A Local Spectral Theory for Operators IV; Invariant Subspaces

JOSEPH G. STAMPFLI

This paper is dedicated to Eberhard Hopf on the occasion of his seventieth birthday.

Introduction. At the Conference on Operator Theory held at the University of New Hampshire in June 1971, Peter Rosenthal raised the question whether an operator $T \in \mathcal{B}(\mathcal{H})$ whose resolvent satisfied the condition

$$\|(\lambda - T)^{-1}\| \leq \frac{K}{\text{dist} [\lambda, \sigma(T)]}$$

and whose spectrum $\sigma(T)$ consisted of two tangent discs, necessarily had an invariant subspace. More generally Rosenthal asked whether “resolvent techniques” would yield invariant subspaces for operators whose spectrum had positive area. The author’s curiosity was further aroused when it became clear that none of the existing techniques (Bartle [1]; Dunford [2] page 588, Lemma 2.5, [3] page 254, Lemma 4; Godement [5] Theorem J; Leaf [6], Schwartz [9], Wermer [11] and Wolf [12]) would work in this new context.

The techniques just alluded to, handle successfully operators whose spectrum lies on a straight line, a circle, or a smooth Jordan curve. We do not wish to involve ourselves in the rather lengthy explanation of why they fail in the case of two disks.

As it turns out, the growth rate on the resolvent needed to produce invariant subspaces depends on the degree of contact between the parts of the spectrum of our operator. We will first prove a general theorem and then apply it to a number of special cases. Throughout the paper we will be considering operators $T \in \mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on Hilbert space. However, the results of the first section are valid in a more general setting as will be pointed out at the appropriate time.

§1. Definition. A Jordan domain D in a bounded open set in the complex plane $\mathbf{C}$ whose boundary is a Jordan curve. When we refer to ∂D we will assume the standard counter clockwise orientation.

I would like to thank M. Radjabalipour for pointing out a gap in the original proof of Theorem 1.

Theorem 1. Let $T \in \mathfrak{B}(\mathcal{H})$. Let D_1, D_2 be Jordan domains with rectifiable boundaries where

- 1) $\sigma(T) \subset D_1 \cup D_2 \cup \{0\}$,
- 2) $\bar{D}_1 \cap \bar{D}_2 = \{0\}$,
- 3) $\sigma(T) \cap D_i \neq \emptyset$ for $i = 1, 2$,
- 4) $\text{dist}[z, \sigma(T)] \leq K \text{dist}[z, D_i]$ for all $z \in \partial D_i$, $i \neq j$ and K is a constant.

For $i = 1, 2$ assume there exist functions $f_i \neq 0$ such that f_i is analytic on D_i and continuous on $\bar{D}_i$. If $\|f_i(\lambda)(\lambda - T)^{-1}\| \leq M$ for $\lambda \in \partial D_i \setminus \{0\}$ for $i = 1, 2$ where M is a constant, then T has a proper invariant subspace.

Proof. We begin by defining

$$A = \int_{\partial D_1} f_1(\lambda)(\lambda - T)^{-1} d\lambda$$

and

$$B = \int_{\partial D_2} f_2(\lambda)(\lambda - T)^{-1} d\lambda.$$

Since the integrand is uniformly bounded on $\partial D_i \setminus \{0\}$, for each i , continuous except at 0 and ∂D_i is rectifiable, the operators A, B exist as Riemann integrals, where the convergence is in the norm topology (see Riesz–Nagy Chapter XI, [8]). We claim that T commutes with A, B and $A \cdot B = B \cdot A = 0$. Indeed

$$\begin{aligned} A \cdot B &= \int_{\partial D_1} f_1(\lambda)(\lambda - T)^{-1} d\lambda \cdot \int_{\partial D_2} f_2(z)(z - T)^{-1} dz \\ &= \int_{\partial D_1} \int_{\partial D_2} f_1(\lambda)f_2(z) \left\{ \frac{(\lambda - T)^{-1} - (z - T)^{-1}}{z - \lambda} \right\} d\lambda dz \\ &= \int_{\partial D_1} f_1(\lambda)(\lambda - T)^{-1} \left[\int_{\partial D_2} \frac{f_2(z)}{z - \lambda} dz \right] d\lambda - \int_{\partial D_2} f_2(z)(z - T)^{-1} \\ &\quad \cdot \left[\int_{\partial D_1} \frac{f_1(\lambda)}{z - \lambda} d\lambda \right] dz = 0. \end{aligned}$$

To justify the interchange in the order of integration we first note that

$$|z - \lambda| \geq \text{dist}[z, D_1] \geq K^{-1} \text{dist}[z, \sigma(T)]$$

for $z \in \partial D_2$ and all $\lambda \in \partial D_1$, and thus

$$|z - \lambda|^{-1} \leq K \|(z - T)^{-1}\|.$$

Hence

$$\begin{aligned} \int_{\partial D_1} |f_1(\lambda)| \|(\lambda - T)^{-1} \| d|\lambda| \cdot \int_{\partial D_2} |f_2(z)| |z - \lambda|^{-1} d|z| \\ \leq \int_{\partial D_1} M d|\lambda| \cdot \int_{\partial D_2} MK d|z| < \infty, \end{aligned}$$

and thus Fubini's Theorem is applicable. The other steps follow from Cauchy's Theorem and the resolvent equation.

As our invariant subspaces we choose $\overline{A\mathfrak{H}}$ and $\overline{B\mathfrak{H}}$. Since T commutes with A and B , these are clearly invariant. It remains to show that they are non-trivial. If $\overline{B\mathfrak{H}} = \mathfrak{H}$, then $A = 0$ since $AB = 0$. Thus it suffices to show that $A \neq 0$ and the rest follows by symmetry. Choose a point $\mu \in \partial\sigma(T) \cap D_1$. We may assume $f_1(\mu) \neq 0$ by replacing f_1 by $g_1(\lambda) = (\lambda - \mu)^{-n}f_1(\lambda)$ if necessary. (This change would result in a new operator, A , but that is immaterial.) Since $\mu \in \partial\sigma(T)$ there exists a unit vector $x \in \mathfrak{H}$ such that $(T - \mu)x = w$ where $\|w\| < \epsilon$. It is easy to see that

$$(\lambda - T)^{-1}x = (\lambda - \mu)^{-1}x - (\lambda - \mu)^{-1}(\lambda - T)^{-1}w.$$

Thus

$$\begin{aligned} Ax &= \int_{\partial D_1} f_1(\lambda)(\lambda - T)^{-1}x \, d\lambda \\ &= \int_{\partial D_1} \frac{f_1(\lambda)}{\lambda - \mu} x \, d\lambda - \int_{\partial D_1} (\lambda - \mu)^{-1}f_1(\lambda)(\lambda - T)^{-1}w \, d\lambda = 2\pi i f(\mu)x - y \end{aligned}$$

where $\|y\| \leq \epsilon C$ and C is a constant independent of x, ϵ . Clearly $Ax \neq 0$ for ϵ sufficiently small and thus $A \neq 0$, which completes the proof.

Corollary. *Let $\mathfrak{A}$ be a Banach algebra with identity e . Let $t \in \mathfrak{A}$ and assume t satisfies the conditions on the operator T of the previous theorem. Then there exists a non-trivial closed left ideal $\mathfrak{J} \subset \mathfrak{A}$ and $\mathfrak{J}$ is invariant under both left and right multiplication by t .*

Proof. We can define elements $a, b \in \mathfrak{A}$ just as we defined A, B . Moreover a, b commute with t and $a \cdot b = b \cdot a = 0$. One can show that a, b are non-zero by the same proof as was used in Theorem 1 where the role of the vector x is replaced by the well known fact about zero divisors in a Banach algebra (see [4], Lemma 4, page 862). Set $\mathfrak{J} = \text{closure } \{qa : q \in \mathfrak{A}\}$. Clearly $\mathfrak{J} \neq \{0\}$. Assume $e \in \mathfrak{J}$. Then there exist elements $q_n \in \mathfrak{A}$ such that $q_n a \rightarrow e$. But in that case $0 = q_n a \cdot b \rightarrow b$ a contradiction. Thus $\mathfrak{J}$ is non-trivial and the other claims are obvious.

Remarks. Several conditions in the theorem can be considerably relaxed.

1. It is only necessary that $f_i \in H^\infty(D_i)$, that is, bounded and analytic in D_i . In that case f_i will have boundary values almost everywhere with respect to arc length, which permits us to define the necessary integrals.

2. The functions $g_i(\lambda) = \|(\lambda - T)^{-1}f_i(\lambda)\|$ need not be bounded. It suffices that they be integrable with respect to arc length.

3. We may permit more contact between $\bar{D}_1$ and $\bar{D}_2$ than the singleton $\{0\}$. However $\bar{D}_1 \cap \bar{D}_2$ should be a set of measure zero with respect to arc length.

Under the above conditions the proof of Theorem 1 goes through virtually unchanged.

4. In Theorem 1 as it stands, condition 4) can be replaced by the weaker

technical condition that

$$\int_{\partial D_1} \int_{\partial D_2} |f_i(z)| |z - \lambda|^{-1} d|\lambda| d|z| < \infty \quad \text{for } i = 1, 2.$$

We shall need this remark in Example 1 below.

5. The Hilbert space can be replaced by a Banach space.

6. The invariant subspaces $\overline{A\mathfrak{H}\mathfrak{C}}$ and $\overline{B\mathfrak{H}\mathfrak{C}}$ which we constructed are actually hyperinvariant (that is, invariant for any operator which commutes with T). This follows immediately from the fact that if S commutes with T this S commutes with A and B .

By working a little harder we can sharpen the form of Theorem 1. Namely we can locate the spectrum of T on the invariant subspaces.

Theorem 1'. *Let the hypothesis of Theorem 1 be in force. Assume $f_i(\lambda) \neq 0$ for $\lambda \in D_i$. Then $\sigma(T | \overline{A\mathfrak{H}\mathfrak{C}}) \subset \bar{D}_1$ and $\sigma(T | \overline{B\mathfrak{H}\mathfrak{C}}) \subset \bar{D}_2$.*

Proof. We will actually prove a little more than is stated in the theorem. Let $\eta \in D_2$. Assuming only that $f_2(\eta) \neq 0$ (not that $f_2(z) \neq 0$ for $z \in D_2$), we will show that $\eta \notin \sigma(T | \overline{A\mathfrak{H}\mathfrak{C}})$. Set

$$C = (2\pi i)^{-1} \int_{\partial D_2} \frac{f_2(\lambda)}{\lambda - \eta} (\lambda - T)^{-1} d\lambda.$$

Clearly $C \in \mathfrak{B}(\mathfrak{H}\mathfrak{C})$. We will first show that $(\eta - T)C = f_2(\eta)I - (2\pi i)^{-1}B$. For R sufficiently large we have

$$\begin{aligned} (\eta - T)C &= (2\pi i)^{-1} \int_{|z|=R} (\eta - z)(z - T)^{-1} dz \cdot (2\pi i)^{-1} \int_{\partial D_2} \frac{f_2(\lambda)}{\lambda - \eta} (\lambda - T)^{-1} d\lambda \\ &= (2\pi i)^{-2} \int_{|z|=R} \int_{\partial D_2} (\eta - z) \frac{f_2(\lambda)}{\lambda - \eta} \frac{1}{\lambda - z} \{(z - T)^{-1} - (\lambda - T)^{-1}\} dz d\lambda \\ &= (2\pi i)^{-2} \int_{|z|=R} (\eta - z)(z - T)^{-1} \left(\int_{\partial D_2} \frac{f_2(\lambda)}{\lambda - \eta} \frac{1}{\lambda - z} d\lambda \right) dz \\ &\quad + (2\pi i)^{-2} \int_{\partial D_2} \frac{f_2(\lambda)}{\lambda - \eta} (\lambda - T)^{-1} \left(\int_{|z|=R} \frac{\eta - z}{\lambda - z} dz \right) d\lambda \\ &= (2\pi i)^{-1} \int_{|z|=R} f_2(\eta)(z - T)^{-1} dz - (2\pi i)^{-1} \int_{\partial D_2} f_2(\lambda)(\lambda - T)^{-1} d\lambda \\ &= f_2(\eta)I - (2\pi i)^{-1}B. \end{aligned}$$

Thus for any element $y \in \overline{A\mathfrak{H}\mathfrak{C}}$ it follows that

$$(\eta - T) \frac{1}{f_2(\eta)} Cy = \frac{1}{f_2(\eta)} C(\eta - T)y = y.$$

Since C commutes with T it follows that $\overline{A\mathfrak{H}\mathfrak{C}}$ is invariant under C and thus $(\eta - T)$ is invertible on $\overline{A\mathfrak{H}\mathfrak{C}}$. Under the hypothesis of the theorem this shows that $\sigma(T | \overline{A\mathfrak{H}\mathfrak{C}}) \subset \bar{D}_1$ which completes the proof.

§2. Applications. In this section we wish to use the results of the last section to exhibit invariant subspaces for more concrete classes of operators. The reader will observe that there is a trade off between the growth rate of the resolvent and the degree of contact between D_1 and D_2 .

Definition. We will say two compact sets $A, B \subset \mathbb{C}$ are separated by the curves $x = \pm f(y)$ for $-1 \leq y \leq 1$ if A lies in the left half plane, B in the right half plane, $A \cap B \subset \{0\}$, and $A \cap S = B \cap S = \{0\}$ where $S = \{x + iy \in \mathbb{C} : -1 \leq y \leq 1 \text{ and } -f(y) \leq x \leq f(y)\}$.

For any operator $T \in \mathcal{B}(\mathcal{H})$, $\|(\lambda - T)^{-1}\| \geq 1/\text{dist}[\lambda, \sigma(T)]$. We begin by considering operators which satisfy the most restrictive growth condition possible (modulo a constant). For convenience let $d_\lambda = \text{dist}[\lambda, \sigma(T)]$ for the rest of this section.

Example 1. Let $T \in \mathcal{B}(\mathcal{H})$ and let $\sigma(T) = A \cup B$ where A, B are separated by the curves $x = \pm e^{-1/|y|^p}$ where $0 < p < 1$. We assume both A, B contain points other than $\{0\}$. Let $\|(\lambda - T)^{-1}\| \leq M d_\lambda^{-1}$ for all $\lambda \notin \sigma(T)$ where M is a constant. Then there exist functions f_i and domains D_i satisfying the hypotheses of Theorem 1 for $i = 1, 2$. Thus the operator T has a non-trivial invariant subspace.

We set $f_2(z) = e^{-Cz^{-p}}$ and $f_1(z) = f_2(-z)$ where $C \cos(p\pi/2) \geq 4$. For D_1 we choose any convex domain with rectifiable boundary which contains $A \setminus \{0\}$ in the interior and has the curve $x = -e^{-2|y|^{-p}}$ for $0 < |y| \leq \delta$ as part of its boundary. The domain D_2 is chosen in a similar manner. Since

$$f_2(z) = e^{-Cr^{-p} \cos p\theta} \cdot e^{iCr^{-p} \sin p\theta}$$

we see that f is bounded and analytic in D_2 . For $z \in \partial D_2$ and z sufficiently small, we have

$$|f_2(z)| \leq e^{-4|y|^{-p}} \quad \text{where } z = x + iy.$$

On the other hand for $z \in \partial D_2$ and z small we see that

$$d_\lambda = \text{dist}[z, \sigma(T)] \geq e^{-2(1+\epsilon)|y|^{-p}}.$$

Thus $F(z) = \|f_2(z)(z - T)^{-1}\| \leq M \exp(-4|y|^{-p} + 2(1+\epsilon)|y|^{-p})$ and since $\lim_{z \rightarrow 0} F(z) = 0$, it follows that $F(z)$ is bounded on $\partial D_2 \setminus \{0\}$.

A similar estimate may be carried out for f_1 on D_1 . It only remains to check condition 4) of Theorem 1. In view of the remarks following that Theorem, it suffices to check instead condition 4'). Note that for any $z = x + iy \in \partial D_2$, $z \neq 0$ and $\lambda \in \partial D_1$ we have $|z - \lambda|^{-1} \leq \exp(2|y|^{-p})$. Thus

$$\begin{aligned} \int_{\partial D_1} \int_{\partial D_2} |f_2(z)| |z - \lambda|^{-1} d|\lambda| d|z| \\ \leq \int_{\partial D_1} \int_{\partial D_2} \exp(-4|y|^{-p} + 2|y|^{-p}) \cdot d|\lambda| d|z| < \infty. \end{aligned}$$

Our verification is complete.

Corollary. Let T be hyponormal and let $\sigma(T) = A \cup B$ where A, B are separated by $x = \pm e^{-|y|^{-p}}$ for $0 < p < 1$. Then T has a proper invariant subspace.

Proof. An operator $T \in \mathfrak{B}(\mathcal{H})$ is hyponormal if $T^*T - TT^* \geq 0$. It is well known that for these operators $\|(\lambda - T)^{-1}\| \leq d_\lambda^{-1}$ for $\lambda \notin \sigma(T)$.

Remark. The above corollary in certain forms has been a folk theorem for years. It was discovered by Allen Shields for T subnormal and A, B tangent disks (unpublished) and by C. R. Putnam for T hyponormal and A, B tangent disks (also unpublished). The same result is valid if we maintain the exponential contact between A, B and weaken our condition on the growth of the resolvent to

$$\|(\lambda - T)^{-1}\| \leq Md_\lambda^{-n}$$

for n a positive integer and

$$|\lambda| \leq \|T\| + 1 \quad \text{where } \lambda \notin \sigma(T).$$

In this case just replace f_1, f_2 by f_1^n, f_2^n respectively.

Counterpoint to Example 1. We will now show that if one employs Theorem 1 to attack an operator whose resolvent has first order growth, then Example 1 represents the maximum amount of contact which can be permitted. More precisely, we will show that there exist sets A, B separated by the curves $x = \pm e^{-|y|^p}$ for $|y| \leq 1$ and $p > 1$ for which there exist no domains D_i and functions f_i satisfying the conditions of Theorem 1. In fact, let A, B be compact sets whose boundaries are $x = \pm e^{-|y|^p}$ for $|y| \leq 1$ respectively and let $T \in \mathfrak{B}(\mathcal{H})$ where $\sigma(T) = A \cup B$ (p is now fixed and greater than 1). Assume $A \subset \bar{D}_1$ and $B \subset \bar{D}_2$ where $\bar{D}_1 \cap \bar{D}_2 = \{0\}$ and assume there exist functions $f_i \neq 0$ for $i = 1, 2$ analytic in D_i and continuous in $\bar{D}_i$, such that

$$\|f_i(\lambda)(\lambda - T)^{-1}\| \leq M \quad \text{for } \lambda \in \partial D_i \setminus \{0\}.$$

We will show that this situation leads to a contradiction. Let $z = x + iy \in \partial D_2$. For y small it is easy to see that

$$d_\lambda = \text{dist} [\lambda, \sigma(T)] \leq e^{-2^{-1}|y|^{-p}}.$$

Since $\|(\lambda - T)^{-1}\| \geq d_\lambda^{-1}$ for any $T \in \mathfrak{B}(\mathcal{H})$ we are led to the conclusion that

$$|f_2(\lambda)| \leq Me^{-2^{-1}|y|^{-p}}$$

for $\lambda \in \partial D_2$ and $|y| \leq \delta$ for δ sufficiently small.

We wish to apply the two constants theorem to conclude that $f_2 \equiv 0$. Fix t where $0 < t < \delta$. Let α_t designate those parts of ∂D_2 bounded by $x = \pm e^{-|y|^p}$ for $|y| \leq t$ and the lines $|\text{Im } \lambda| = t$. Fix $z_0 = x_0 \in D_2$ and assume for the moment that the harmonic measure $w(x_0, \alpha_t, D_2) \geq Kt$ for some constant $K > 0$ which is independent of t for $0 < t \leq \delta_1 < \delta$. Then by the two constants theorem

$$\begin{aligned} |f_2(x_0)| &\leq (M \exp(-2^{-1}|t|^{-p})^{w(x_0, \alpha_t, D_2)} M_1^{1-w(x_0, \alpha_t, D_2)} \\ &\leq M_2 \exp(-2^{-1}K|t|^{1-p}). \end{aligned}$$

Letting $t \rightarrow 0$ we see that $f_2(x_0) = 0$ since $p > 1$. Clearly this implies that $f_2(z) \equiv 0$ in D_2 . Now we return to the problem of showing that the harmonic measure of $w(x_0, \alpha_t, D_2)$ is comparable with t . Fix t where $0 < t \leq \delta_1$.

Let D_2^* be the simply connected domain with boundary $\beta_t \cup \gamma_t \cup \rho_t$ where

$$\begin{aligned}\beta_t &= \{-\exp(-|t|^{-p}) + iy : 0 \leq |y| \leq t\}, \\ \gamma_t &= \{x \pm it : 0 \leq |x| \leq \exp(-|t|^{-p})\}, \\ \rho_t &= \text{any smooth convex Jordan arc with end points } \exp(-|t|^{-p}) \pm it \text{ so} \\ &\quad \text{chosen that } \rho_t \subset \bar{D}_2 \text{ and } D_2^* \text{ now contains } x_0.\end{aligned}$$

It follows from Carleman's Theorem on extension of domain (see [7] page 68) that $w(x_0, \alpha_t, D_2) \geq w(x_0, \beta_t, D_2^*)$. By applying Carleman's Theorem on cross cuts ([7] page 72) we see that

$$t^{-1}w(x_0, \gamma_t, D_2^*) \rightarrow 0 \quad \text{as } t \rightarrow 0$$

so we may replace β_t by $\beta_t \cup \gamma_t$ in our estimates. Set $\eta_t = \{\exp(-|s|^{-p}) + is; -t \leq s \leq t\}$ and let D_2^{**} be the domain bounded by $\eta_t \cup \rho_t$. Clearly $w(x_0, \beta_t \cup \gamma_t, D_2^*) \geq \frac{1}{2}w(x_0, \eta_t, D_2^{**})$. Combining the foregoing inequalities we discover that

$$2w(x_0, \alpha_t, D_2) \geq w(x_0, \eta_t, D_2^{**}) - \epsilon(t)$$

where $\epsilon(t)t^{-1} \rightarrow 0$ as $t \rightarrow 0$. It is now easy to show either by conformal mapping techniques (D_2^{**} has a smooth boundary so we may map it onto the unit disc and then compare harmonic measures via the Poisson kernel and Kellogg's Theorem [10] page 71) or by directly constructing a harmonic minorant, that $w(x_0, \eta_t, D_2^{**}) \geq Kt$ where $K > 0$ is independent of t for t small. Thus $w(x_0, \alpha_t, D_2)$ is comparable with t as claimed. I am grateful to Glenn Schober for suggesting the use of the two constants theorem in this problem.

Example 2. Let $\sigma(T) \subset S_\alpha = \{re^{i\theta} : |\theta| \leq \alpha \text{ or } |\theta - \pi| \leq \alpha\}$ where $\alpha < \pi/2$. (Actually $\sigma(T) \cap \{|z| \leq r\} \subset S_2$ for r small is just as good.) Assume $\|(\lambda - T)^{-1}\| \leq e^{d_\lambda^{-p}}$ in a neighborhood of $\sigma(T)$ where $p > 0$ and $p\alpha < \pi/2$. Then Theorem 1 is applicable. Indeed choose ρ, β such that $\alpha < \beta, p < \rho$ and $\beta\rho < \pi/2$. Then set

$$D_1 = \{re^{i\theta} : |\theta - \pi| < \beta\} \cap \{|z| \leq R\}$$

and

$$D_2 = \{re^{i\theta} : |\theta| < \beta\} \cap \{|z| \leq R\}$$

where R is chosen so large that $\sigma(T) \subset \bar{D}_1 \cup \bar{D}_2$. Finally set $f_2(z) = e^{-z^{-\rho}}$ and $f_1(z) = f_2(-z)$. Clearly there exists a constant $K > 0$ such that $d_\lambda \geq K |\operatorname{Im} \lambda|$ for $\lambda \in \partial D_i, i = 1, 2$. For $z = re^{i\theta}$ and $z \in \partial D_i, |f_i(z)| = e^{-(\cos \rho\beta)r^{-\rho}}$. Since $\rho\beta < \pi/2$ we see that

$$\|f_i(\lambda)(\lambda - T)^{-1}\| \leq e^{(K_1 r^{-2} - K r^{-\rho})}.$$

Since the last expression tends to zero as $r \rightarrow 0$ it follows that $\|f_i(\lambda)(\lambda - T)\|^{-1}$ is uniformly bounded on ∂D_i for $i = 1, 2$. Condition 4) is obviously satisfied.

We note that given an operator $T \in \mathfrak{B}(\mathfrak{H})$ where $\sigma(T)$ is real and $T \in C_p$ (the Schatten p class) then $\|(\lambda - T)^{-1}\| \leq \exp(Kd_\lambda^{-(p+1)})$ where K is a constant.

We also note that at first glance it might seem preferable to choose D_1, D_2 so that the approach to 0 is made along a path close to the imaginary axis, thus staying as far away as possible from $\sigma(T)$ and thus in turn keeping $\|(\lambda - T)^{-1}\|$ small, comparatively speaking. While such a selection of D_i 's does accomplish the foregoing, it is then impossible to find a function f_i , analytic in D_i , which vanishes rapidly enough at 0 to enable us to apply the theorem.

Example 3. In this example we permit the resolvent of the operator to assume the largest growth rate for which we can produce invariant subspaces with positive area. Namely we assume that

$$\|(\lambda - T)^{-1}\| \leq \exp \exp d_\lambda^{-p} \quad \text{for } |\lambda| \leq \|T\| + 1 \quad \text{where } 0 < p < 1.$$

If $\sigma(T) \subset E$ where $E = \{z = x + iy : |x| \leq \|T\| \text{ and } -|x|^{q+1} \leq y \leq |x|^{q+1}\}$ and $p < q/(q+1)$, then the theorem is applicable.

To begin with, we choose q' such that $q' < q$ and $p < q'/(q' + 1)$. Then $\{|z| \leq r\} \cap \sigma(T) \subset E_1$

$$= \left\{ z = x + iy : |x| \leq \|T\| \quad \text{and} \quad \frac{-\pi}{5q'} |x|^{q'+1} \leq y \leq \frac{\pi}{5q'} |x|^{q'+1} \right\}$$

for r small. We will assume $\sigma(T) \subset E_1$ for convenience. Moreover we will designate q' by q from here on.

We take

$$D_1 = \left\{ z = x + iy : -\|T\| < x < 0 \quad \text{and} \quad \frac{-\pi}{4q} |x|^{q+1} < y < \frac{\pi}{4q} |x|^{q+1} \right\}$$

and

$$D_2 = \left\{ z = x + iy : 0 < x < \|T\| \quad \text{and} \quad \frac{-\pi}{4q} |x|^{q+1} < y < \frac{\pi}{4q} |x|^{q+1} \right\}.$$

It is easy to see that $\text{dist}[z, E_1] \geq K |\text{Im}z|$ for $z \in \partial D_i$ where K is a positive constant. Thus $\|(\lambda - T)^{-1}\| \leq \exp \exp (K^{-2}|y|^{-p})$ for $\lambda \in \partial D_i$ and $\lambda = x + iy$. We let $f_2(z) = \exp[-\exp(z^{-q})]$ and $f_1(z) = f_2(-z)$. Thus if $z = re^{i\theta}$, then

$$|f_2(z)| = \exp[-\cos(r^{-q} \sin q\theta) \cdot e^{o(s)} r^{-q}].$$

To show that f_2 is analytic in D_2 and continuous on $\bar{D}_2$, it is only necessary to examine the behavior of $\cos(r^{-q} \sin q\theta)$ for z near 0. Under these conditions (with $z = re^{i\theta} = x + iy \neq 0$ and small)

$$\begin{aligned} |r^{-q} \sin q\theta| &\leq (1 + \epsilon)x^{-q} |q\theta| \leq (1 + \epsilon)^2 x^{-q} |y/x| \\ &\leq (1 + \epsilon)^2 x^{-q} q \frac{\pi}{4q} x^{q+1}/x \leq \pi/3 \quad \text{for } \epsilon \text{ small.} \end{aligned}$$

Thus $\cos(r^{-q} \sin q\theta) > K_1 > 0$ for $re^{i\theta} \in \bar{D}_2$ and r small. It follows that f_2 is

continuous in $\bar{D}_2$ as claimed. Finally for $\lambda \in \partial D_2 \setminus \{0\}$ we see that

$$\begin{aligned} A(\lambda) &= \|f_2(\lambda)(\lambda - T)^{-1}\| \\ &\leq \exp \exp (K^{-p} |y|^{-p}) \cdot \exp [-\cos (r^{-q} \sin q\theta) e^{+\cos q\theta r^{-q}}] \\ &\leq \exp \exp (K_2^{-p} x^{-p(q+1)}) \cdot \exp [-K_1 e^{K_3 x^{-q}}] \end{aligned}$$

where $K_1, K_3 > 0$. Thus

$$\lim_{\lambda \rightarrow 0} A(\lambda) = 0 \quad \text{for } \lambda \in \partial D_2 \setminus \{0\}$$

since $p(q+1) < q$ and hence $A(\lambda)$ is bounded on ∂D_2 . The term involving f_1 is handled in the same way. In this example as in the previous one, condition 4) is automatically satisfied.

Added in proof. We would like to draw the reader's attention to a paper by J. I. Ljubic and V. I. Macaev (*Operators with separable spectrum*, Amer. Soc. Trans., Ser. 2, **47** (1965), 89–129). Even though they never consider operators whose spectrum has positive measure and although there are a number of technical differences between our approach and theirs, still their paper is much closer to ours in spirit than any of the earlier references. Also their paper begins with a very interesting example. Namely, an operator $T \in \mathcal{B}(\mathcal{H})$ where the spectrum of T is the interval $[0, 1]$ and $\sigma(T|_M) = [0, 1]$ for every invariant subspace M of T .

REFERENCES

1. R. G. BARTLE, *Spectral localization of operators in Banach space*, Math. Ann. **153** (1964), 261–269.
2. N. DUNFORD, *Spectral theory II. Resolutions of the identity*, Pacific J. Math. **2** (1952), 559–614.
3. ———, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc. **64** (1958), 217–274.
4. N. DUNFORD & J. SCHWARTZ, *Linear Operators, II*, Interscience, New York, 1963.
5. R. GODEMENT, *Théorème Taubérienne et théorie spectrales*, Ann. Sci. École Norm. Sup. (3) **64** (1947), 119–138.
6. G. K. LEAF, *A spectral theory for a class of linear operators*, Pacific J. Math. **13** (1963), 141–155.
7. R. NEVANLINNA, *Analytic functions*, Springer-Verlag, New York, 1970.
8. F. RIESZ & B. SZ.-NAGY, *Functional analysis*, Ungar, New York, 1955.
9. J. SCHWARTZ, *Subdiagonalization of operators in Hilbert space with compact imaginary part*, Comm. Pure Appl. Math. **15** (1962), 159–172.
10. M. TSUJI, *Potential theory in modern function theory*, Maruzen Tokyo, 1959.
11. J. WERMER, *The existence of invariant subspaces*, Duke Math. J. **19** (1952), 615–622.
12. F. WOLF, *Operators in Banach space which admit a generalized spectral decomposition*, Nederl. Akad. Wetensch. Proc. Ser. A **60** = Indag. Math. **19** (1957), 302–311.

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Indiana University

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