v(x, r) \leq e^{-1} Q M^{p-1} r^{n-\alpha p} \quad \text{for } 0 < r \leq r_0, \quad 0 < \alpha p < n$$

and

$$v(x, r) \leq e^{-1} Q M^{p-1} \left(\log \frac{r_0}{r} \right)^{1-p} \quad \text{for } 0 < r \leq r_0, \quad \alpha p = n;$$

Q is independent of μ , M and r .

Proof. Use Theorems 3.1 and 3.2.

Q.E.D.

Theorem 3.4. Let $k = h_\alpha$ and $u \leq M$. Then

$$u(x_0) \leq Q_1 M^{((2-p)n-\alpha)/(n-\alpha p)} \int_0^\infty (r^{\alpha p - n} v(x_0, r))^{(n-\alpha)/(n-\alpha p)} \frac{dr}{r}, \quad 1 < p < 2 - \alpha/n;$$

$$u(x_0) \leq Q_1 \int_0^\infty \left(r^{\alpha p - n} v(x_0, r) \cdot \log \frac{QM^{p-1}}{r^{\alpha p - n} v(x_0, r)} \right)^{1/(p-1)} \frac{dr}{r}, \quad p = 2 - \alpha/n.$$

Q_1 is independent of μ and M ; Q is the constant of Lemma 3.1 (i).

Proof. Let $0 < a = a(r) < 1$, which is to be properly determined. If we consider the first integral in Proposition 3.3, we must estimate

$$\int_0^r s^{\alpha(p-1)^2} (s^{\alpha p - n} v(x_0, 2r, 2s))^{2-p} \frac{ds}{s} = \int_0^{ar} \dots \frac{ds}{s} + \int_{ar}^r \dots \frac{ds}{s} = I' + I''.$$

We will denote constants independent of μ and M by Q_1 . From Lemma 3.1

$$I' \leq Q_1 M^{(p-1)(2-p)} (ar)^{\alpha(p-1)^2},$$

while

$$I'' \leq \frac{1}{(2-p)n - \alpha} v^{2-p}(x_0, 4r) (ar)^{\alpha - (2-p)n}, \quad 1 < p < 2 - \alpha/n;$$

$$I'' \leq v^{2-p}(x_0, 4r) \log \frac{1}{a}, \quad p = 2 - \alpha/n.$$

Set

$$a = \left(\frac{(4r)^{\alpha p - n} v(x_0, 4r)}{QM^{p-1}} \right)^{1/(n - \alpha p)}, \quad v(x_0, 4r) \neq 0,$$

where Q is the constant of Lemma 3.1 and

$$a = \frac{1}{2} \quad \text{otherwise.}$$

One can now easily show that the first integral in Proposition 3.3 has the desired bound.

As we have seen in the proof of Theorem 3.3, the second integral in Proposition 3.3 is bounded by

$$\begin{aligned} Q_1 \int_0^\infty (r^{\alpha p - n} v(x_0, r))^{1/(p-1)} \frac{dr}{r} \\ \leq Q_1 M^{((2-p)n - \alpha)/(n - \alpha p)} \int_0^\infty (r^{\alpha p - n} v(x_0, r))^{(n - \alpha)/(n - \alpha p)} \frac{dr}{r} \end{aligned}$$

by Lemma 3.1. The desired inequality is now immediate. Q.E.D.

We define

$$v(\mu, r) = v(r) = \sup_{\mathbf{R}^n} v(x, r).$$

Theorem 3.5. Let $k = h_\alpha$. Then

$$u(x_0) \leq Q \left(\int_0^\infty (r^{\alpha p - n} v(x_0, r))^{1/(p-1)} \frac{dr}{r} \right)^{p-1} \left(\int_0^\infty (r^{\alpha p - n} v(r))^{1/(p-1)} \frac{dr}{r} \right)^{2-p},$$

$$1 < p \leq 2 - \alpha/n;$$

Q is independent of μ .

Proof. Q will stand for constants independent of μ . From Proposition 3.3

$$\begin{aligned} u(x_0) \leq Q \int_0^\infty r^{\alpha p - n} v(x_0, 2r) r^{-\alpha(p-1)} & \left(\int_0^r s^{\alpha(p-1)^2} (s^{\alpha p - n} v(2s))^{2-p} \frac{ds}{s} \right)^{1/(p-1)} \frac{dr}{r} \\ & + Q \int_0^\infty r^\alpha \left(\int_r^\infty t^{\alpha - n} v(x_0, 2t) \frac{dt}{t} \right)^{1/(p-1)} \frac{dr}{r} = Q(I + II). \end{aligned}$$

By Hölder's Inequality

$$\begin{aligned} I \leq & \left(\int_0^\infty (r^{\alpha p - n} v(x_0, 2r))^{1/(p-1)} \frac{dr}{r} \right)^{p-1} \\ & \cdot \left(\int_0^\infty r^{-\alpha(p-1)/(2-p)} \left(\int_0^r s^{\alpha(p-1)^2} (s^{\alpha p - n} v(2s))^{2-p} \frac{ds}{s} \right)^{1/(p-1)(2-p)} \frac{dr}{r} \right)^{2-p}. \end{aligned}$$

If we apply Hardy's Inequality (see p. 245, [4]) to the second factor we get the required bound for I . The bound on II follows also from Hardy's Inequality just as before. Q.E.D.

We now discuss kernels which are bounded in the neighborhood of 0 by Riesz kernels.

Theorem 3.6. *Let*

$$k(r) \leq Q_1 h_\alpha(r) \quad \text{for } 0 < r \leq 1.$$

Let $\nu = (1 - \chi_{B(x_0, r_0)})\mu$. *Then*

$$u(x_0) \leq Q \int_0^{4r_0} (r^{\alpha p - n} v(x_0, r))^{1/(p-1)} \frac{dr}{r} + Qu_\nu(x_0), \quad 2 - \alpha/n < p \leq n/\alpha;$$

Q is independent of μ . $u_\nu(x_0)$ can be estimated by Proposition 3.2.

Proof. In the course of this proof we will denote constants independent of μ by Q . Clearly

$$u(x_0) \leq Qu_\lambda(x_0) + Qu_\nu(x_0).$$

Let $\varphi = \varphi(r)$ be a continuously differentiable non-increasing function of r such that $\varphi(r) \equiv 1$ for $0 \leq r \leq \frac{3}{2}r_0$ and $\varphi(r) \equiv 0$ for $r \geq 2r_0$. Then

$$k = \varphi k + (1 - \varphi)k = k_0 + k_1.$$

Now

$$k_0 \leq Q\varphi h_\alpha.$$

Therefore

$$\begin{aligned} u_\lambda(x_0) \leq Q\varphi h_\alpha * (\varphi h_\alpha * \lambda)^{1/(p-1)}(x_0) & + Qk * (k_1 * \lambda)^{1/(p-1)}(x_0) \\ & + Qk_1 * (k_0 * \lambda)^{1/(p-1)}(x_0) = Q(I + II + III). \end{aligned}$$

By the reasoning of Theorem 3.3 one can show

$$I \leq Q \int_0^{4r_0} (r^{\alpha p - n} v(x_0, r))^{1/(p-1)} \frac{dr}{r}.$$

$k_1 * \lambda(x) = 0$ if $|x - x_0| \leq r_0$. Hence, by Hölder's and Young's Inequalities,

$$\begin{aligned} II &\leq \left(\int_{\{|x| > r_0/2\}} k^{p'}(x) dx \right)^{1/p'} \left(\int_{\{|x| > 3r_0/2\}} k^{p'}(x) dx \right)^{1/p} \|\lambda\|_1^{1/(p-1)} \\ &\leq n\omega_n \int_{r_0/2}^{\infty} k^{p'}(t) t^{n-1} dt \cdot \|\lambda\|_1^{1/(p-1)}. \end{aligned}$$

For $|x - x_0| > 3r_0/2$

$$k_0 * \lambda(x) \leq k(|x| - r_0/2) \|\lambda\|_1,$$

and these are the only values we need consider in estimating *III*, since $k_1(x - x_0) = 0$ for $|x - x_0| \leq 3r_0/2$. By Hölder's Inequality

$$III \leq Q \int_{r_0}^{\infty} k^{p'}(t) t^{n-1} dt \cdot \|\lambda\|_1^{1/(p-1)}.$$

Since

$$II + III \leq Q \int_{r_0}^{4r_0} (r^{\alpha p - n} v(x_0, r))^{1/(p-1)} \frac{dr}{r},$$

the result follows. Q.E.D.

Theorem 3.7. *Let*

$$Q_1^{-1} h_{\alpha}(r) \leq k(r) \leq Q_1 h_{\alpha}(r) \quad \text{for } 0 < r \leq 8r_0.$$

Let $u = u_{\mu} \leq M$ *and let* $\nu = (1 - \chi_{B(x_0, r_0)})\mu$. *Then*

$$u(x_0) \leq Q_2 M^{((2-p)n - \alpha)/(n - \alpha p)} \int_0^{8r_0} (r^{\alpha p - n} v(x_0, r))^{(n - \alpha)/(n - \alpha p)} \frac{dr}{r} + Q_2 u_{\nu}(x_0),$$

$$1 < p < 2 - \alpha/n;$$

$$u(x_0) \leq Q_2 \int_0^{8r_0} \left(r^{\alpha p - n} v(x_0, r) \cdot \log \frac{QM^{p-1}}{r^{\alpha p - n} v(x_0, r)} \right)^{1/(p-1)} \frac{dr}{r} + Q_2 u_{\nu}(x_0),$$

$$p = 2 - \alpha/n.$$

Q_2 is independent of μ and M and Q is the constant of Lemma 3.1 (ii) with r_0 replaced by $8r_0$. $u_{\nu}(x_0)$ can be estimated by Proposition 3.3.

Proof. Use the techniques of Theorems 3.4 and 3.6. Q.E.D.

Theorem 3.8. *Let*

$$k(r) \leq Q_1 h_{\alpha}(r) \quad \text{for } 0 < r \leq 1.$$

Let $\nu = (1 - \chi_{B(x_0, r_0)})\mu$. Then

$$u(x_0) \leq Q \left(\int_0^{4r_0} (r^{\alpha p - n} \nu(x_0, r))^{1/(p-1)} \frac{dr}{r} \right)^{p-1} \\ \cdot \left(\int_0^{4r_0} (r^{\alpha p - n} \nu(r))^{1/(p-1)} \frac{dr}{r} \right)^{2-p} + Qu_\nu(x_0), \quad 1 < p \leq 2 - \alpha/n;$$

Q is independent of μ .

Proof. Use the techniques of Theorems 3.5 and 3.6.

Q.E.D.

Theorem 3.9. Let $u = g_\alpha * (g_\alpha * \mu)^{1/(p-1)}$ and $u' = g_\beta * (g_\beta * \mu)^{1/(q-1)}$ where $\mu \in \mathfrak{M}_0^+$. Then u' bounded implies u bounded under either one of the following conditions:

- (i) $\beta q < \alpha p$.
- (ii) $\beta q = \alpha p = w \leq n$, $1 + (1 - w/n)^{1/2} < p < q$.

Proof. In (i) we may as well restrict to the case $\alpha p \leq n$, for in the contrary case u is always bounded. (i) now follows by Lemma 3.1 and Theorems 3.6, 3.8. To prove (ii) use Lemma 3.1 and Theorems 3.2, 3.6.

Q.E.D.

Theorem 3.9 implies that in cases (i) and (ii), if $B_{\alpha,p}(E) = 0$, then $B_{\beta,q}(E) = 0$. This is contained in the more general results relating Bessel capacities, to be found in [1]. Whether Theorem 3.9 can be extended to other cases is at present unknown.

We will give two examples concerning the accuracy of the upper and lower bounds in case $1 < p < 2 - \alpha/n$, as given in Theorems 3.1 and 3.4. Specifically, we are interested in the exponents, $1/(p-1)$ in the lower bound and $(n-\alpha)/(n-\alpha p)$ in the upper bound. The forms of these bounds are crucial to our method of deriving the Wiener Criteria. The difference in the exponents causes a difference in the necessary and the sufficient criteria for these values of p ; see Theorems 5.1 and 5.4. We give the example concerning the lower bound here. For the other, see Example 5.1 at the end of that section.

Example 3.1. Let $1 < p \leq 2 - \alpha/n$. Let μ be the function

$$\mu(x) = |x|^{-\alpha p} \left(\log \frac{1}{|x|} \right)^{-(p-1)/\theta} \quad \text{for } 0 \leq |x| \leq r_0, \\ \mu(x) = 0 \quad \text{for } |x| > r_0;$$

here $0 < \theta < 1$ and r_0 is chosen so small that $\mu(x)$ is decreasing in $|x|$. Now

$$Q_1^{-1} r^{n-\alpha p} \left(\log \frac{1}{r} \right)^{-(p-1)/\theta} \leq v(0, r) \leq Q_1 r^{n-\alpha p} \left(\log \frac{1}{r} \right)^{-(p-1)/\theta} \quad \text{for } 0 \leq r \leq r_0, \\ v(0, r) = v(0, r_0) \quad \text{for } r > r_0.$$

Clearly

$$\int_0^\infty (r^{\alpha p - n} v(0, r))^{1/(p-1)} \frac{dr}{r} < \infty,$$

and since μ is decreasing in $|x|$

$$v(x, r) \leq v(0, r) \quad \text{for all } x \text{ and } r.$$

This implies, from Theorem 3.5, that $h_\alpha * (h_\alpha * \mu)^{1/(p-1)} = u$ is a bounded function. However,

$$\int_0^\infty (r^{\alpha p - n} v(0, r))^{\theta/(p-1)} \frac{dr}{r} = \infty.$$

Let $\varphi: [0, +\infty) \rightarrow [0, +\infty)$ be increasing and $\varphi(+\infty) = \lim_{t \rightarrow +\infty} \varphi(t) = \infty$; then from the lower bound

$$\varphi\left(\int_0^\infty (r^{\alpha p - n} v(x_0, r))^{\theta/(p-1)} \frac{dr}{r}\right) \leq u(x_0),$$

we infer

$$\sup u < \infty \quad \text{implies} \quad \int_0^\infty (r^{\alpha p - n} v(x_0, r))^{\theta/(p-1)} \frac{dr}{r} < \infty,$$

which is false.

4. Equivalence of capacities. We will investigate inequalities between $C_{k,p}$ and various capacities, with a view toward proving them comparable. This will permit us to interchange them in most circumstances.

Proposition 4.1. *Let h and k be kernels. Let $h_0 = h_0(x)$ with $h_0 \in \mathfrak{L}_p^+$ and let $Q > 0$. If*

$$k \leq Qh + h_0,$$

then

$$C_{h,p}(E) \leq Q^p (1 - \|h_0\|_p C_{k,p}^{1/p}(E))^{-p} C_{k,p}(E),$$

provided $C_{k,p}(E) < \|h_0\|_p^{-p}$.

Proof. We may as well assume $C_{k,p}(E) < \infty$. Let $\epsilon > 0$ and let f be a test function for $C_{k,p}(E)$ with

$$\int f^p dx \leq C_{k,p}(E) + \epsilon.$$

Since

$$h_0 * f(x) \leq \|h_0\|_p \|f\|_p,$$

$$h * f(x) \geq Q^{-1} (1 - \|h_0\|_p (C_{k,p}(E) + \epsilon)^{1/p}) \quad \text{for } x \in E.$$

The result follows immediately.

Q.E.D.

Proposition 4.2. *If h and k are kernels and*

$$Q_1^{-1}h(r) \leq k(r) \leq Q_1h(r) \quad \text{for } 0 < r \leq r_0,$$

then there exists $r_1 > 0$ and Q_2 , independent of E , such that

$$Q_2^{-1}C_{h,p}(E) \leq C_{k,p}(E) \leq Q_2C_{h,p}(E),$$

provided $\text{diam } E < r_1$.

Proof. Clearly

$$k \leq Q_1h + h_0,$$

where $h_0 = \chi_0(k + r^{-n})$, χ_0 = indicator function of $(r_0, +\infty)$. Now if $k \notin \mathfrak{L}_{p'}$, then $C_{k,p}(\{0\}) = 0$, so that

$$C_{k,p}(E) < \|h_0\|_{p'}^{-p}$$

for $\text{diam } E$ sufficiently small. The first inequality follows very easily from Proposition 4.1, while the second follows in the same fashion. The case $k \in \mathfrak{L}_{p'}$ is even simpler. Q.E.D.

Proposition 4.3. *We have*

$$C_{k,p}(E) = \inf_{\mu} \int (k * \mu)^{p'} dx,$$

where $\mu \in \mathfrak{M}^+$ and

$$u_{\mu}(x) \geq 1 \quad \text{for all } x \in E.$$

Proof. Temporarily call the above infimum $C'_{k,p}(E)$. Obviously

$$C_{k,p}(E) \leq C'_{k,p}(E).$$

On the other hand, if $C_{k,p}(E) < \infty$, then, given $\epsilon > 0$, there exists G (open) $\supset E$ such that

$$C_{k,p}(G) \leq C_{k,p}(E) + \epsilon.$$

Now

$$u^G = k * (k * \mu)^{1/(p-1)},$$

and since $u^G(x) \geq 1$ for all $x \in G$, we get

$$C'_{k,p}(E) \leq \int (k * \mu)^{p'} dx \leq C_{k,p}(E) + \epsilon.$$

The equality now follows. Q.E.D.

Corollary 4.1. *If $u_{\mu} \leq M$, $\mu \in \mathfrak{L}_1^+$, and $u_{\mu}(x) \geq 1$ on E , then*

$$C_{k,p}(E) \leq M \|\mu\|_1.$$

Definition 4.1. If $E \subset \mathbb{R}^n$ we define

$$\tilde{C}_{k,p}(E) = \inf_{\mu} \|\mu\|_1,$$

where $\mu \in \mathcal{L}_1^+$ and

$$(4.1) \quad u_{\mu}(x) \geq 1 \quad \text{for all } x \in E.$$

Proposition 4.4. $\tilde{C}_{k,p}$ is an outer capacity and

$$\tilde{C}_{k,p} \leq C_{k,p}.$$

If $2 \leq p < \infty$, then we also have

$$(4.2) \quad \tilde{Q}^{-1} C_{k,p} \leq \tilde{C}_{k,p},$$

where $\tilde{Q}$ is the constant of Proposition 1.5.

Proof. The proof that $\tilde{C}_{k,p}$ is an outer capacity is straightforward and the proof of the first inequality is the same as $C'_{k,p} \leq C_{k,p}$ in Proposition 4.3. Since both capacities are outer, it suffices to prove the second inequality for E a G_{δ} -set. Let K (compact) $\subset E$. Then unless $k \equiv 0$, in which case everything is trivially true, $C_{k,p}(K) < \infty$. Assuming $C_{k,p}(K) < \infty$, we let $u^K = u_{\lambda}$. If μ satisfies (4.1), then

$$\begin{aligned} \|\lambda\|_1 &\leq \int k * (k * \mu)^{1/(p-1)} d\lambda = \int (k * \mu)^{1/(p-1)} k * \lambda dx \\ &\leq \left(\int k * \mu (k * \lambda)^{1/(p-1)} dx \right)^{1/(p-1)} \left(\int (k * \lambda)^{p'} dx \right)^{(p-2)/(p-1)} \\ &= \left(\int k * (k * \lambda)^{1/(p-1)} d\mu \right)^{1/(p-1)} \left(\int (k * \lambda)^{p'} dx \right)^{(p-2)/(p-1)}. \end{aligned}$$

Thus,

$$C_{k,p}(K) \leq \tilde{Q}^{1/(p-1)} C_{k,p}^{(p-2)/(p-1)}(K) \|\mu\|_1^{1/(p-1)}$$

or

$$C_{k,p}(K) \leq \tilde{Q} \|\mu\|_1,$$

from which the inequality follows by use of Proposition 1.3.

Q.E.D.

Inequality (4.2) is not generally true when $1 < p < 2$. Suppose $\int_0^1 k^{1/(p-1)}(t) t^{n-1} dt = \infty$. Then set

$$u = k * (k * \delta_0)^{1/(p-1)} = k * k^{1/(p-1)}.$$

Clearly $u \equiv +\infty$ on some ball $B(0, r_0)$. Hence, $\tilde{C}_{k,p}(B(0, r_0)) = 0$ but $C_{k,p}(B(0, r_0)) > 0$.

Definition 4.2. Let $\mathfrak{A}$ be the σ -algebra of all sets measurable with respect to every measure in $\mathfrak{M}^+$. If $A \in \mathfrak{A}$, then

$$\mathfrak{C}_{k,p}(A) = \sup_{\mu} \|\mu\|_1,$$

where μ varies over the set of all $\mu \in \mathfrak{L}_1^+$ which are carried by A and

$$u_\mu(x) \leq 1 \quad \text{for all } x.$$

Proposition 4.5. $\mathfrak{C}_{k,p}$ is a capacity. If A is an analytic set, then

$$\mathfrak{C}_{k,p}(A) \leq C_{k,p}(A) \leq \tilde{Q}^{p-1} \cdot \mathfrak{C}_{k,p}(A).$$

For a proof see Corollary 2.2, [1].

Corollary 4.2. If $\mu \in \mathfrak{M}^+$ and $u_\mu \leq M$, then for every analytic set A ,

$$\mu(A) \leq M^{p-1} C_{k,p}(A).$$

Proposition 4.6. If E is an analytic set with $C_{k,p}(E) < \infty$ and $u^E = u_\mu$, then

$$\mu(G) \leq \tilde{Q}^{p-1} C_{k,p}(G \cap E),$$

for every open set G .

Proof. We know from (iii) of Proposition 1.5 that there exists a sequence $\{K_i\}$ of compact subsets of E with $u^{K_i} = u_{\mu_i}$ such that $\mu_i \rightarrow \mu$ weakly. From Corollary 4.2

$$\mu_i(G) = \mu_i(G \cap E) \leq \tilde{Q}^{p-1} C_{k,p}(G \cap E).$$

Since $\mu(G) \leq \liminf_{i \rightarrow \infty} \mu_i(G)$, the inequality follows. Q.E.D.

The interesting part of Proposition 4.6 is that μ need not be carried by E .

5. Wiener Criteria for thinness. Throughout this section we will assume that the kernel k satisfies

$$(5.1) \quad Q_1^{-1} h_\alpha(r) \leq k(r) \leq Q_1 h_\alpha(r) \quad \text{for } 0 < r \leq r_0.$$

We also introduce the following notation: if $E \subset \mathbb{R}^n$, then $E(x_0, r) = B(x_0, r) \cap E$.

Theorem 5.1. If E is $\mathcal{O}_{k,p}$ -thin at x_0 and $2 \leq p < \infty$, then

$$\int_0^1 (r^{\alpha p - n} B_{\alpha,p}(E(x_0, r)))^{1/(p-1)} \frac{dr}{r} < \infty.$$

Proof. The cases $\alpha p > n$ and $x_0 \notin \bar{E}$ are trivial. In the contrary case, Proposition 2.5 implies that given $0 < \epsilon, \delta < 1$ there exists $u = u_\mu \in \mathcal{O}_{k,p}$ such that

$$\liminf_{\substack{x \rightarrow x_0 \\ x \in E}} u(x) \geq 1,$$

$$u(x_0) \leq \epsilon,$$

$$\text{supp } \mu \subset B(x_0, \delta).$$

Throughout this proof, Q will denote constants independent of μ , ϵ and ρ (to be introduced presently) if ρ is sufficiently small.

From Theorem 3.2

$$(5.2) \quad \int_0^{r_0/2} (r^{\alpha p - n} v(x_0, r))^{1/(p-1)} \frac{dr}{r} \leq Q\epsilon.$$

We must now bound $v(x_0, r)$ below by a capacity. For each fixed $\rho > 0$, let u' and u'' be the non-linear potentials of $\chi_{B(x_0, 2\rho)} \cdot \mu$ and $(1 - \chi_{B(x_0, 2\rho)}) \cdot \mu$ respectively. Let $|x - x_0| < \rho < \delta$; then from Theorem 3.6

$$\begin{aligned} u''(x) &\leq Q \int_\rho^{8\delta} (r^{\alpha p - n} v(x, r))^{1/(p-1)} \frac{dr}{r} \\ &\leq Q \int_0^{8\delta} (r^{\alpha p - n} v(x_0, 2r))^{1/(p-1)} \frac{dr}{r}. \end{aligned}$$

Therefore, from (5.2), if δ is sufficiently small

$$u''(x) \leq Q\epsilon \quad \text{for } |x - x_0| < \rho < \delta.$$

Since $u \leq u' + u''$ and ϵ can be chosen arbitrarily small, we can satisfy

$$u'(x) \geq 1 - Q\epsilon \geq \frac{1}{2}$$

for $x \in E(x_0, \rho)$, $\rho < \delta$. From Proposition 4.4 we then have

$$C_{k,p}(E(x_0, \rho)) \leq Qv(x_0, 2\rho), \quad \rho < \delta.$$

Proposition 4.2 then implies

$$B_{\alpha,p}(E(x_0, \rho)) \leq Qv(x_0, 2\rho)$$

for ρ sufficiently small. The result is now an immediate consequence of (5.2).
Q.E.D.

Theorem 5.2. *If E is $\mathfrak{F}_{k,p}$ -thin at x_0 and $1 < p < 2$, then*

$$\int_0^1 (r^{\alpha p - n} B_{\alpha,p}(E(x_0, r)))^{1/(p-1)} \frac{dr}{r} < \infty.$$

Proof. In case $2 - \alpha/n < p < 2$ the proof is essentially the same as that of Theorem 5.1 except that we must use Corollary 4.1 in place of Proposition 4.4. In case $1 < p \leq 2 - \alpha/n$ however, the upper and lower bounds assume a different form and we must apparently do something else to prove u'' small on $B(x_0, \rho)$. We outline a procedure which yields the proper estimate.

Proposition 2.5 implies that for some finite M and every $\epsilon, \delta > 0$ there exists $u \in \mathfrak{F}_{k,p}$ such that

$$u \leq M,$$

$$\liminf_{\substack{x \rightarrow x_0 \\ x \in E}} u(x) \geq 1,$$

$$u(x_0) \leq \epsilon, \quad \|\mu\|_1 \leq \epsilon^p$$

and

$$\text{supp } \mu \subset B(x_0, \delta).$$

If we inspect the contributions from the ‘tails’ of the kernels it follows, using (5.1), that for $\delta < r_0/2$

$$(5.3) \quad Q^{-1}w''(x) - Q\epsilon \leq u''(x) \leq Qw''(x) + Q\epsilon \quad \text{for } |x - x_0| < r_0/2;$$

here w is the Riesz potential ($k = h_\alpha$) of μ and Q is independent of μ , ϵ , ρ and x . Hence for ϵ sufficiently small it will follow that

$$w''(x_0) \leq Q\epsilon.$$

By the Harnack Inequality (see Theorem 6.1) we then have

$$w''(x) \leq Q\epsilon \quad \text{for } |x - x_0| < \rho,$$

which by (5.3) implies

$$u''(x) \leq Q\epsilon \quad \text{for } |x - x_0| < \rho.$$

The rest is done as before.

Q.E.D.

Definition 5.1. For $E \subset \mathbb{R}^n$ we define

$$D_{\alpha,p}(E, x_0) = \lim_{r \rightarrow 0} B_{\alpha,p}(E(x_0, r)) / B_{\alpha,p}(B(0, r)),$$

called the $B_{\alpha,p}$ -capacitary density of E at x_0 .

Corollary 5.1. If E is $\mathcal{O}_{k,p}$ -thin ($2 \leq p < \infty$) at x_0 or $\mathcal{F}_{k,p}$ -thin ($1 < p < 2$) at x_0 , then

$$D_{\alpha,p}(E, x_0) = 0.$$

Proof. Of course only the case $\alpha p \leq n$ need be considered. By the monotonicity of $B_{\alpha,p}(E(x_0, r))$ and the convergence of the integral, we infer that as $r \rightarrow 0$

$$B_{\alpha,p}(E(x_0, r)) = o(r^{n-\alpha p}), \quad \alpha p < n,$$

or

$$B_{\alpha,p}(E(x_0, r)) = o\left(\log \frac{1}{r}\right)^{1-p}, \quad \alpha p = n.$$

But it is known that $r^{n-\alpha p}$ and $(\log(1/r))^{1-p}$ are, for small r , comparable to $B_{\alpha,p}(B(0, r))$ in the respective cases (see Lemmas 7 and 8, [7]). Q.E.D.

Theorem 5.3. If $2 - \alpha/n < p < \infty$ and

$$(5.4) \quad \int_0^1 (r^{\alpha p - n} B_{\alpha,p}(E(x_0, r)))^{1/(p-1)} \frac{dr}{r} < \infty,$$

then

$$u^{E \cap V}(x_0) \rightarrow 0 \quad \text{as} \quad C_{k,p}(V) \rightarrow 0;$$

V here denotes any set. Therefore (5.4) implies that E is $\mathfrak{F}_{k,p}$ -thin at x_0 .

Proof. The case $\alpha p > n$ is trivial. In case $\alpha p \leq n$ assume to begin with that E and V are G_δ -sets. Fix $\delta_0 > 0$ so that $C_{k,p}(A)$ and $B_{\alpha,p}(A)$ are comparable for $\text{diam } A \leq \delta_0$. Define $\lambda = \chi_{B(x_0, \delta_0/4)} \mu$ and $\nu = \mu - \lambda$. As a consequence of Proposition 4.6

$$v(\lambda, x_0, r) \leq Q B_{\alpha,p}(E(x_0, r)) \quad \text{for} \quad 0 < r \leq \delta_0;$$

Q denotes, as it will in the course of the proof, constants independent of V and δ (to be introduced later). Hence, from Theorem 3.6

$$\begin{aligned} u_\lambda(x_0) &\leq Q \int_0^{\delta_0} (r^{\alpha p - n} B_{\alpha,p}(E \cap V(x_0, r)))^{1/(p-1)} \frac{dr}{r} \\ &\leq Q \int_0^{\delta} (r^{\alpha p - n} B_{\alpha,p}(E(x_0, r)))^{1/(p-1)} \frac{dr}{r} + Q \cdot \int_\delta^{\delta_0} r^{(\alpha p - n)/(p-1)} \frac{dr}{r} \cdot C_{k,p}^{1/(p-1)}(V). \end{aligned}$$

Hence, $u_\lambda(x_0)$ can be made as small as we wish by making δ and $C_{k,p}(V)$ small. On the other hand $u_\nu(x_0)$ can be made as small as we wish by making $C_{k,p}(V)$ small, from Lemma 2.2. Hence $u^{E \cap V}(x_0)$ can be made as small as we wish.

To handle the general case we first carry out a preliminary construction. For each positive rational s determine a G_δ -set A_s so that

$$x_0 \notin A_s \supset E(x_0, s)$$

and

$$B_{\alpha,p}(A_s) = B_{\alpha,p}(E(x_0, s)).$$

Define

$$A = \bigcap_s (A_s \cup (\sim B(x_0, s))).$$

A is a G_δ -set and for all $r > 0$,

$$B_{\alpha,p}(A(x_0, r)) = B_{\alpha,p}(E(x_0, r)).$$

Choose a G_δ -set

$$W \supset E \cap V$$

with

$$C_{k,p}(W) = C_{k,p}(E \cap V).$$

It follows easily that when $C_{k,p}(E \cap V) < \infty$

$$u^{A \cap W} = u^{E \cap V}.$$

That $u^{E \cap V}(x_0) \rightarrow 0$ now follows from the case of G_δ -sets.

To prove the thinness of E at x_0 , let V be a neighborhood of x_0 so small that

$$u^{E \cap V}(x_0) < 1.$$

Since $u^{E \cap V}(x) \geq 1$ $C_{k,p}$ -a.e. on E , the thinness follows from Propositions 2.6 and 2.7. Q.E.D.

Theorem 5.4. *If*

$$(5.5) \quad \int_0^1 \left(r^{\alpha p - n} B_{\alpha,p}(E(x_0, r)) \cdot \log \frac{Q}{r^{\alpha p - n} B_{\alpha,p}(E(x_0, r))} \right)^{1/(p-1)} \frac{dr}{r} < \infty,$$

$$p = 2 - \alpha/n,$$

where $Q = e \sup_{0 < r \leq 1} r^{\alpha p - n} B_{\alpha,p}(B(0, r))$, or if

$$(5.6) \quad \int_0^1 (r^{\alpha p - n} B_{\alpha,p}(E(x_0, r)))^{(n-\alpha)/(n-\alpha p)} \frac{dr}{r} < \infty, \quad 1 < p < 2 - \alpha/n,$$

then

$$u^{E \cap V}(x_0) \rightarrow 0 \quad \text{as} \quad C_{k,p}(V) \rightarrow 0.$$

Therefore (5.5) or (5.6) implies that E is $\mathfrak{F}_{k,p}$ -thin at x_0 .

Proof. The proof is essentially the same as that of Theorem 5.3 except that one must use the upper bounds of Theorem 3.7. Q.E.D.

Example 5.1. We will show that the exponent $(n - \alpha)/(n - \alpha p)$ in the sufficiency condition for $\mathfrak{F}_{k,p}$ -thinness, $1 < p < 2 - \alpha/n$, is the largest possible. For the sake of simplicity we deal with Riesz kernels.

Suppose that a set E is $\mathfrak{F}_{h_{\alpha,p}}$ -thin at $x_0 \in \bar{E}$. Then there exists $u = u_\mu \in \mathfrak{F}_{h_{\alpha,p}}$ such that

$$u(x_0) < \liminf_{\substack{x \rightarrow x_0 \\ x \in E}} u(x).$$

If we let $f = (h_\alpha * \mu)^{1/p-1}$, then $f \in \mathcal{L}_p^+$ and $u = h_{\alpha/2} * (h_{\alpha/2} * f)$. Hence, from the case of exponent 2, which is in fact classical, the condition

$$(5.7) \quad \int_0^1 r^{\alpha - n} B_{\alpha/2,2}(E(x_0, r)) \frac{dr}{r} < \infty$$

is necessary. We will now show that (5.7) is not implied by any condition

$$(5.8) \quad \int_0^1 (r^{\alpha p - n} B_{\alpha,p}(E(x_0, r)))^{\theta(n-\alpha)/(n-\alpha p)} \frac{dr}{r} < \infty$$

for $\theta > 1$. In particular in the range $1 < p < 2 - \alpha/n$, (5.7) is a necessary condition independent of the condition of Theorem 5.2. Let $E(x_0, r, s) = E(x_0, s) \sim E(x_0, r)$. Statement (5.7) is equivalent to

$$(5.9) \quad \sum_{i=0}^{\infty} 2^{i(n-\alpha)} B_{\alpha/2,2}(E(x_0, 2^{-i-1}, 2^{-i})) < \infty.$$

Statement (5.8) is equivalent to

$$(5.10) \quad \sum_{i=0}^{\infty} (2^{i(n-\alpha p)} B_{\alpha,p}(E(x_0, 2^{-i})))^{\theta(n-\alpha)/(n-\alpha p)} < \infty.$$

Construct a set E as follows. Let $y_i = x_0 + (2^{-i-2} + 2^{-i-1}, 0, \dots, 0)$ for $j = 0, 1, \dots$. Let

$$E = \bigcup_{i=0}^{\infty} B(y_i, a_i \cdot 2^{-i-2}),$$

where $0 < a_i < 1$ and will be determined. Then

$$E(x_0, 2^{-i-1}, 2^{-i}) = B(y_i, a_i \cdot 2^{-i-2}).$$

Since the capacity of a ball of radius r is of the order of $r^{n-\alpha p}$ (see Lemma 7, [7]) it follows that (5.9) is equivalent to

$$(5.11) \quad \sum_{i=0}^{\infty} a_i^{n-\alpha} < \infty.$$

On the other hand, by the subadditivity of the capacity, inequality (5.10) is implied by

$$(5.12) \quad \sum_{j=0}^{\infty} (2^{j(n-\alpha p)} \sum_{i=j}^{\infty} a_i^{(n-\alpha p)} 2^{-i(n-\alpha p)})^{\theta(n-\alpha)/(n-\alpha p)} < \infty.$$

If the sequence of a_i is decreasing it is clear that (5.12) and hence (5.10) is implied by

$$(5.13) \quad \sum_{j=0}^{\infty} a_j^{\theta(n-\alpha)} < \infty.$$

If we set $a_j = (j+1)^{-1/(n-\alpha)}$, then (5.13) holds, but (5.11) does not, which concludes the example.

The example also shows that the exponent $(n-\alpha)/(n-\alpha p)$, $1 < p < 2 - \alpha/n$, in the upper bound of Theorem 3.4 cannot be increased; otherwise the exponent would carry over to the Wiener Criterion by the methods of this section.

It is worthwhile to consider the exponents as functions of p for capacities of fixed weight, $\alpha p = w < n$, as depicted by the graph on page 196.

6. The Harnack Inequality.

Theorem 6.1. *Let k satisfy*

$$(6.1) \quad k(r) \leq Q_1 k(2r), \quad 0 < r \leq 1; \quad k(r) \leq Q_1 k(r+1), \quad 1 < r < \infty.$$

Then for $d_0 = \text{dist}(x_0, \text{supp } \mu) \leq d$, we have

$$u(x_0) \leq Qu(x), \quad |x - x_0| \leq d_0/2;$$

Q depends only on d and k .

The condition (6.1) prevents k from having a compact support unless $k \equiv 0$; thus the choice of $r = 1$ as the point of division between the two conditions is arbitrary.

Proof. In the course of the proof any constant depending only on d and k will be denoted by Q .

Let $w = k * \mu$ and suppose $|x - x_0| \leq 3d_0$. If $d_0 \leq |x_0 - y|$, then

$$|x_0 - y| \geq \frac{1}{4} |x - y|,$$

which implies

$$k(x_0 - y) \leq Qk(x - y) \quad \text{for } y \in B(x_0, 2d) \cap \text{supp } \mu.$$

On the other hand,

$$|x_0 - y| + 3d \geq |x - y|.$$

Therefore

$$k(x_0 - y) \leq Qk(x - y) \quad \text{for } y \in \text{supp } \mu \sim B(x_0, 2d).$$

We conclude

$$(6.2) \quad w(x_0) \leq Qw(x) \quad \text{for } |x - x_0| \leq 3d_0.$$

Now let $f = (k * \mu)^{1/(p-1)}$ which we may assume is in $\mathfrak{M}^+$, for otherwise $u \equiv \infty$ and there is nothing to prove. Now we assume $d_x = |x - x_0| \leq d_0/2$. From (6.2)

$$(6.3) \quad \int_{\sim(B(x_0, d_x/3) \cup B(x, d_x/3))} k(x_0 - y)f(y) dy \\ \leq Q \int_{\sim(B(x_0, d_x/3) \cup B(x, d_x/3))} k(x - y)f(y) dy.$$

Further,

$$\int_{B(x_0, d_x/3)} k(x_0 - y)f(y) dy = \int_{B(x, d_x/3)} k(x - z)f(x_0 - x + z) dz.$$

From 6.2

$$f(x_0 - x + z) \leq Qf(z) \quad \text{for } z \in B(x, d_x/3).$$

Thus

$$(6.4) \quad \int_{B(x_0, d_x/3)} k(x_0 - y)f(y) dy \leq Q \int_{B(x, d_x/3)} k(x - y)f(y) dy.$$

By the same kind of reasoning used to get (6.4),

$$(6.5) \quad \int_{B(x, d_x/3)} k(x_0 - y)f(y) dy \leq Q \int_{B(x_0, d_x/3)} k(x - y)f(y) dy.$$

The inequality now follows from (6.3)–(6.5).

Q.E.D.

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