The Lebesgue Set of a Function Whose Distribution Derivatives are p -th Power Summable

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This paper is dedicated to Eberhard Hopf on the occasion of his seventieth birthday.

1. Introduction. Assuming that $1 \leq p < n$, $s = np/(n - p)$ and f is a locally $\mathcal{L}^n$ summable real valued function on $\mathbf{R}^n$ whose distribution derivatives are p -th power locally summable, we prove here the existence of a set E with Hausdorff dimension at most $n - p$ such that to each point a in $\mathbf{R}^n \sim E$ corresponds a real number z for which

$$r^{-n} \int_{\{x: |x-a| < r\}} |f(x) - z|^s d\mathcal{L}^n x \rightarrow 0 \quad \text{as } r \downarrow 0.$$

Indeed we show that the functional capacity of degree p of the set E equals 0. For the special case $p = 1$ this result was announced in [5] and proved in [6, §4.5.9].

Various definitions and properties of functional capacity may be found in [1], [4], [7], [9], [10], [11], [12], [13]. However, we present here a simple and direct treatment of this topic, including several fundamental new facts.

Our method is based on geometric measure theory. We use the terminology of [6], which contains glossaries of basic notation on pages 669 to 671.

2. The space $\mathcal{Y}_p(\mathbf{R}^n)$ *is the class of all $\bar{\mathbf{R}}$ valued $\mathcal{L}^n$ measurable functions f such that*

$$\int |f|^s d\mathcal{L}^n < \infty \quad \text{with} \quad \frac{1}{n} < \frac{1}{s} + \frac{1}{n} = \frac{1}{p} \leq 1$$

and there exist $\bar{\mathbf{R}}$ valued $\mathcal{L}^n$ measurable functions $u_1, \dots, u_n$ satisfying the conditions

$$\int |u_i|^p d\mathcal{L}^n < \infty \quad \text{and} \quad \int \varphi u_i d\mathcal{L}^n = - \int f D_i \varphi d\mathcal{L}^n$$

for all $i \in \{1, \dots, n\}$ and $\varphi \in \mathcal{D}^0(\mathbf{R}^n)$. It is a well known consequence that at

$\mathcal{E}^n$ almost all points x the function f has an approximate differential, denoted $apDf(x)$ or $\nabla f(x)$, such that

$$\langle v, \nabla f(x) \rangle = \sum_{i=1}^n v_i u_i(x) \quad \text{for } v \in \mathbf{R}^n.$$

This may be verified by use of regularization and [6, §3.1.4], or deduced from [6, §4.5.16 and 4.5.9(26)] because the distribution $\mathcal{E}^n \llcorner f \in \mathfrak{D}_0(\mathbf{R}^n)$ corresponds to the current $\mathbf{E}^n \llcorner f \in \mathbf{N}_n^{\text{loc}}(\mathbf{R}^n)$ with

$$\begin{aligned} (-1)^i [\partial(\mathbf{E}^n \llcorner f)] \llcorner (DX_1 \wedge \cdots \wedge DX_{i-1} \wedge DX_{i+1} \wedge \cdots \wedge DX_n) \\ = D_i(\mathcal{E}^n \llcorner f) = \mathcal{E}^n \llcorner u_i \end{aligned}$$

for $i \in \{1, \dots, n\}$, hence

$$\partial(\mathbf{E}^n \llcorner f) = -\mathbf{E}^n \llcorner \nabla f \quad \text{and} \quad \|\partial(\mathbf{E}^n \llcorner f)\| = \mathcal{E}^n \llcorner |\nabla f|.$$

On the vector space $\mathcal{Y}_p(\mathbf{R}^n)$ we define the seminorms μ and ν by the formulae

$$\mu(f) = \left(\int |f|^s d\mathcal{E}^n \right)^{1/s} \quad \text{and} \quad \nu(f) = \left(\int |\nabla f|^p d\mathcal{E}^n \right)^{1/p}.$$

From the completeness of $\mathbf{L}_s(\mathcal{E}^n)$ and $\mathbf{L}_p(\mathcal{E}^n)$ we infer that

$$\mathcal{Y}_p(\mathbf{R}^n) \text{ is complete with respect to } \mu + \nu.$$

Next we verify that

$$\mathfrak{D}^0(\mathbf{R}^n) \text{ is } \mu + \nu \text{ dense in } \mathcal{Y}_p(\mathbf{R}^n).$$

Choosing $\psi_j \in \mathfrak{D}^0(\mathbf{R}^n)$ for each positive integer j so that

$$\begin{aligned} \psi_j(x) &= 1 \text{ in case } |x| \leq j, \psi_j(x) = 0 \text{ in case } |x| \geq 3j, \\ 0 &\leq \psi_j(x) \leq 1 \text{ and } |D\psi_j(x)| \leq 1/j \text{ whenever } x \in \mathbf{R}^n, \end{aligned}$$

and observing that

$$\int |D\psi_j|^n d\mathcal{E}^n \leq j^{-n} \mathcal{E}^n\{x : j < |x| < 3j\} = \mathcal{E}^n\{x : 1 < |x| < 3\}.$$

we see that if $f \in \mathcal{Y}_p(\mathbf{R}^n)$, then $\psi_j f \in \mathcal{Y}_p(\mathbf{R}^n)$ with

$$\begin{aligned} \nabla(\psi_j f)(x) &= \psi_j(x) \nabla f(x) + f(x) D\psi_j(x) \quad \text{for } \mathcal{E}^n \text{ almost all } x, \\ \mu(f - \psi_j f) &\leq \left(\int_{\{|x| > j\}} |f|^s d\mathcal{E}^n \right)^{1/s}, \\ \nu(f - \psi_j f) &\leq \left(\int_{\{|x| > j\}} |\nabla f|^p d\mathcal{E}^n \right)^{1/p} \\ &\quad + \left(\int_{\{|x| > j\}} |f|^s d\mathcal{E}^n \right)^{1/s} \left(\int |D\psi_j|^n d\mathcal{E}^n \right)^{1/n}, \end{aligned}$$

hence $\mu(f - \psi_j f) + \nu(f - \psi_j f) \rightarrow 0$ as $j \rightarrow \infty$. Regularizing the functions $\psi_j f$, which have compact supports, we find $\mu + \nu$ approximations in $\mathfrak{D}^0(\mathbf{R}^n)$.

We adopt the convention that $c_1, c_2, \dots$ are suitably large finite positive numbers, which depend only on n and p . As in [6, §4.5.15] we derive the functional isoperimetric inequality

$$\mu(f) \leq c_1 \nu(f) \quad \text{for } f \in \mathcal{Y}_p(\mathbf{R}^n).$$

Thus ν and $\mu + \nu$ are equivalent seminorms on $\mathcal{Y}_p(\mathbf{R}^n)$.

We define $\mathcal{Y}_p^+(\mathbf{R}^n) = \mathcal{Y}_p(\mathbf{R}^n) \cap \{f : f \geq 0\}$.

The space $\mathcal{Y}_p(\mathbf{R}^n)$ is a lattice:

If $f \in \mathcal{Y}_p(\mathbf{R}^n)$, $g \in \mathcal{Y}_p(\mathbf{R}^n)$ and $h = \sup\{f, g\}$, then $h \in \mathcal{Y}_p(\mathbf{R}^n)$ and

$\nabla h(x) = \nabla f(x)$ for $\mathcal{L}^n$ almost all those x where $f(x) \geq g(x)$,

$\nabla h(x) = \nabla g(x)$ for $\mathcal{L}^n$ almost all those x where $f(x) \leq g(x)$.

If $f \in \mathcal{Y}_p(\mathbf{R}^n)$ and $0 \leq t \in \mathbf{R}$, then $\inf\{f, t\} \in \mathcal{Y}_p(\mathbf{R}^n)$.

If F is a countable subset of $\mathcal{Y}_p^+(\mathbf{R}^n)$ and

$$w = \sup\{f : f \in F\}, \quad \lambda = \sup\{|\nabla f| : f \in F\}, \quad \int \lambda^p d\mathcal{L}^n < \infty,$$

then $w \in \mathcal{Y}_p^+(\mathbf{R}^n)$ and $|\nabla w(x)| \leq \lambda(x)$ for $\mathcal{L}^n$ almost all x .

To prove the last statement we arrange $F = \{f_1, f_2, f_3, \dots\}$ and observe that $h_i = \sup\{f_1, \dots, f_i\} \uparrow w$ as $i \uparrow \infty$,

$$\left(\int w^p d\mathcal{L}^n\right)^{1/p} = \lim_{i \rightarrow \infty} \mu(h_i) \leq \limsup_{i \rightarrow \infty} c_1 \nu(h_i) \leq c_1 \left(\int \lambda^p d\mathcal{L}^n\right)^{1/p},$$

$$\begin{aligned} S(\varphi) &= \sum_{i=1}^n [D_i(\mathcal{L}^n \lfloor w)](\varphi_i) = \sum_{i=1}^n - \int w D_i \varphi_i d\mathcal{L}^n \\ &= \lim_{i \rightarrow \infty} \sum_{i=1}^n - \int h_i D_i \varphi_i d\mathcal{L}^n = \lim_{i \rightarrow \infty} \int \langle \varphi, \nabla h_i \rangle d\mathcal{L}^n \leq \int |\varphi| \lambda d\mathcal{L}^n \end{aligned}$$

whenever $\varphi_1, \dots, \varphi_n \in \mathcal{D}^0(\mathbf{R}^n)$ and $\varphi(x) = (\varphi_1(x), \dots, \varphi_n(x))$ for $x \in \mathbf{R}^n$, hence the distribution $S \in \mathcal{D}'(\mathbf{R}^n, \mathbf{R}^n)$ is representable as an integral

$$S(\varphi) = \int \langle \varphi, k \rangle d\mathcal{L}^n$$

for some function $k \in \mathbf{L}_p[\mathcal{L}^n, \text{Hom}(\mathbf{R}^n, \mathbf{R})]$ with $|k| \leq \lambda$, and it follows that $w \in \mathcal{Y}_p^+(\mathbf{R}^n)$ with $\nabla w(x) = k(x)$ for $\mathcal{L}^n$ almost all x .

Similarly one sees that if $p > 1$, $0 \leq M < \infty$,

$h_i \in \mathcal{Y}_p^+(\mathbf{R}^n)$ with $\nu(h_i) \leq M$ for each positive integer i ,

and $h_i \uparrow w$ as $i \uparrow \infty$, then $w \in \mathcal{Y}_p^+(\mathbf{R}^n)$ with $\nu(w) \leq M$.

We also take note of the following criterion:

In case $p > 1$ a sequence of functions f_i in $\mathcal{Y}_p(\mathbf{R}^n)$ is $\mu + \nu$ convergent if and

only if

$$\lim_{(i, j) \rightarrow (\infty, \infty)} \nu[(f_i + f_j)/2] \in \mathbf{R}.$$

In fact, since $L_p[\mathcal{L}^n, \text{Hom}(\mathbf{R}^n, \mathbf{R})]$ is uniformly convex according to [3], the above condition implies that the approximate differentials ∇f_i form an $\mathcal{L}^n_{(p)}$ Cauchy sequence, hence the functions f_i form a $\mu + \nu$ Cauchy sequence.

3. Functional capacity of degree p . Whenever $A \subset \mathbf{R}^n$ we define

$$\Gamma_p(A)$$

as the *infimum of all numbers* $\int |\nabla f|^p d\mathcal{L}^n$ *corresponding to those functions* $f \in \mathcal{Y}^+_p(\mathbf{R}^n)$ *for which* $A \subset \text{Int} \{x : f(x) \geq 1\}$. We verify that Γ_p is a (outer) measure over $\mathbf{R}^n$:

$$\text{If } B \subset \bigcup_{i=1}^{\infty} A_i \subset \mathbf{R}^n, \text{ then } \Gamma_p(B) \leq \sum_{i=1}^{\infty} \Gamma_p(A_i).$$

In fact, for any $f_i \in \mathcal{Y}^+_p(\mathbf{R}^n)$ with $A_i \subset \text{Int} \{x : f_i(x) \geq 1\}$, the function $\lambda = \sup \{|\nabla f_i| : i = 1, 2, 3, \dots\}$ satisfies the condition

$$\int \lambda^p d\mathcal{L}^n \leq \int \sum_{i=1}^{\infty} |\nabla f_i|^p d\mathcal{L}^n = \sum_{i=1}^{\infty} \int |\nabla f_i|^p d\mathcal{L}^n,$$

and, in case the last sum is finite,

$w = \sup \{f_i : i = 1, 2, 3, \dots\} \in \mathcal{Y}^+_p(\mathbf{R}^n)$, $|\nabla w(x)| \leq \lambda(x)$ for $\mathcal{L}^n$ almost all x ,

$$B \subset \text{Int} \{x : w(x) \geq 1\}, \quad \Gamma_p(B) \leq \int |\nabla w|^p d\mathcal{L}^n \leq \int \lambda^p d\mathcal{L}^n.$$

The following two statements are evident:

If $A \subset \mathbf{R}^n$, then $\Gamma_p(A) = \inf \{\Gamma_p(U) : U \text{ is open and } A \subset U\}$.

If $A_1 \supset A_2 \supset A_3 \supset \dots$ are compact subsets of $\mathbf{R}^n$, then

$$\Gamma_p\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i \rightarrow \infty} \Gamma_p(A_i).$$

The approximation procedure of §2 yields the proposition:

If A is a compact subset of $\mathbf{R}^n$, then

$$\Gamma_p(A) = \inf \left\{ \int |D\varphi|^p d\mathcal{L}^n : 0 \leq \varphi \in \mathcal{D}^0(\mathbf{R}^n), \varphi(x) \geq 1 \text{ for } x \in A \right\}.$$

Next we observe that if ψ is a diffeomorphism of class 1 mapping $\mathbf{R}^n$ onto $\mathbf{R}^n$ and

$$\xi = \sup \{ \|D\psi(x)\|^p / J_n \psi(x) : x \in \mathbf{R}^n \} < \infty,$$

then $f \in \mathcal{Y}_p(\mathbf{R}^n)$ implies $f \circ \psi \in \mathcal{Y}_p(\mathbf{R}^n)$ with

$$\int |\nabla(f \circ \psi)|^p d\mathcal{L}^n \leq \xi \int (|\nabla f|^p \circ \psi) J_n \psi d\mathcal{L}^n = \xi \int |\nabla f|^p d\mathcal{L}^n,$$

hence $\Gamma_p[\psi^{-1}(B)] \leq \xi \Gamma_p(B)$ for $B \subset \mathbf{R}^n$; moreover

$$\begin{aligned} \xi &\leq \sup \{ \|D\psi(x)\|^{p-1} \|D\psi(x)^{-1}\|^{n-1} : x \in \mathbf{R}^n \} \\ &\leq [\text{Lip}(\psi)]^{p-1} [\text{Lip}(\psi^{-1})]^{n-1}. \end{aligned}$$

The inequality involving ξ holds because $|\det T| \geq \|T\| \cdot \|T^{-1}\|^{1-n}$ for every linear automorphism T of $\mathbf{R}^n$, as may be verified by use of the last paragraph of [6, §1.7.3].

Clearly Γ_p is invariant under isometries. If $0 < r < \infty$ and $\psi(x) = r^{-1}x$ for $x \in \mathbf{R}^n$, then $\Gamma_p[\psi^{-1}(B)] = r^{n-p} \Gamma_p(B)$ for $B \subset \mathbf{R}^n$. It follows that

$$\Gamma_p[\mathbf{B}(b, r)] = r^{n-p} c_2 \quad \text{for } b \in \mathbf{R}^n \text{ and } 0 < r < \infty,$$

where $c_2 = \Gamma_p[\mathbf{B}(0, 1)]$, hence

$$\Gamma_p(A) \leq c_3 \mathcal{H}^{n-p}(A) \quad \text{for } A \subset \mathbf{R}^n,$$

where $c_3 = c_2 \alpha(n-p)^{-1} 2^{n-p}$.

We conclude this section by proving the following proposition:

If $p > 1$, $A \subset \mathbf{R}^n$ and $\mathcal{H}^{n-p}(A) < \infty$, then $\Gamma_p(A) = 0$.

Choosing γ so that $\mathcal{S}^{n-p}(A) \leq 2^{n-p} \mathcal{H}^{n-p}(A) < \gamma < \infty$ we show first that for each neighborhood V of A there exists a neighborhood W of A and a function $g \in \mathcal{Y}_p^+(\mathbf{R}^n)$ such that

$$W \subset \{x : g(x) = 1\} \subset \{x : g(x) > 0\} \subset V \quad \text{and} \quad \int |\nabla g|^p d\mathcal{L}^n < c_4 \gamma,$$

where $c_4 = \alpha(n)(2^n - 1)/\alpha(n-p)$. Using the fact that

$$\sum_{i \in \mathbb{Z}} \mathcal{S}^{n-p}[A \cap \{x : 2^{-i} < \text{dist}(x, \mathbf{R}^n \setminus V) \leq 2^{1-i}\}] < \gamma$$

in case $V \neq \mathbf{R}^n$, we construct a sequence of balls $\mathbf{U}(b_i, r_i)$ with $\mathbf{U}(b_i, 2r_i) \subset V$,

$$A \subset W = \bigcup_{i=1}^{\infty} \mathbf{U}(b_i, r_i) \quad \text{and} \quad \sum_{i=1}^{\infty} \alpha(n-p)(r_i)^{n-p} < \gamma,$$

define $h_i \in \mathcal{Y}_p^+(\mathbf{R}^n)$ so that

$$\begin{aligned} h_i(x) &= 1 \quad \text{if } |x - b_i| \leq r_i, \quad h_i(x) = 0 \quad \text{if } |x - b_i| \geq 2r_i, \\ h_i(x) &= 2 - |x - b_i|/r_i \quad \text{if } r_i < |x - b_i| < 2r_i, \end{aligned}$$

hence $\int |\nabla h_i|^p d\mathcal{L}^n = (r_i)^{-p} \alpha(n)[(2r_i)^n - (r_i)^n] = c_4 \alpha(n-p)(r_i)^{n-p}$, and take

$$g = \sup \{h_i : i = 1, 2, 3, \dots\} \quad \text{with} \quad \int |\nabla g|^p d\mathcal{L}^n \leq \int \sum_{i=1}^{\infty} |\nabla h_i|^p d\mathcal{L}^n \leq c_4 \gamma.$$

Starting with $V_1 = \mathbf{R}^n$ we then inductively define neighborhoods V_k of A and functions $g_k \in \mathcal{Y}_p^+(\mathbf{R}^n)$ such that

$$V_{k+1} \subset \{x : g_k(x) = 1\} \subset \{x : g_k(x) > 0\} \subset V_k \quad \text{and} \quad \int |\nabla g_k|^p d\mathcal{L}^n < c_4 \gamma$$

for every positive integer k . Since

$$\mathcal{L}^n[\{x : \nabla g_k(x) \neq 0\} \sim (V_k \sim V_{k+1})] = 0$$

and the sets $V_k \sim V_{k+1}$ are disjoint, we infer that

$$\int \left(\sum_{k=1}^{\infty} k^{-1} |\nabla g_k| \right)^p d\mathcal{L}^n = \int \sum_{k=1}^{\infty} k^{-p} |\nabla g_k|^p d\mathcal{L}^n < c_4 \gamma \delta$$

where $\delta = \sum_{k=1}^{\infty} k^{-p} < \infty$, hence

$$f = \sum_{k=1}^{\infty} k^{-1} g_k \in \mathcal{Y}_x^+(\mathbf{R}^n) \quad \text{with} \quad \int |\nabla f|^p d\mathcal{L}^n < c_4 \gamma \delta.$$

Observing that $f(x) \geq \sum_{k=1}^N k^{-1}$ for $x \in V_N$, we conclude

$$\Gamma_x(A) \leq \left(\sum_{k=1}^N k^{-1} \right)^{-p} c_4 \gamma \delta \downarrow 0 \quad \text{as} \quad N \uparrow \infty.$$

4. Functional capacity of degree 1. In case $p = 1$ the analytic theory is made more difficult by the irreflexivity of $\mathbf{L}_1(\mathcal{L}^n)$, but we circumvent this obstacle and gain geometric insight through use of the theory of normal and integral currents.

We shall employ three lemmas:

If U is a bounded open subset of $\mathbf{R}^n$ and $\text{Bdry } U$ can be covered by a finite family of $n - 1$ dimensional planes, then

$$\Gamma_1(\text{Clos } U) \leq \mathcal{H}^{n-1}(\text{Bdry } U).$$

In fact, letting $f_\delta(x) = [1 - \delta^{-1} \text{dist}(x, U)]^+$ for $0 < \delta < \infty$ and $x \in \mathbf{R}^n$, one readily sees that $\text{Lip}(f_\delta) = \delta^{-1}$, $\text{spt}(f_\delta)$ is compact and

$$\int |Df_\delta| d\mathcal{L}^n = \delta^{-1} \mathcal{L}^n\{x : \text{dist}(x, U) < \delta\} \rightarrow \mathcal{H}^{n-1}(\text{Bdry } U) \quad \text{as} \quad \delta \downarrow 0.$$

If P is an $\mathcal{L}^n$ measurable set, $\mathcal{L}^n(P) < \infty$ and

$$Q = \{b : \Theta^{*n}(\mathcal{L}^n \llcorner P, b) > 1/4\},$$

then Q can be covered by a sequence of balls $\mathbf{B}(b_i, r_i)$ with $b_i \in Q$ and

$$\sum_{i=1}^{\infty} (r_i)^{n-1} \leq c_5 \mathbf{M}[\partial(\mathbf{E}^n \llcorner P)].$$

This is a slight generalization of [6, §4.5.4] with $\tau = 1/4$, and can be proved by an almost identical argument.

If g is a nonnegative function for which

$$\int g^* d\mathcal{L}^n < \infty \quad \text{and} \quad \mathbf{M}[\partial(\mathbf{E}^n \llcorner g)] < \infty,$$

then there exist y and X such that

$$0 < y < 1, X = \{x : g(x) \geq y\}, \mathcal{L}^n(X) < \infty,$$

$$\mathbf{M}[\partial(\mathbf{E}^n \llcorner X)] \leq \mathbf{M}[\partial(\mathbf{E}^n \llcorner g)].$$

Clearly $\mathcal{L}^n\{x : g(x) \geq y\} \leq y^{-s} \int g^s d\mathcal{L}^n < \infty$ for all $y > 0$; moreover [6, §4.5.9(13)] implies that

$$\int_0^1 \mathbf{M}[\partial(\mathbf{E}^n \llcorner \{x : g(x) \geq y\})] d\mathcal{L}^1 y \leq \mathbf{M}[\partial(\mathbf{E}^n \llcorner g)].$$

The following proposition gives two alternate characterizations of Γ_1 , in terms of functions whose distribution derivatives are representable by measures which need not be absolutely continuous with respect to $\mathcal{L}^n$, and in terms of sets with finite perimeter.

If $A \subset \mathbf{R}^n$, $s = n/(n-1)$ and

$\alpha =$ the infimum of the set of numbers $\mathbf{M}[\partial(\mathbf{E}^n \llcorner w)]$ corresponding to all nonnegative functions w such that $\int w^s d\mathcal{L}^n < \infty$ and

$$\Theta^{*n}[\mathcal{L}^n \llcorner \{x : w(x) \geq 1\}, a] > 0 \text{ for } \mathcal{H}^{n-1} \text{ almost all } a \text{ in } A,$$

$\beta =$ the infimum of the set of numbers $\mathbf{M}[\partial(\mathbf{E}^n \llcorner B)]$ corresponding to all $\mathcal{L}^n$ measurable sets B such that $\mathcal{L}^n(B) < \infty$ and $A \subset \text{Int } B$,

then $\alpha = \beta = \Gamma_1(A)$.

First we note that $\alpha \leq \beta$, because w can be taken as the characteristic function of B .

Next we verify that $\beta \leq \Gamma_1(A)$. For any $f \in \mathcal{Y}_1^+(\mathbf{R}^n)$ such that

$$A \subset \text{Int } \{x : f(x) \geq 1\}$$

we apply our third lemma with $g = f$, take $B = X$ and conclude

$$A \subset \text{Int } B, \quad \beta \leq \mathbf{M}[\partial(\mathbf{E}^n \llcorner B)] \leq \mathbf{M}[\partial(\mathbf{E}^n \llcorner f)] = \int |\nabla f| d\mathcal{L}^n.$$

Finally we prove that $\Gamma_1(A) \leq \alpha$. Given $\epsilon > 0$ and any nonnegative function w such that $\int w^s d\mathcal{L}^n < \infty$, $\mathbf{M}[\partial(\mathbf{E}^n \llcorner g)] < \alpha + \epsilon$ and

$$\Theta^{*n}[\mathcal{L}^n \llcorner \{x : w(x) \geq 1\}, a] > 0 \text{ for } \mathcal{H}^{n-1} \text{ almost all } a \text{ in } A,$$

we apply our third lemma with $g = w$ to obtain an $\mathcal{L}^n$ measurable set X such that $\mathbf{M}(\mathbf{E}^n \llcorner X) = \mathcal{L}^n(X) < \infty$, $\mathbf{M}[\partial(\mathbf{E}^n \llcorner X)] < \alpha + \epsilon$ and

$$\Theta^{*n}(\mathcal{L}^n \llcorner X, a) > 0 \text{ for } \mathcal{H}^{n-1} \text{ almost all } a \text{ in } A.$$

Letting $u(x) = |x|$ for $x \in \mathbf{R}^n$, we see from [6, §4.2.1] that

$$\int_0^\infty \mathbf{M}(\mathbf{E}^n \llcorner X, u, \rho +) d\mathcal{L}^1 \rho \leq \mathbf{M}(\mathbf{E}^n \llcorner X) < \infty$$

and choose a finite positive number ρ for which

$$|\partial(\mathbf{E}^n \llcorner X)| \{x : |x| > \rho\} + \mathbf{M}(\mathbf{E}^n \llcorner X, u, \rho) < \epsilon,$$

hence $\mathbf{M}(\partial[(\mathbf{E}^n \llcorner X) \llcorner \{x : |x| > \rho\}]) < \epsilon$ and, by [6, §4.2.16(2)],

$$R = (\mathbf{E}^n \llcorner X) \llcorner \{x : |x| \leq \rho\} \in \mathbf{I}_n(\mathbf{R}^n) \quad \text{with} \quad \mathbf{M}(\partial R) < \alpha + 2\epsilon.$$

From [6, §4.2.20] we obtain $S \in \mathcal{O}_n(\mathbf{R}^n)$ and a diffeomorphism ψ of class 1 mapping $\mathbf{R}^n$ onto $\mathbf{R}^n$ such that

$$\mathbf{N}(S - \psi_\# R) \leq \epsilon, \quad \text{Lip}(\psi) \leq 1 + \epsilon, \quad \text{Lip}(\psi^{-1}) \leq 1 + \epsilon.$$

Applying our first lemma with $U = \text{Int}(\text{spt } S)$ we find that

$$\begin{aligned} \Gamma_1(\text{spt } S) &\leq \mathcal{H}^{n-1}[\text{Bdry}(\text{spt } S)] \leq \mathbf{M}(\partial S) \\ &\leq \mathbf{M}(\psi_\# \partial R) + \epsilon \leq (1 + \epsilon)^{n-1}(\alpha + 2\epsilon) + \epsilon, \end{aligned}$$

and infer from §3 that

$$\Gamma_1[\psi^{-1}(\text{spt } S)] \leq (1 + \epsilon)^{n-1} \Gamma_1(\text{spt } S) \leq (1 + \epsilon)^{2n-2}(\alpha + 3\epsilon).$$

Then we choose an integer valued $\mathcal{L}_n$ measurable function h such that

$$\mathbf{E}^n \llcorner h = (\mathbf{E}^n \llcorner X) - \psi_\#^{-1} S = (\mathbf{E}^n \llcorner X) \llcorner \{x : |x| > \rho\} + \psi_\#^{-1}(\psi_\# R - S)$$

and let $P = \{x : h(x) = 1\}$. Since $P = \{x : h(x) \geq y\}$ for $0 < y \leq 1$, it follows from [6, §4.5.9(13)] that

$$\mathbf{M}[\partial(\mathbf{E}^n \llcorner P)] \leq \mathbf{M}[\partial(\mathbf{E}^n \llcorner h)] \leq \epsilon + (1 + \epsilon)^{n-1} \epsilon.$$

We also observe that $\mathcal{L}^n[(X \sim P) \sim \psi^{-1}(\text{spt } S)] = 0$, hence

$$\Theta^{*n}(\mathcal{L}^n \llcorner P, b) > 0 \text{ for } \mathcal{H}^{n-1} \text{ almost all } b \text{ in } A \sim \psi^{-1}(\text{spt } S).$$

Using [6, §4.5.6] with A replaced by P we see that

$$\Theta^*(\mathcal{L}^n \llcorner P, b) \in \{1, 0, 1/2\} \text{ for } \mathcal{H}^{n-1} \text{ almost all } b \text{ in } \mathbf{R}^n,$$

$$\Theta^*(\mathcal{L}^n \llcorner P, b) \geq 1/2 \text{ for } \mathcal{H}^{n-1} \text{ almost all } b \text{ in } A \sim \psi^{-1}(\text{spt } S),$$

and infer from our second lemma that $\mathcal{H}^{n-1}$ almost all of $A \sim \psi^{-1}(\text{spt } S)$ can be covered by a sequence of balls $\mathbf{B}(b_i, r_i)$ with

$$\sum_{i=1}^{\infty} (r_i)^{n-1} \leq c_5 \mathbf{M}[\partial(\mathbf{E}^n \llcorner P)] \leq c_5 (1 + \epsilon)^{n-1} 2\epsilon.$$

Recalling §3 we conclude $\Gamma_1(A) \leq (1 + \epsilon)^{2n-2}(\alpha + 3\epsilon) + c_2 c_5 (1 + \epsilon)^{n-1} 2\epsilon$.

For a sufficiently solid set A the minimizing sequences arising from the above definition of β have subsequences which converge to an extremal set with finite perimeter:

If $A \subset \mathbf{R}^n$, $\Gamma_1(A) < \infty$ and $\Theta^{*n}(A, a) > 0$ for $\mathcal{H}^{n-1}$ almost all a in A , then there exists an $\mathcal{L}^n$ measurable set P such that $\mathcal{L}^n(P) < \infty$, $\Gamma_1(A) = \mathbf{M}[\partial(\mathbf{E}^n \llcorner P)]$ and $\Theta^*(P, a) \geq 1/2$ for $\mathcal{H}^{n-1}$ almost all a in A .

To prove this we choose a sequence of $\mathcal{L}^n$ measurable sets B_i such that $\mathcal{L}^n(B_i) < \infty$, $A \subset \text{Int } B_i$ and

$$\lim_{i \rightarrow \infty} \mathbf{M}[\partial(\mathbf{E}^n \llcorner B_i)] = \Gamma_1(A),$$

hence $\limsup_{i \rightarrow \infty} \mathbf{M}(\mathbf{E}^n \llcorner B_i) = \limsup_{i \rightarrow \infty} \mathcal{L}^n(B_i) \leq [c_1 \Gamma_1(A)]^n$. From [6, §4.3.16] and [8, §3.4] we infer that the sequence of currents $\mathbf{E}^n \llcorner B_i \in \mathbf{I}_n^{1,00}(\mathbf{R}^n) \subset \mathcal{F}_n^{1,00}(\mathbf{R}^n)$ has a convergent subsequence with limit $\mathbf{E}^n \llcorner P$ for some $\mathcal{L}^n$ measurable set P . The characteristic functions of the corresponding sets B_i converge $\mathcal{L}^n$ almost everywhere to the characteristic function of P , and $\mathcal{L}^n(A \sim P) = 0$ because $A \subset B_i$. The lower semicontinuity of mass implies

$$\mathbf{M}[\partial(\mathbf{E}^n \llcorner P)] \leq \Gamma_1(A) \text{ and } \mathcal{L}^n(P) = \mathbf{M}(\mathbf{E}^n \llcorner P) \leq [c_1 \Gamma_1(A)]^n.$$

Using [6, §4.5.6] with A replaced by P we obtain

$$0 < \Theta^{*n}(A, a) \leq \Theta^n(P, a) \in \{1, 0, 1/2\} \text{ for } \mathcal{H}^{n-1} \text{ almost all } a \in A.$$

Finally we see that $\Gamma_1(A) = \alpha \leq M[\partial(\mathbf{E}^n \llcorner P)]$ because w can be taken as the characteristic function of P .

Extremal sets fail to exist for many sets $A \subset \mathbf{R}^n$ with $\Gamma_1(A) < \infty$, for example when A is a line segment contained in $\mathbf{R}^2$.

From the second and third lemmas and §3 we infer the proposition:

For $A \subset \mathbf{R}^n$, $\Gamma_1(A) = 0$ if and only if $\mathcal{H}^{n-1}(A) = 0$.

5. Functional capacity of the union of an increasing sequence.

If $A_1 \subset A_2 \subset A_3 \subset \dots$ are subsets of $\mathbf{R}^n$, then

$$\Gamma_p\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \rightarrow \infty} \Gamma_p(A_i).$$

To prove this we suppose

$$\gamma = \lim_{i \rightarrow \infty} \Gamma_p(A_i) < \infty \quad \text{and} \quad 0 < \epsilon < \infty.$$

We choose $f_i \in \mathcal{Y}_p^+(\mathbf{R}^n)$ so that

$$A_i \subset \text{Int } \{x : f_i(x) \geq 1\} \quad \text{and} \quad \int |\nabla f_i|^p d\mathcal{L}^n \leq \Gamma_p(A_i) + \epsilon 2^{-i},$$

let $h_i = \sup \{f_1, \dots, f_i\}$ and observe that, when $j > 1$,

$$h_i = \sup \{h_{i-1}, f_i\}, \quad A_{i-1} \subset \text{Int } \{x : \inf \{h_{i-1}, f_i\}(x) \geq 1\},$$

$$\begin{aligned} \int |\nabla h_i|^p d\mathcal{L}^n &+ \Gamma_p(A_{i-1}) \\ &\leq \int |\nabla \sup \{h_{i-1}, f_i\}|^p d\mathcal{L}^n + \int |\nabla \inf \{h_{i-1}, f_i\}|^p d\mathcal{L}^n \end{aligned}$$

$$\begin{aligned}
&= \int |\nabla h_{i-1}|^p d\mathcal{L}^n + \int |\nabla f_i|^p d\mathcal{L}^n \\
&\leq \int |\nabla h_{i-1}|^p d\mathcal{L}^n + \Gamma_p(A_i) + \epsilon 2^{-i}.
\end{aligned}$$

Since $h_1 = f_1$ we inductively estimate

$$\int |\nabla h_i|^p d\mathcal{L}^n \leq \Gamma_p(A_i) + \sum_{i=1}^i \epsilon 2^{-i} < \gamma + \epsilon.$$

Defining w so that $h_i \uparrow w$ as $j \uparrow \infty$, we infer

$$\bigcup_{i=1}^{\infty} A_i \subset \text{Int } \{x : w(x) \geq 1\}.$$

In case $p > 1$ it follows from §2 that $w \in \mathcal{Y}_p^+(\mathbf{R}^n)$ and

$$\Gamma_p\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \int |\nabla w|^p d\mathcal{L}^n \leq \gamma + \epsilon.$$

In case $p = 1$ it can happen that $w \notin \mathcal{Y}_1(\mathbf{R}^n)$, but it is true that

$$\left(\int w^* d\mathcal{L}^n\right)^{1/s} = \lim_{i \rightarrow \infty} \mu(h_i) \leq \limsup_{i \rightarrow \infty} c_1 \nu(h_i) \leq c_1(\gamma + \epsilon),$$

and since $\mathbf{E}^n \lfloor h_i \rightarrow \mathbf{E}^n \lfloor w$ in $\mathfrak{D}_n(\mathbf{R}_n)$ as $j \rightarrow \infty$ the proposition proved in §4 implies

$$\Gamma_p\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \mathbf{M}[\partial(\mathbf{E}^n \lfloor w)] \leq \liminf_{i \rightarrow \infty} \mathbf{M}[\partial(\mathbf{E}^n \lfloor h_i)] = \liminf_{i \rightarrow \infty} \nu(h_i) \leq \gamma + \epsilon.$$

From the preceding limit theorem it follows immediately that *the equation*

$$\Gamma_p(A) = \sup \{ \Gamma_p(K) : K \text{ is compact and } K \subset A \}$$

holds for every open subset A of $\mathbf{R}^n$. Furthermore it can be inferred by a classical argument, presented in [2, §1] and similar to the proof of [6, §2.2.12], that *this equation is true for every Suslin subset A of $\mathbf{R}^n$* . However, we shall not need this result here.

6. Approximation by medians. Suppose $b \in \mathbf{R}^n$ and $0 < r < \infty$. The set

$$\text{med}(f, b, r)$$

of $\mathcal{L}^n$ medians of an $\bar{\mathbf{R}}$ valued $\mathcal{L}^n$ measurable function f with respect to the ball $\mathbf{U}(b, r)$ consists of all $t \in \bar{\mathbf{R}}$ satisfying the two conditions

$$\mathcal{L}^n\{x : |x - b| < r, f(x) > t\} \leq \alpha(n)r^n/2,$$

$$\mathcal{L}^n\{x : |x - b| < r, f(x) < t\} \leq \alpha(n)r^n/2.$$

We note that $\text{med}(f, b, r)$ is a subinterval of $\bar{\mathbf{R}}$, and that

$$\inf \{y : \mathcal{L}^n\{x : |x - b| < r, f(x) > y\} < \alpha(n)r^n/2\},$$

$$\sup \{z : \mathcal{L}^n\{x : |x - b| < r, f(x) < z\} < \alpha(n)r^n/2\}$$

are the greatest and least elements of $\text{med}(f, b, r)$.

If $f(x) \in \mathbf{R}$ for $\mathcal{L}^n$ almost all x in $\mathbf{U}(b, r)$, then $\text{med}(f, b, r) \subset \mathbf{R}$.

The concept of median is important because of the estimate:

If $f \in \mathcal{Y}_p(\mathbf{R}^n)$ and $t \in \text{med}(f, b, r)$, then

$$\left(\int_{\mathbf{U}(b, r)} |f - t|^s d\mathcal{L}^n \right)^{1/s} \leq c_6 \left(\int_{\mathbf{U}(b, r)} |\nabla f|^p d\mathcal{L}^n \right)^{1/p}.$$

To verify this inequality we consider the function

$$\varphi = \text{sign}(f - t) \cdot |f - t|^{s(n-1)/n} \in \mathcal{Y}_1(\mathbf{R}^n)$$

with $\nabla \varphi(x) = \text{sign}[f(x) - t] \cdot (s - s/n) |f(x) - t|^{s-s/n-1} \nabla f(x)$ for $\mathcal{L}^n$ almost all x . Choosing q so that $1/p + 1/q = 1$, hence $1/s + 1/n + 1/q = 1$, we apply [6, §4.5.9(19)] with f and t replaced by φ and 0, as well as Hölder's inequality, to obtain

$$\begin{aligned} \left(\int_{\mathbf{U}(b, r)} |f - t|^s d\mathcal{L}^n \right)^{1-1/n} &= \left(\int_{\mathbf{U}(b, r)} |\varphi|^{n/(n-1)} d\mathcal{L}^n \right)^{(n-1)/n} \\ &\leq \sigma \|\partial(\mathbf{E}^n \lfloor \varphi)\| \mathbf{U}(b, r) = \sigma \int_{\mathbf{U}(b, r)} |\nabla \varphi| d\mathcal{L}^n \\ &\leq \sigma(s - s/n) \left(\int_{\mathbf{U}(b, r)} |\nabla f|^p d\mathcal{L}^n \right)^{1/p} \cdot \left(\int_{\mathbf{U}(b, r)} |f - t|^s d\mathcal{L}^n \right)^{1/q}. \end{aligned}$$

Thus we can take $c_6 = \sigma s(n-1)/n = \sigma p(n-1)/(n-p)$.

Next we prove an extension theorem:

For each $g \in \mathcal{Y}_p^+(\mathbf{R}^n)$ there exists an $h \in \mathcal{Y}_p^+(\mathbf{R}^n)$ such that

$$\begin{aligned} h|_{\mathbf{B}(b, r)} &= g|_{\mathbf{B}(b, r)}, \quad \text{spt } h \subset \mathbf{B}(b, 3r/2), \\ \nu(h) &\leq c_7 \left(\int_{\mathbf{U}(b, r)} |\nabla g|^p d\mathcal{L}^n \right)^{1/p} + c_8 \left(\int_{\mathbf{U}(b, r)} g^s d\mathcal{L}^n \right)^{1/s}. \end{aligned}$$

Assuming $b = 0$ we define $k \in \mathcal{Y}_p^+(\mathbf{R}^n)$ so that

$$k(x) = (2r^{-1}|x| - 1)g(x) \quad \text{if } |x| \geq r/2, \quad k(x) = 0 \quad \text{if } |x| < r/2,$$

and infer for $\mathcal{L}^n$ almost all x in $\mathbf{B}(0, r) \sim \mathbf{U}(0, r/2)$ that

$$\langle v, \nabla k(x) \rangle = (2r^{-1}|x| - 1)\langle v, \nabla g(x) \rangle + 2r^{-1}(|x|^{-1}x \bullet v)g(x)$$

whenever $v \in \mathbf{R}^n$, hence $|\nabla k(x)| \leq |\nabla g(x)| + 2r^{-1}g(x)$. It follows from the inequalities of Minkowski and Hölder that

$$\left(\int_{\mathbf{U}(0, r)} |\nabla k|^p d\mathcal{L}^n \right)^{1/p} \leq \left(\int_{\mathbf{U}(0, r)} |\nabla g|^p d\mathcal{L}^n \right)^{1/p} + 2 \left(\int_{\mathbf{U}(0, r)} g^s d\mathcal{L}^n \right)^{1/s} [\alpha(n)]^{1/n}.$$

We also define the diffeomorphism

$$\psi : \mathbf{R}^n \cap \{x : r \leq |x| \leq 3r/2\} \simeq \mathbf{R}^n \cap \{x : r/2 \leq |x| \leq r\},$$

$$\psi(x) = (2r |x|^{-1} - 1)x \quad \text{whenever } r \leq |x| \leq 3r/2,$$

note that $\|D\psi\| = 1$ and $J_n \psi \geq 3^{1-n}$, let

$$h(x) = k[\psi(x)] \quad \text{if } r \leq |x| \leq 3r/2,$$

$$h(x) = g(x) \quad \text{if } |x| < r, \quad h(x) = 0 \quad \text{if } |x| > 3r/2,$$

conclude that

$$\begin{aligned} \int_{\{x : r \leq |x| \leq 3r/2\}} |\nabla h|^p d\mathcal{L}^n &\leq 3^{n-1} \int_{\{x : r \leq |x| \leq 3r/2\}} (|\nabla k|^p \circ \psi) J_n \psi d\mathcal{L}^n \\ &= 3^{n-1} \int_{\{x : r/2 \leq |x| \leq r\}} |\nabla k|^p d\mathcal{L}^n, \end{aligned}$$

and take $c_7 = 1 + 3^{(n-1)/p}$, $c_8 = 3^{(n-1)/p} 2[\alpha(n)]^{1/n}$.

We will use the following corollary:

If $f \in \mathcal{Y}_p(\mathbf{R}^n)$ and $t \in \text{med}(f, b, r)$, then there exists $h \in \mathbf{R}_p^+(\mathbf{R}^n)$ such that

$$h | \mathbf{B}(b, r) = (t - f)^+ | \mathbf{B}(b, r) \quad \text{and} \quad \nu(h) \leq c_9 \left(\int_{\mathbf{B}(b, r)} |\nabla f|^p d\mathcal{L}^n \right)^{1/p}.$$

To verify the corollary, with $c_9 = c_7 + c_8 c_6$, we construct $g \in \mathcal{Y}_p^+(\mathbf{R}^n)$ so that $g | \mathbf{B}(b, r) = (t - f)^+ | \mathbf{B}(b, r)$ and combine the extension theorem with the estimate which precedes it.

7. Medians and functional capacity are related by the fundamental proposition:

If $f \in \mathcal{Y}_p^+(\mathbf{R}^n)$, $0 < \epsilon < \infty$ and

$$A = \{b : \inf [\text{med}(f, b, r)] > \epsilon \text{ for some } r\},$$

then A is open and $\Gamma_p(A) \leq c_{10} \epsilon^{-p} \int |\nabla f|^p d\mathcal{L}^n$.

To prove this we consider the relation

$$\begin{aligned} P &= \{(b, r) : \inf [\text{med}(f, b, r)] > \epsilon\} \\ &= \{(b, r) : \mathcal{L}^n\{x : |x - b| < r, f(x) > \epsilon\} > \alpha(n)r^n/2\}. \end{aligned}$$

Clearly P is open in $\mathbf{R}^n \times \mathbf{R}$, hence $A = \{b : (b, r) \in P \text{ for some } r\}$ is open in $\mathbf{R}^n$. Moreover $\{r : (b, r) \in P \text{ for some } b\}$ is bounded, because

$$\alpha(n)r^n/2 < \mathcal{L}^n\{x : f(x) > \epsilon\} \leq \epsilon^{-s} \int f^s d\mathcal{L}^n \quad \text{for } (b, r) \in P.$$

Applying [6, §2.8.9 to 2.8.14] with $X = \mathbf{R}^n$, $\xi = \infty$ and suitable constants ξ, η, ζ, τ determined by n , we obtain subsets Q and $Q_1, Q_2, \dots, Q_{2\tau+1}$ of P

such that

$$A \subset \bigcup_{(b,r) \in Q} \mathbf{B}(b, r), \quad Q = \bigcup_{i=1}^{2\zeta+1} Q_i$$

and for each $j \in \{1, 2, \dots, 2\zeta + 1\}$ the balls $\mathbf{B}(b, r)$ corresponding to the pairs (b, r) in Q_j are disjoint, hence Q_j is countable. For each $(b, r) \in Q$ we use the corollary of the extension theorem in §6 with t replaced by

$$t_{b,r} = \inf [\text{med } (f, b, r)] > \epsilon$$

to find $h_{b,r} \in \mathcal{Y}_p^+$ such that $h_{b,r} \mid \mathbf{B}(b, r) = (t_{b,r} - f)^+ \mid \mathbf{B}(b, r)$ and

$$\int |\nabla h_{b,r}|^p d\mathcal{L}^n \leq (c_8)^p \int_{\mathbf{U}(b,r)} |\nabla f|^p d\mathcal{L}^n.$$

We observe that $f(x) + h_{b,r}(x) \geq t_{b,r} > \epsilon$ whenever $x \in \mathbf{B}(b, r)$. Letting

$$w = \sup \{h_{b,r} : (b, r) \in Q\}, \quad \lambda = \sup \{|\nabla h_{b,r}| : (b, r) \in Q\},$$

we infer that $f(x) + w(x) > \epsilon$ whenever $x \in A$,

$$\begin{aligned} \int \lambda^p d\mathcal{L}^n &\leq \int \sum_{(b,r) \in Q} |\nabla h_{b,r}|^p d\mathcal{L}^n \\ &\leq \sum_{i=1}^{2\zeta+1} \sum_{(b,r) \in Q_i} (c_8)^p \int_{\mathbf{U}(b,r)} |\nabla f|^p d\mathcal{L}^n \\ &\leq (2\zeta + 1)(c_8)^p \int |\nabla f|^p d\mathcal{L}^n, \end{aligned}$$

hence $w \in \mathcal{Y}_p^+(\mathbf{R}^n)$ with $|\nabla w(x)| \leq \lambda(x)$ for $\mathcal{L}^n$ almost all x , and

$$\begin{aligned} \Gamma_p(A) &\leq \int |\nabla[\epsilon^{-1}(f + w)]|^p d\mathcal{L}^n \leq 2^p \epsilon^{-p} \int (|\nabla f|^p + \lambda^p) d\mathcal{L}^n \\ &\leq 2^p [1 + (2\zeta + 1)(c_8)^p] \epsilon^{-p} \int |\nabla f|^p d\mathcal{L}^n. \end{aligned}$$

8. The Lebesgue set of degree s for an $\mathcal{L}^n$ measurable function f consists of all points b in $\mathbf{R}^n$ such that

$$\lim_{r \downarrow 0} r^{-n} \int_{\mathbf{U}(b,r)} |f(x) - f(b)|^s d\mathcal{L}^n = 0.$$

The exceptional set of degree s for f is the complement with respect to $\mathbf{R}^n$ of the Lebesgue set of degree s for f .

It is well known that membership of b in the Lebesgue set of degree s for f implies convergence of regularization at b ; in fact

$$\lim_{\epsilon \downarrow 0} \int f(b + \epsilon v) \Phi(v) d\mathcal{L}^n v = f(b)$$

for every real valued $\mathcal{L}^n$ measurable function Φ with compact support such that $\int \Phi d\mathcal{L}^n = 1$ and $\int |\Phi|^{s/(s-1)} d\mathcal{L}^n < \infty$.

Classical differentiation theory shows that in case $\int |f|^s d\mathcal{L}^n < \infty$ the exceptional set of degree s for f has $\mathcal{L}^n$ measure 0. Here and in §9 it will be proved that in case $f \in \mathcal{Y}_p(\mathbf{R}^n)$ there exists a function which agrees $\mathcal{L}^n$ almost everywhere with f and for which the exceptional set of degree s has Hausdorff dimension at most $n - p$, and even Γ_p measure 0.

We need a lemma concerning densities:

If γ is a Radon measure over $\mathbf{R}^n$ and γ is absolutely continuous with respect to $\mathcal{L}^n$, then

$$\mathcal{H}^m\{b: \Theta^{*m}(\gamma, b) > 0\} = 0 \text{ for } 0 \leq m < n.$$

To prove this we fix $m < n$, define $B_i = \{b: |b| < i, \Theta^{*m}(\gamma, b) > i^{-1}\}$ for each positive integer i , and infer from [6, §2.10.19(3)] that

$$i^{-1}\mathcal{H}^m(B_i) \leq \gamma(B_i).$$

Since B_i is bounded $\gamma(B_i) < \infty$, $\mathcal{H}^m(B_i) < \infty$, $\mathcal{H}^n(B_i) = 0$, $\gamma(B_i) = 0$ and $\mathcal{H}^m(B_i) = 0$.

Now we will establish a connection between densities and medians. If $f \in \mathcal{Y}_p(\mathbf{R}^n)$ and

$$P_m = \{b: \Theta^{*m}(\mathcal{L}^n \llcorner |\nabla f|^p, b) > 0\} \quad \text{for } 0 \leq m < \infty,$$

then the following five statements hold:

- (1) $P_{n-p} \subset \bigcap \{P_m : m > n - p\}$.
- (2) $\mathcal{H}^{n-p}(P_{n-p}) = 0$.
- (3) If $b \in \mathbf{R}^n \sim P_{n-p}$, then

$$\limsup_{r \downarrow 0} \left\{ r^{-n} \int_{\mathbf{U}(b, r)} |f - t|^s d\mathcal{L}^n : t \in \text{med}(f, b, r) \right\} = 0.$$

- (4) $\mathcal{H}^\kappa(\bigcap \{P_m : m > n - p\}) = 0$ for $n - p < \kappa < \infty$.

(5) If $b \in \mathbf{R}^n \sim \bigcap \{P_m : m > n - p\}$, then there exists $z \in \mathbf{R}$ with the two properties

$$\limsup_{r \downarrow 0} \{|t - z| : t \in \text{med}(f, b, r)\} = 0,$$

$$\lim_{r \downarrow 0} r^{-n} \int_{\mathbf{U}(b, r)} |f - z|^s d\mathcal{L}^n = 0.$$

The assertion (1) is trivial. Applying our lemma with $\gamma = \mathcal{L}^n \llcorner |\nabla f|^p$ we find that $\mathcal{H}^m(P_m) = 0$ for $0 \leq m < n$, hence infer (2) and (4). The statement (3) holds because the estimate in §6 implies

$$\left(r^{-n} \int_{\mathbf{U}(b, r)} |f - t|^s d\mathcal{L}^n \right)^{1/s} \leq c_6 (r^{p-n} \gamma[\mathbf{U}(b, r)])^{1/p}$$

for $0 < r < \infty$ and $t \in \text{med}(f, b, r)$. To prove (5) we assume that $b \in \mathbf{R}^n \sim P_m$ with $n - p < m < \infty$, hence $\omega = (m + p - n)/p > 0$, then choose $\delta > 0$ so that $(1 + 2^{n/s}) \delta^\omega < 1 - 2^{-\omega}$ and

$$\gamma[\mathbf{U}(b, r)] \leq [\alpha(n)]^{p/s} (c_6)^{-p} r^m \quad \text{for } 0 < r \leq \delta,$$

which implies

$$\left(\alpha(n)^{-1} r^{-n} \int_{\mathbf{U}(b, r)} |f - t|^s d\mathcal{L}^n \right)^{1/s} \leq r^\omega \quad \text{for } t \in \text{med}(f, b, r).$$

Consequently, if $0 < \rho/2 \leq r \leq \rho \leq \delta$, $t \in \text{med}(f, b, r)$ and $\tau \in \text{med}(f, b, \rho)$, then

$$\begin{aligned} |t - \tau| &= \left([\alpha(n)]^{-1} r^{-n} \int_{\mathbf{U}(b, r)} |t - \tau|^s d\mathcal{L}^n \right)^{1/s} \\ &\leq \left([\alpha(n)]^{-1} r^{-n} \int_{\mathbf{U}(b, r)} |f - t|^s d\mathcal{L}^n \right)^{1/s} \\ &\quad + \left([\alpha(n)]^{-1} r^{-n} \int_{\mathbf{U}(b, r)} |f - \tau|^s d\mathcal{L}^n \right)^{1/s} \\ &\leq r^\omega + (\rho/r)^{n/s} \rho^\omega \leq (1 + 2^{n/s}) \rho^\omega. \end{aligned}$$

Letting $\rho_i = 2^{-i} \delta$ and $\tau_i \in \text{med}(f, b, \rho_i)$ for each nonnegative integer i we infer that if $\rho_{i+1} \leq r \leq \rho_i$ and $t \in \text{med}(f, b, r)$ for some integer $j \geq i$, then

$$|t - \tau_i| \leq |t - \tau_j| + \sum_{k=i}^{j-1} |\tau_{k+1} - \tau_k| \leq \sum_{k=i}^j (1 + 2^{n/s}) (\rho_k)^\omega \leq 2^{-i\omega}.$$

We conclude the existence of $z = \lim_{i \rightarrow \infty} \tau_i \in \mathbf{R}$ with the first property in (5), which we then combine with (1) and (3) to deduce the second property in (5).

We derive the corollary that

$$\Gamma_p(A) = 0 \text{ implies } \mathcal{H}^s(A) = 0 \text{ for } A \subset \mathbf{R}^n \text{ and } n - p < \kappa < \infty,$$

because there exist $\varphi_i \in \mathcal{Y}_p^+(\mathbf{R}^n)$ with $A \subset \text{Int} \{x: \varphi_i(x) \geq 1\}$ and

$$\sum_{i=1}^{\infty} \nu(\varphi_i) < \infty, \quad \text{hence} \quad f = \sum_{i=1}^{\infty} \varphi_i \in \mathcal{Y}_p^+(\mathbf{R}^n),$$

$$A \subset \bigcap_{i=1}^{\infty} \text{Int} \{x: f(x) \geq i\} \subset \{b: \liminf_{r \downarrow 0} [\text{med}(f, b, r)] = \infty\},$$

so that (5) and (4) yield $A \subset \bigcap \{P_m: m > n - p\}$ and $\mathcal{H}^s(A) = 0$.

The result of this corollary is optimal in case $p > 1$, but not in case $p = 1$, as shown by the propositions at the end of §3 and §4, respectively.

9. The main theorem. *For every function $f \in \mathcal{Y}_p(\mathbf{R}^n)$ there exists a function $g \in \mathcal{Y}_p(\mathbf{R}^n)$ such that $g(x) = f(x)$ for $\mathcal{L}^n$ almost all x and the exceptional set of degree s for g has Γ_p measure 0.*

To prove this we define

$$M = \{b: \Theta^{*n-p}(\mathcal{L}^n \llcorner |\nabla f|^p, b) > 0\}$$

and infer from §8(2), §3 that $\mathcal{H}^{n-p}(M) = 0$, $\Gamma_p(M) = 0$. For each positive integer i we choose

$$\varphi_i \in \mathcal{D}^0(\mathbf{R}^n) \quad \text{with} \quad c_{10} 2^{ip} \int |\nabla(f - \varphi_i)|^p d\mathcal{L}^n < 2^{-i},$$

let

$$N_i = \{b: \inf [\text{med}(|f - \varphi_i|, b, r)] > 2^{-i} \text{ for some } r\},$$

use §7 to estimate $\Gamma_p(N_i) < 2^{-i}$, and observe that

$$\{b: \Theta^{*n-p}(\mathcal{L}^n \llcorner |\nabla|f - \varphi_i||^p, b) > 0\} \subset M$$

because $D\varphi_i$ is continuous.

If $b \in \mathbf{R}^n \sim (M \cup N_i)$, $0 < r < \infty$ and $t \in \text{med}(f, b, r)$, then

$$0 \leq \tau = \inf [\text{med}(|f - \varphi_i|, b, r)] \leq 2^{-i},$$

and $|t - \varphi_i(b)| \leq |f(x) - t| + |\varphi_i(x) - \varphi_i(b)| + ||f(x) - \varphi_i(x)| - \tau| + \tau$ for $\mathcal{L}^n$ almost all x , hence

$$\begin{aligned} |t - \varphi_i(b)| &\leq \left([\alpha(n)]^{-1} r^{-n} \int_{\mathbf{U}(b, r)} |f - t|^s d\mathcal{L}^n \right)^{1/s} \\ &\quad + \left([\alpha(n)]^{-1} r^{-n} \int_{\mathbf{U}(b, r)} |\varphi_i - \varphi_i(b)|^s d\mathcal{L}^n \right)^{1/s} \\ &\quad + \left([\alpha(n)]^{-1} r^{-n} \int_{\mathbf{U}(b, r)} ||f - \varphi_i| - \tau|^s d\mathcal{L}^n \right)^{1/s} + 2^{-i}. \end{aligned}$$

As r tends to 0, the first and third of these $\mathcal{L}^n$ means over $\mathbf{U}(b, r)$ approach 0 by virtue of §8(3), since $b \notin M$; the second mean approaches 0 because φ_i is continuous. Consequently

$$\limsup_{r \downarrow 0} \sup \{|t - \varphi_i(b)| : t \in \text{med}(f, b, r)\} \leq 2^{-i} \quad \text{for } b \in \mathbf{R}^n \sim (M \cup N_i).$$

It follows that $|\varphi_i(b) - \varphi_i(b)| \leq 2^{-i} + 2^{-i}$ for $b \in \mathbf{R}^n \sim (M \cup N_i \cup N_i)$.

For each positive integer k we consider the set

$$E_k = M \cup \bigcup_{i=k}^{\infty} N_i \quad \text{with} \quad \Gamma_p(E_k) \leq \sum_{i=k}^{\infty} 2^{-i} = 2^{1-k},$$

and infer for each $b \in \mathbf{R}^n \sim E_k$ the existence of

$$g(b) = \lim_{i \rightarrow \infty} \varphi_i(b) \in \mathbf{R}$$

with $\limsup_{r \downarrow 0} \{|t - g(b)| : t \in \text{med}(f, b, r)\} = 0$; since $b \notin M$, it follows from §8(3)

that

$$\lim_{r \downarrow 0} r^{-n} \int_{\mathbf{U}(b, r)} |f - g(b)|^s d\mathcal{L}^n = 0.$$

Now classical differentiation theory implies that

$$g(b) = f(b) \text{ for } \mathcal{L}^n \text{ almost all } b \text{ in } \mathbf{R}^n \sim E_k.$$

Finally we define

$$E = \bigcap_{k=1}^{\infty} E_k \text{ with } \Gamma_p(E) = 0, \text{ hence } \mathcal{L}^n(E) = 0,$$

and conclude that the domain of g contains $\mathbf{R}^n \sim E$, $g(x) = f(x)$ for $\mathcal{L}^n$ almost all x , $g \in \mathcal{Y}_p(\mathbf{R}^n)$ and the exceptional set of degree s for g is contained in E .

We also note that, since $\sup \{|\varphi_i(b) - g(b)| : b \in \mathbf{R}^n \sim E_k\} \leq 2^{-i}$ when $i \geq k$, the restriction $g|_{(\mathbf{R}^n \sim E_k)}$ is continuous for each k .

10. Quasicontinuity. We say that a function f on $\mathbf{R}^n$ is Γ_p quasicontinuous if and only if for every positive number θ there exists a closed subset C of $\mathbf{R}^n$ such that $f|_C$ is continuous and $\Gamma_p(\mathbf{R}^n \sim C) < \theta$. The utility of this concept is due to the proposition:

If $f \in \mathcal{Y}_p(\mathbf{R}^n)$, f is Γ_p quasicontinuous and $0 < \epsilon < \infty$, then

$$\Gamma_p\{x: f(x) \geq \epsilon\} \leq \epsilon^{-p} \int_{\{x: f(x) > 0\}} |\nabla f|^p d\mathcal{L}^n.$$

To prove this proposition we suppose $0 < \theta < \epsilon$, choose C as above, select $h \in \mathcal{Y}_p^+(\mathbf{R}^n)$ with $\mathbf{R}^n \sim C \subset \{x: h(x) \geq 1\}$ and $\int |\nabla h|^p d\mathcal{L}^n < \theta$, infer that

$$k = \sup \{(\epsilon - \theta)^{-1} f^+, h\} \in \mathcal{Y}_p^+(\mathbf{R}^n),$$

$$U = [C \cap \{x: f(x) > \epsilon - \theta\}] \cup [\mathbf{R}^n \sim C] \text{ is open in } \mathbf{R}^n,$$

$$\{x: f(x) \geq \epsilon\} \subset U \subset \{x: k(x) \geq 1\},$$

$$\begin{aligned} \Gamma_p\{x: f(x) \geq \epsilon\} &\leq \int |\nabla k|^p d\mathcal{L}^n \\ &\leq \int [\sup \{(\epsilon - \theta)^{-1} |\nabla f^+|, |\nabla h|\}]^p d\mathcal{L}^n \\ &\leq (\epsilon - \theta)^{-p} \int_{\{x: f(x) > 0\}} |\nabla f|^p d\mathcal{L}^n + \theta, \end{aligned}$$

and let θ approach 0. Replacing f by $f - g$ we deduce the corollary:

If f and g are Γ_p quasicontinuous functions in $\mathcal{Y}_p(\mathbf{R}^n)$, then

$$\mathcal{L}^n\{x: f(x) > g(x)\} = 0 \text{ implies } \Gamma_p\{x: f(x) > g(x)\} = 0.$$

Next we establish a characterization of quasicontinuity by local behavior. For each $f \in \mathcal{Y}_p(\mathbf{R}^n)$ the following three properties are equivalent:

- (1) f is Γ_p quasicontinuous.
- (2) f is $\mathcal{L}^n$ approximately continuous Γ_p almost everywhere.
- (3) The exceptional set of degree s for f has Γ_p measure 0.

Clearly (3) implies (2). To prove that (2) implies (1), and that (1) implies (3), we will use the function g constructed in §9, observing that g is Γ_p quasicontinuous. If (2) holds, then $f(x) = g(x)$ for Γ_p almost all x , hence (1) holds. If (1) holds the above corollary shows that $f(x) = g(x)$ for Γ_p almost all x , hence (3) holds.

There is a useful connection between ν convergence and Γ_p almost everywhere convergence. In the special case of a sequence of Γ_p quasicontinuous functions $f_i \in \mathcal{Y}_p(\mathbf{R}^n)$ for which

$$\sum_{i=1}^{\infty} [2^i \nu(f_{i+1} - f_i)]^p < \infty$$

we consider the set $H_j = \bigcup_{i=j}^{\infty} \{x: |f_{i+1}(x) - f_i(x)| \geq 2^{-i}\}$ with

$$\Gamma_p(H_j) \leq \sum_{i=j}^{\infty} 2^{ip} \int |\nabla(f_{i+1} - f_i)|^p d\mathcal{L}^n \downarrow 0 \quad \text{as } j \uparrow \infty,$$

and infer that $g(x) = \lim_{i \rightarrow \infty} f_i(x) \in \mathbf{R}$ for $x \in \mathbf{R}^n \sim \bigcap_{i=1}^{\infty} H_i$. For each j the convergence is uniform on H_j , hence $g|_{H_j}$ is continuous. Consequently g is Γ_p quasicontinuous. The functions f_i converge to g both Γ_p almost everywhere and with respect to ν , as i approaches ∞ . From this special result one readily deduces the following proposition:

Every ν convergent sequence of Γ_p quasicontinuous functions in $\mathcal{Y}_p(\mathbf{R}^n)$ has a subsequence which converges Γ_p almost everywhere to a Γ_p quasicontinuous function in $\mathcal{Y}_p(\mathbf{R}^n)$.

We infer that in case $p > 1$ the minimizing sequences which arise from the definition of $\Gamma_p(A)$ converge to an extremal.

If $p > 1$, $A \subset \mathbf{R}^n$ and $\Gamma_p(A) < \infty$, then there exists a Γ_p quasicontinuous function $g \in \mathcal{Y}_p^+(\mathbf{R}^n)$ such that

$$\Gamma_p(A) = \int |\nabla g|^p d\mathcal{L}^n \text{ and } g(x) \geq 1 \text{ for } \Gamma_p \text{ almost all } x \text{ in } A.$$

Any two such functions agree Γ_p almost everywhere.

In fact $\Gamma_p(A) = \inf \{ \int |\nabla f|^p d\mathcal{L}^n : f \in \Phi \}$, where Φ consists of all Γ_p quasicontinuous functions $f \in \mathcal{Y}_p^+(\mathbf{R}^n)$ such that $f(x) \geq 1$ for Γ_p almost all x in A . The criterion at the end of §2 shows that every sequence of functions $f_i \in \Phi$ satisfying the condition

$$\Gamma_p(A) = \lim_{i \rightarrow \infty} \int |\nabla f_i|^p d\mathcal{L}^n$$

is ν convergent, because $(f_i + f_i)/2 \in \Phi$, hence

$$[\Gamma_p(A)]^{1/p} \leq \nu[(f_i + f_i)/2] \leq [\nu(f_i) + \nu(f_i)]/2 \rightarrow [\Gamma_p(A)]^{1/p}$$

as i and j approach ∞ . Then a suitable function g is obtained as the limit of a Γ_p almost everywhere convergent subsequence. Any two such functions have ν distance 0, hence agree $\mathcal{L}^n$ and Γ_p almost everywhere.

If also $B \subset A$ and h is a Γ_p quasicontinuous function in $\mathcal{Y}_p^+(\mathbf{R}^n)$ such that $\Gamma_p(B) = \int |\nabla h|^p d\mathcal{L}^n$ and $h(x) \geq 1$ for Γ_p almost all x in B , then $h(x) \leq g(x)$ for Γ_p almost all x . Considering the Γ_p quasicontinuous functions $u = \sup \{g, h\}$ and $v = \inf \{g, h\}$ we find that

$$\Gamma_p\{x: x \in A, u(x) < 1\} = 0, \quad \Gamma_p\{x: x \in B, v(x) < 1\} = 0,$$

$$\Gamma_p\{x: u(x) \neq x\} = 0 \quad \text{or} \quad \Gamma_p(A) < \int |\nabla u|^p d\mathcal{L}^n,$$

$$\Gamma_p\{x: v(x) \neq h(x)\} = 0 \quad \text{or} \quad \Gamma_p(B) < \int |\nabla v|^p d\mathcal{L}^n,$$

and the second alternatives are excluded by the fact that

$$\Gamma_p(A) + \Gamma_p(B) = \int (|\nabla g|^p + |\nabla h|^p) d\mathcal{L}^n = \int (|\nabla u|^p + |\nabla v|^p) d\mathcal{L}^n.$$

In case $p = 1$ extremals need not exist. (See §4.)

For each open subset Ω of $\mathbf{R}^n$ we define

$$\mathcal{Y}_p(\Omega) = \text{the } \nu \text{ closure of } \mathcal{D}^0(\mathbf{R}^n) \cap \{\varphi: \text{spt } \varphi \subset \Omega\} \text{ in } \mathcal{Y}_p(\mathbf{R}^n).$$

We will show that a Γ_p quasicontinuous function f in $\mathcal{Y}_p(\mathbf{R}^n)$ belongs to $\mathcal{Y}_p(\Omega)$ if and only if $f(x) = 0$ for Γ_p almost all x in $\mathbf{R}^n \sim \Omega$. If $f \in \mathcal{Y}_p(\Omega)$, then there exists a sequence of functions $\varphi_i \in \mathcal{D}^0(\mathbf{R}^n)$ with $\text{spt } \varphi_i \subset \Omega$ which converges Γ_p almost everywhere to f , hence $f(x) = 0$ for Γ_p almost all x in $\mathbf{R}^n \sim \Omega$. To prove the converse in case f is nonnegative we choose δ_i for each positive integer i so that $0 < \delta_i < (c_1 i^{-1})^*$ and

$$\int_S |\nabla f|^p d\mathcal{L}^n < i^{-p}$$

whenever S is an $\mathcal{L}^n$ measurable set with $\mathcal{L}^n(S) \leq \delta_i$, select a closed set C_i such that $f|_{C_i}$ is continuous,

$$\{x: f(x) > 0\} \sim \Omega \subset \mathbf{R}^n \sim C_i, \quad \Gamma_p(\mathbf{R}^n \sim C_i) < (ic_1)^{-p}(\delta_i)^{p/s},$$

take $g_i \in \mathcal{Y}_p(\mathbf{R}^n)$ so that $0 \leq g_i(x) \leq 1$ for $x \in \mathbf{R}^n$,

$$\mathbf{R}^n \sim C_i \subset \{x: g_i(x) = 1\}, \quad \nu(g_i) < (ic_1)^{-1}(\delta_i)^{1/s} < i^{-2},$$

let $T_i = \{x: g_i(x) \geq i^{-1}\}$ and define

$$h_i = \inf \{f, i\} - \inf \{f, i^{-1}\} \in \mathcal{Y}_p^+(\mathbf{R}^n).$$

We infer that $0 \leq h_i(x) \leq i$ and $|\nabla h_i(x)| \leq |\nabla f(x)|$ for $\mathcal{L}^n$ almost all x ,

$$[i^{-s}\mathcal{L}^n(T_i)]^{1/s} \leq \mu(g_i) \leq c_1\nu(g_i) \leq i^{-1}(\delta_i)^{1/s},$$

$$\mathcal{L}^n(T_i) \leq \delta_i, \quad \int_{T_i} |\nabla h_i|^p d\mathcal{L}^n \leq \int_{T_i} |\nabla f|^p d\mathcal{L}^n < i^{-p}.$$

Since g_i and h_i are bounded we find that $g_i h_i \in \mathcal{Y}_p^+(\mathbf{R}^n)$ with

$$\nabla(g_i h_i)(x) = h_i(x)\nabla g_i(x) + g_i(x)\nabla h_i(x) \text{ for } \mathcal{L}^n \text{ almost all } x,$$

$$\nu(g_i h_i) \leq i\nu(g_i) + i^{-1}\nu(h_i) + \left(\int_{T_i} |\nabla h_i|^p d\mathcal{L}^n\right)^{1/p}$$

$$\leq i^{-1} + i^{-1}\nu(f) + i^{-1} = i^{-1}[2 + \nu(f)],$$

$$\{x: [1 - g_i(x)]h_i(x) \neq 0\} \subset C_i \cap \{x: f(x) \geq 1/i\} \subset \Omega,$$

hence $\text{spt}(h_i - g_i h_i) \subset \Omega$ and the approximation method of §2 (multiplication by ψ_i and regularization) shows that $h_i - g_i h_i$ belongs to $\mathcal{Y}_p(\Omega)$. Finally we observe that

$$\nu(h_i - f) = \left(\int_{\{x: f(x) > i \text{ or } f(x) < 1/i\}} |\nabla f|^p d\mathcal{L}^n\right)^{1/p},$$

$$\nu(h_i - g_i h_i - f) \leq \nu(h_i - f) + \nu(g_i h_i) \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

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