Nerve Axon Equations: II Stability at Rest

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1. Introduction. In a previous paper [3] the behavior of solutions $W(x, t)$ to equations of the form

$$(1) \quad \frac{\partial W}{\partial t} = \begin{bmatrix} \frac{\partial^2 W^0}{\partial x^2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} g^0(W) \\ \vdots \\ g^n(W) \end{bmatrix}$$

is studied. Equations of this type arise in neurophysiology in the study of the nerve impulse. The reader is referred to [3] and the references in that paper for background.

Two features required by physiological considerations in a system such as (1) are one, a resting state where $g^0 = \cdots = g^n = 0$ which we take to be at $W = 0$ and two, a nonzero solution to (1) of the form $\phi(x - vt)$ which represents a nerve impulse traveling with velocity v . If $\phi(x - vt)$ is a solution of (1), then $\phi(y)$ satisfies the ordinary differential equation

$$(2) \quad -v \frac{d\phi}{dy} - \begin{bmatrix} \frac{d^2 \phi^0}{dy^2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} g^0(\phi) \\ \vdots \\ g^n(\phi) \end{bmatrix}.$$

The linearization of (1) about the resting state $W = 0$ is given by

$$(3) \quad \frac{\partial W}{\partial t} = \begin{bmatrix} \frac{\partial^2 W^0}{\partial x^2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} a & r \\ c & B \end{bmatrix} W$$

where B is an $n \times n$ matrix, $\begin{pmatrix} a & r \\ c & B \end{pmatrix} = (\partial g^i / \partial W^i)_{i,i=0}^*$ and W of course is the transpose of $(W^0, \dots, W^n)$ and the linearization of (2) about $\phi = 0$ is conveniently given in the form

$$(4) \quad \frac{d}{dy} \begin{pmatrix} \frac{d\phi^0}{dy} \\ \phi \end{pmatrix} = \begin{pmatrix} -v & -a & -r \\ 1 & 0 & 0 \\ 0 & -\frac{1}{v}c & -\frac{1}{v}B \end{pmatrix} \begin{pmatrix} \frac{d\phi^0}{dy} \\ \phi \end{pmatrix}$$

for $v \neq 0$.

In this paper we study the stability of (3) and (4). When combined with the results of [3] we achieve a characterization of the stability at rest of (1) as defined in [3] in terms of properties of the matrix $\begin{pmatrix} a & r \\ c & B \end{pmatrix}$. Also, asymptotic properties of solutions $\phi(y)$ to (2) which tend to the resting state as y tends to infinity are derived. All of the results obtained here are used in an essential way in a paper following this one when the linearization of (1) about a non zero ϕ is treated as an operator perturbation of (3) in a study of the stability of the nerve impulse.

In Section 2 some preliminaries and definitions are given and in Section 3 the main results are stated. Some mathematical background is given in Section 4 and in Section 5 the main results are proved.

2. Preliminaries and definitions. For $\hat{W}(\theta, t)$ the transpose of $(\hat{W}^0(\theta, t), \dots, \hat{W}^n(\theta, t))$ where $\hat{W}^i(\theta, t) = \int_{-\infty}^{\infty} e^{-i\theta x} W^i(x, t) dx$ is the Fourier transform in x of $W^i(x, t)$ when this integral exists we have formally that

$$(5) \quad \frac{\partial \hat{W}}{\partial t} = \begin{pmatrix} a - \theta^2 & r \\ c & B \end{pmatrix} \hat{W}$$

if W is a solution of (3).

In agreement with the preceding paper we say that a system such as (3) (or (5)) is exponentially stable at rest if there are constants $P, \alpha > 0$ such that if W is a solution of (3) (or (5)) $\|W(\cdot, t)\|_{\infty} \leq P \|W(\cdot, 0)\|_{\infty} e^{-\alpha t}$ for $t \geq 0$. Here $W(\cdot, t)$ is that function whose value at x is $W(x, t)$ and $\|W(\cdot, t)\|_{\infty}$ is the maximum for $i = 0, \dots, n$ of the least upper bound in x of $|W^i(x, t)|$.

We make no distinction between real and complex solutions to the linear systems (3) and (5).

3. Main results. The main results consist of two theorems.

Theorem 1. (i). *The linearized system (3) is exponentially stable at rest if and only if the system (5) given by the Fourier transform is exponentially stable at rest.*

(ii). *The system (5) given by the Fourier transform of (3) is exponentially stable*

at rest if and only if there is a negative least upper bound for real θ for the real parts of the eigenvalues of $\begin{pmatrix} a-\theta & r \\ c & B \end{pmatrix}$.

Theorem 2. *If the linear system (3) is exponentially stable at rest, then the matrix*

$$\begin{pmatrix} -v & -a & -r \\ 1 & 0 & 0 \\ 0 & -\frac{1}{v}c & -\frac{1}{v}B \end{pmatrix}$$

appearing in (4) has one negative eigenvalue and $n + 1$ eigenvalues with positive real parts for all $v > 0$.

Corollary. *If the system (1) is exponentially stable, then for each fixed $v > 0$ in (2) there are two solutions $\phi^+(y)$ and $\phi^-(y)$ to (2) which tend to zero as y goes to infinity such that any solution $\phi(y)$ to (2) for this fixed v tending to zero as y goes to infinity is equal for some real η to either $\phi^+(y - \eta)$ or $\phi^-(y - \eta)$ for all y .*

The first theorem with the results of the previous paper [3] characterizes the axon equations which are exponentially stable at rest. We note that the Hodgkin-Huxley equations are exponentially stable at rest. See the study of the linearized Hodgkin-Huxley equations in [7].

Theorem 2 and the corollary show that there are only two possible candidates for a nerve impulse at a fixed v . These have the same exponential asymptotic behavior as y tends to infinity but approach zero from opposite directions. This uniqueness of asymptotic behavior was used in [2] and [5] in the numerical search for solutions to (2) in the case of the Hodgkin-Huxley equations.

4. Mathematical background. The existence and uniqueness of solutions of (1) (and consequently of (3)) is given in [4]. We will use the fact that if $W_0(x, t)$, $W_1(x, t)$, $\dots$ are solutions of (3) with $\|W_k(\cdot, 0)\|_\infty \leq M < \infty$ for $k = 0, 1, \dots$ and with the property that $W_k(x, 0)$ converges uniformly to $W_0(x, 0)$ for x in any bounded set as k goes to infinity, then $W_k(x, t)$ converges to $W_0(x, t)$ for $0 \leq t$ and all x . This is a consequence of Theorem 3 of [4].

We also use the bound $\|W(\cdot, t)\|_\infty \leq e^{Lt} \|W(\cdot, 0)\|_\infty$ for W a solution to (3) given in [3] where $L/(n + 1)$ is the maximum of $|\partial g^i / \partial \phi^j(0)|$ for $0 \leq i, j \leq n$.

For a vector W we denote by $\|W\|$ the maximum of the magnitudes of the entries, and for a matrix A we set $\|A\|$ equal to the least upper bound of $\|AW\|$ where W is any vector of appropriate dimension with $\|W\| = 1$ so that $\|AD\| \leq \|A\| \|D\|$ for matrices A and D of appropriate dimension. For a square matrix A we denote by A^{-1} the inverse of A if it exists, by $\det A$ the determinant of A and we define the exponential of the matrix e^A by $e^A = I + A + A^2/2! + \dots$. We use the fact that for a square matrix A , if z is complex number with

$$|z| > \|A\|, \text{ then } (zI - A)^{-1} = \frac{1}{z} \left(I + \frac{1}{z} A + \frac{1}{z^2} A^2 + \cdots \right)$$

where I is the identity matrix of appropriate dimension. For a differentiable curve Γ in the complex plane which surrounds the eigenvalues of a square matrix A we use the fact that

$$e^{tA} = \frac{1}{2\pi i} \int_{\Gamma} e^{tz} (zI - A)^{-1} dz$$

where integrals (or derivatives) of a matrix are always entry by entry integrals (or derivatives). We note also that the system of constant coefficient ordinary differential equations $(dy/dt)(t) = Ay(t)$ has as its unique solution $y(t) = e^{tA}y(0)$. This standard material on matrices is found in [6].

We make repeated use of the fact that

$$\frac{\partial}{\partial y} \int_{-\infty}^{\infty} A(x, y) dx = \int_{-\infty}^{\infty} \frac{\partial A}{\partial y}(x, y) dx$$

if $A(x, y)$ and $(\partial A/\partial y)(x, y)$ are matrices of integrable functions in x with $\|(\partial A/\partial y)(x, u)\| \leq h(x)$ for all u in some interval about y and all x where $h(x)$ is integrable. This can be verified directly by use of the mean value theorem and the dominated convergence theorem.

If D is a matrix of integrable functions of x , then the Fourier transform $\hat{D}$ of this matrix is defined by

$$\hat{D}(\theta) = \int_{-\infty}^{\infty} e^{-i\theta x} D(x) dx$$

and the inverse Fourier transform $\check{D}$ by

$$\check{D}(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\theta x} D(x) dx.$$

If $D, \partial D/\partial x, \dots, \widehat{\partial^m D/\partial x^m}$ are matrices of integrable functions of x , then it can be verified directly using integration by parts that $(\partial^m D/\partial x^m)(\theta) = (i\theta)^m \hat{D}(\theta)$. We distinguish a class S of infinitely differentiable functions with the property that the least upper bound for f in S of $|x|^k |(d^j f/dx^j)(x)|$ is finite for $j, k \geq 0$. For f in S we have that $\check{f}$ is in S and that $\check{\check{f}} = f$. This material on Fourier transforms is found in chapter 1 of [8].

5. Proof of the theorems.

Proof of Theorem 1. The proof of the theorem is based on the following propositions which characterize the matrix $e^{tA(\theta^2)}$ and solutions to (3) where

$$A(\theta^2) = \begin{bmatrix} a - \theta^2 & r \\ c & B \end{bmatrix}$$

is the matrix appearing in the theorem. We state the propositions, give a proof of the theorem using the propositions and then prove the propositions.

Proposition 1. *A complex number λ is an eigenvalue of $A(\theta^2)$ if and only if $(\alpha_0 - \lambda) \cdots (\alpha_n - \lambda) = \theta^2(\beta_1 - \lambda) \cdots (\beta_n - \lambda)$ where $\alpha_0, \dots, \alpha_n$ are the eigenvalues of $A(0)$ and $\beta_1, \dots, \beta_n$ are the eigenvalues of B .*

Proposition 2. *For θ^2 sufficiently large there is an eigenvalue $\lambda(\theta^2)$ of $A(\theta^2)$ with the property that $1/z\lambda(1/z)$ can be extended to a complex analytic function in a neighborhood of $z = 0$ with value -1 at $z = 0$. All other eigenvalues of $A(\theta^2)$ have as a limit one of the $\beta_1, \dots, \beta_n$ as θ^2 tends to infinity.*

Proposition 3. *For $-\alpha$ greater than an upper bound for real θ of the real parts of the eigenvalues of $A(\theta^2)$ and for $\delta > 0$ there are constants $M_{i,j}^\delta$ for $i, j \geq 0$ such that*

$$\left\| \frac{\partial^{i+j}}{\partial \theta^i \partial t^j} F(\theta, t) \right\| \leq M_{i,j}^\delta \left(\frac{1}{1 + \theta^2} \right) e^{-\alpha t}$$

for $t \geq \delta$ and all real θ where

$$F(\theta, t) = e^{tA(\theta^2)} - \begin{bmatrix} 0 & \cdots \\ \vdots & \\ \vdots & e^{tB} \end{bmatrix}.$$

Also, $F(\theta, t)$ is uniformly bounded by a constant K_1 for $0 \leq t \leq 1$ and θ real.

Proposition 4. *For any solution W to (3) with initial conditions $W(x, 0) = \phi(x)$ where $\|\phi(x)\|_\infty$ is finite, the representation*

$$W(x, t) = \int_{-\infty}^{\infty} \tilde{F}(x - y, t) \phi(y) dy + \begin{bmatrix} 0 & \cdots \\ \vdots & \\ \vdots & e^{tB} \end{bmatrix} \phi(x)$$

holds, where $\tilde{F}$ is the inverse Fourier transform in θ of F .

Proof of (ii). If $-\alpha < 0$ is greater than an upper bound in θ of the real parts of the eigenvalues of $A(\theta^2)$ for θ real, we have by Proposition 2 that $-\alpha$ is greater than the real parts of the eigenvalues of B so there exists a constant K_2 such that $\|e^{tB}\| \leq K_2 e^{-\alpha t}$ for $t \geq 0$. This bound along with Proposition 3 easily gives exponential stability for (5) with constants P, α where P is the maximum of $K_2 + M_{00}^1$ and $K_1 e^\alpha + K_2$. We recall that solutions to (5) with initial conditions $\psi(\theta)$ are given by

$$e^{tA(\theta^2)} \psi(\theta) = F(\theta, t) \psi(\theta) + \begin{bmatrix} 0 & \cdots \\ \vdots & \\ \vdots & e^{tB} \end{bmatrix} \psi(\theta).$$

The condition stated in (ii) is clearly necessary for the exponential stability of (5) since if for some fixed θ it happens that $A(\theta^2) \hat{W}(\theta, 0) = \lambda \hat{W}(\theta, 0)$ where $\hat{W}(\theta, 0) \neq 0$, then $\hat{W}(\theta, t) = e^{\lambda t} \hat{W}(\theta, 0)$ for $t \geq 0$ in (5).

Proof of (i). If (5) is exponentially stable, then by (ii) and Proposition 3 there is an $\alpha > 0$ with

$$||F(\theta, t)|| \leq \frac{M_{00}^1}{1 + \theta^2} e^{-\alpha t}, \quad \left\| \frac{\partial F}{\partial \theta}(\theta, t) \right\| \leq \frac{M_{10}^1}{1 + \theta^2} e^{-\alpha t}, \dots$$

for $t \geq 1$ so that

$$||\tilde{F}(x, t)|| \leq \frac{e^{-\alpha t}}{2\pi} \int_{-\infty}^{\infty} \frac{M_{00}^1}{1 + \theta^2} d\theta$$

and

$$\left\| \frac{\partial^2 \tilde{F}}{\partial \theta^2}(x, t) \right\| = ||x^2 \tilde{F}(x, t)|| \leq \frac{e^{-\alpha t}}{2\pi} \int_{-\infty}^{\infty} \frac{M_{20}^1}{1 + \theta^2} d\theta$$

for all x . Combining these two bounds we have that $(1 + x^2) ||\tilde{F}(x, t)|| \leq K_3 e^{-\alpha t}$ where

$$K_1 = \frac{(M_{00}^1 + M_{20}^1)}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta}{1 + \theta^2}.$$

Again for some constant K_2 , $||e^{tB}|| \leq K_2 e^{-\alpha t}$ because $-\alpha$ is greater than the real parts of the eigenvalues of B so it follows from Proposition 4 that if W is a solution to (3) with initial conditions $||W(x, 0)|| \leq 1$, then for all x and for $t \geq 1$

$$||W(x, t)|| \leq e^{-\alpha t} \int_{-\infty}^{\infty} \frac{K_3}{1 + y^2} dy + K_2 e^{-\alpha t} = e^{-\alpha t} \left[\int_{-\infty}^{\infty} \frac{K_3}{1 + y^2} dy + K_2 \right].$$

Now since $||W(x, t)|| \leq e^{Lt}$ for $0 \leq t \leq 1$ by the remarks in Section 4 we see that (3) is exponentially stable with constants P, α where P is the maximum of $e^{L+\alpha}$ and $\int_{-\infty}^{\infty} (K_3/(1 + y^2)) dy + K_2$.

If (5) is not exponentially stable, then by (ii) for any $\alpha > 0$ there is $a - \alpha < \text{Real } \lambda$ such that $A(\theta^2)\psi = \lambda\psi$ for some fixed θ and some vector ψ with $||\psi|| = 1$. In this case $W(x, t) = e^{i\theta x + \lambda t}\psi$ is a solution of (3) so that (3) cannot be exponentially stable.

We now give the proofs of the propositions.

Proof of Proposition 1. A complex number λ is an eigenvalue of $A(\theta^2)$ if and only if

$$\begin{aligned} 0 &= \det \begin{pmatrix} a - \theta^2 - \lambda & r \\ c & B - \lambda I \end{pmatrix} = \det \begin{pmatrix} a - \lambda & r \\ c & B - \lambda I \end{pmatrix} - \theta^2 \det(B - \lambda I) \\ &= (\alpha_0 - \lambda) \cdots (\alpha_n - \lambda) - \theta^2 (\beta_1 - \lambda) \cdots (\beta_n - \lambda). \end{aligned}$$

Proof of Proposition 2. We define $z(\xi)$ a complex function of a complex variable by

$$z(\xi) = \frac{\xi(\xi\beta_1 - 1) \cdots (\xi\beta_n - 1)}{(\xi\alpha_0 - 1) \cdots (\xi\alpha_n - 1)}$$

so that $z(0) = 0$ and $(dz/d\xi)(0) = -1$. In a neighborhood of zero there is an inverse function $f(z)$ such that $f(z(\xi)) = \xi$, $z(f(\xi)) = \xi$ with $f(0) = 0$ and

$$\frac{df}{dz}(0) = \frac{1}{\frac{dz}{d\xi}(0)} = -1.$$

We set $\lambda(\theta^2) = 1/f(1/\theta^2)$ for θ^2 sufficiently large so that $1/z\lambda(1/z) = f(z)/z$ has the properties stated in the proposition. We have

$$\begin{aligned} \theta^2 &= \frac{1}{z\left(f\left(\frac{1}{\theta^2}\right)\right)} = \frac{\left(\frac{1}{\lambda(\theta^2)}\alpha_0 - 1\right) \cdots \left(\frac{1}{\lambda(\theta^2)}\alpha_n - 1\right)}{\frac{1}{\lambda(\theta^2)}\left(\frac{1}{\lambda(\theta^2)}\beta_1 - 1\right) \cdots \left(\frac{1}{\lambda(\theta^2)}\beta_n - 1\right)} \\ &= \frac{(\alpha_0 - \lambda(\theta^2)) \cdots (\alpha_n - \lambda(\theta^2))}{(\beta_1 - \lambda(\theta^2)) \cdots (\beta_n - \lambda(\theta^2))} \end{aligned}$$

and $\lambda(\theta^2)$ is an eigenvalue of $A(\theta^2)$ by Proposition 1.

The last statement in Proposition 2 can be shown by several standard methods in complex analysis. For example

$$\frac{\frac{d}{d\lambda} \det(A(\theta^2) - \lambda I)}{\det(A(\theta^2) - \lambda I)} = \frac{-\sum_{i=0}^n \prod_{j \neq i} (\alpha_j - \lambda) + \theta^2 \sum_{i=1}^n \prod_{j \neq i} (\beta_j - \lambda)}{\prod_{i=0}^n (\alpha_i - \lambda) - \theta^2 \prod_{i=1}^n (\beta_i - \lambda)}$$

and the limit of this expression as θ^2 goes to infinity is $\sum_{i=1}^n 1/(\lambda - \beta_i)$ provided $\lambda \neq \beta_1, \dots, \beta_n$. Thus the winding number of $\det(A(\theta^2) - \lambda I)$ about the origin for θ^2 sufficiently large and for λ on a small circle about one of the β_i is exactly the multiplicity of the eigenvalue β_i giving this many solutions to $\det(A(\theta^2) - \lambda I) = 0$ counting multiplicities near β_i for $i = 1, \dots, n$.

Proof of Proposition 3. We state and prove four lemmas. Some of the lemmas involve a complex line integral with domain Γ which we now define. By Proposition 2 there is a finite $\beta > 0$ which is greater than the magnitudes of the imaginary parts of the eigenvalues of $A(\theta^2)$ for all real θ . For α as in Proposition 3 and any $r > 0$, let Γ_r be the oriented curve consisting of straight line segments from $(-\alpha, -\beta)$ to $(-\alpha, \beta)$ to $(-\alpha - r, \beta + r)$ to $(-\alpha - r, -\beta - r)$ and back to $(-\alpha, -\beta)$. We define Γ as the limit of the curve Γ_r as r goes to infinity.

Lemma 1. For θ^2 sufficiently large $A(\theta^2) = \lambda(\theta^2)P(\theta^2) + Q(\theta^2)$ where $P(\theta^2)$ is of rank one

$$(6) \quad P(\theta^2)P(\theta^2) = P(\theta^2),$$

$$(7) \quad A(\theta^2)P(\theta^2) = P(\theta^2)A(\theta^2) = \lambda(\theta^2)P(\theta^2)$$

and $P(1/z)$ and $Q(1/z)$ can be extended to complex analytic functions in a neighborhood of $z = 0$ with

$$(8) \quad P\left(\frac{1}{z}\right)\Big|_{z=0} = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & & \\ \vdots & & \end{bmatrix}$$

and

$$(9) \quad Q\left(\frac{1}{z}\right)\Big|_{z=0} = \begin{bmatrix} 0 & \cdots \\ \vdots & \\ \vdots & B \end{bmatrix}.$$

Lemma 2. For $z \neq 0$ not an eigenvalue of $A(\theta^2)$

$$(zI - A(\theta^2))^{-1} = \frac{1}{z - \lambda(\theta^2)} P(\theta^2) + (I - P(\theta^2))(zI - Q(\theta^2))^{-1}.$$

Lemma 3. There exist constants $M_1, M_2, \dots$ such that

$$\left\| \frac{\partial^j}{\partial \theta^j} (zI - A(\theta^2))^{-1} \right\| \leq \frac{M_j}{1 + \theta^2}, \quad j = 1, 2, \dots$$

for z on Γ and all real θ .

Lemma 4.

$$\frac{\partial^{i+j}}{\partial t^i \partial \theta^j} F(\theta, t) = \frac{1}{2\pi i} \int_{\Gamma} z^i e^{tz} \frac{\partial^j}{\partial \theta^j} \cdot \left[(zI - A(\theta^2))^{-1} - \begin{bmatrix} 0 & \cdots \\ \vdots & (zI - B)^{-1} \end{bmatrix} \right] dz \quad \text{for } i, j \geq 0.$$

Proof of Lemma 1. For θ^2 sufficiently large there is a vector $\begin{pmatrix} b \\ d \end{pmatrix} \neq 0$ such that

$$\begin{bmatrix} a - \theta^2 & r \\ c & B \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \lambda(\theta^2) \begin{bmatrix} b \\ d \end{bmatrix} \quad \text{or}$$

$$(10) \quad (a - \theta^2)b + rd = \lambda(\theta^2)b,$$

$$(11) \quad cb + Bd = \lambda(\theta^2)d.$$

Now $|\lambda(\theta^2)| > \|B\|$ for θ^2 sufficiently large and we have

$$(12) \quad (\lambda(\theta^2)I - B)^{-1} = \frac{1}{\lambda(\theta^2)} \left(I + \frac{1}{\lambda(\theta^2)} B + \left(\frac{1}{\lambda(\theta^2)} B \right)^2 + \cdots \right)$$

and from (11) that

$$(13) \quad d = b(\lambda(\theta^2)I - B)^{-1}c.$$

We take $b = 1$ so that

$$p(\theta^2) = \begin{bmatrix} 1 \\ (\lambda(\theta^2)I - B)^{-1}c \end{bmatrix}$$

is an eigenvector of $A(\theta^2)$ with eigenvalue $\lambda(\theta^2)$, and equation (10) becomes

$$(14) \quad a - \theta^2 = \lambda(\theta^2) - r(\lambda(\theta^2)I - B)^{-1}c.$$

Similarly the row vector $q(\theta^2) = (1, r(\lambda(\theta^2)I - B)^{-1})$ satisfies $q(\theta^2)A(\theta^2) = \lambda(\theta^2)q(\theta^2)$. We now define $P(\theta^2)$ by

$$(15) \quad \begin{aligned} P(\theta^2) &= \frac{p(\theta^2)q(\theta^2)}{q(\theta^2)p(\theta^2)} \\ &= \frac{1}{1 + r(\lambda(\theta^2)I - B)^{-2}c} \begin{bmatrix} 1 & r(\lambda(\theta^2)I - B)^{-1} \\ (\lambda(\theta^2)I - B)^{-1}c & (\lambda(\theta^2)I - B)^{-1}cr(\lambda(\theta^2)I - B)^{-1} \end{bmatrix} \end{aligned}$$

Obviously $P(\theta^2)$ is of rank 1 and satisfies (6) and (7). Substituting $1/z$ for θ^2 in the definition (15) and using (12) and Proposition 2 it is seen that (8) holds with $P(1/z)$ analytic near $z = 0$.

On setting $Q(\theta^2) = A(\theta^2) - \lambda(\theta^2)P(\theta^2)$ and using (14) we have that

$$\begin{aligned} Q(\theta^2) &= \begin{bmatrix} \lambda(\theta^2) - r(\lambda(\theta^2)I - B)^{-1}c & r \\ c & B \end{bmatrix} - \frac{1}{1 + r(\lambda(\theta^2)I - B)^{-2}c} \\ &\quad \cdot \begin{bmatrix} \lambda(\theta^2) & r\lambda(\theta^2)(\lambda(\theta^2)I - B)^{-1} \\ \lambda(\theta^2)(\lambda(\theta^2)I - B)^{-1}c & \lambda(\theta^2)(\lambda(\theta^2)I - B)^{-1}cr(\lambda(\theta^2)I - B)^{-1} \end{bmatrix} \end{aligned}$$

and again using (12) and Proposition 2 it is seen that (9) holds with $Q(1/z)$ analytic in a neighborhood of $z = 0$.

Proof of Lemma 2. The eigenvalues of $Q(\theta^2)$ are zero and the eigenvalues of $A(\theta^2)$ not equal to $\lambda(\theta^2)$ so $(zI - Q(\theta^2))^{-1}$ exists. Now $(zI - A(\theta^2))P(\theta^2) = (z - \lambda(\theta^2))P(\theta^2)$ and $(zI - A(\theta^2))(I - P(\theta^2)) = zI - zP(\theta^2) - Q(\theta^2) = (I - P(\theta^2))(zI - Q(\theta^2))$ since $P(\theta^2)Q(\theta^2) = 0$. Thus

$$(zI - A(\theta^2)) \left[\frac{1}{z - \lambda(\theta^2)} P(\theta^2) + (I - P(\theta^2))(zI - Q(\theta^2))^{-1} \right]$$

is $P(\theta^2) + (I - P(\theta^2))$ or I , and the lemma is proved.

Proof of Lemma 3. We have that

$$\frac{\partial}{\partial \theta} (zI - A(\theta^2))^{-1} = -(zI - A(\theta^2))^{-1} \frac{\partial A(\theta^2)}{\partial \theta} (zI - A(\theta^2))^{-1},$$

$$\frac{\partial A(\theta^2)}{\partial \theta} = \begin{bmatrix} -2\theta & 0 & \cdots \\ 0 \\ \vdots \end{bmatrix},$$

$$\frac{\partial^2 A(\theta^2)}{\partial \theta^2} = \begin{bmatrix} -2 & 0 & \cdots \\ 0 & & \\ \vdots & & \end{bmatrix} \quad \text{and} \quad \frac{\partial^j A(\theta^2)}{\partial \theta^j} = 0 \quad \text{for } j \geq 3.$$

Combining these facts we see by induction that

$$(16) \quad \frac{\partial^j}{\partial \theta^j} (zI - A(\theta^2))^{-1} = \sum_{\substack{i_1 + \cdots + i_k = j \\ i_1, \dots, i_k = 1, 2}} K_{i_1, \dots, i_k} (zI - A(\theta^2))^{-1} \cdot \prod_{m=1}^k \left[\begin{bmatrix} -2\theta^{(2-i_m^-)} & 0 & \cdots \\ 0 & & \\ \vdots & & \end{bmatrix} (zI - A(\theta^2))^{-1} \right]$$

for constants $K_{i_1, \dots, i_k}$ and $j = 1, 2, \dots$. We see from (16) that for θ restrained to a bounded set, $(\partial^j/\partial \theta^j)(zI - A(\theta^2))^{-1}$ goes to zero uniformly as z goes to infinity on Γ so the lemma is proved if it can be shown that the bounds in the lemma hold uniformly in z for large θ^2 . In the product on the right hand side of (16) only terms from the first row or first column of $(zI - A(\theta^2))^{-1}$ are multiplied by non-zero terms, and by the nature of (16) we see that the lemma is proved if it is shown that there is a constant N such that the magnitudes of terms in the first row and first column of $(zI - A(\theta^2))^{-1}$ are bound by $N/(1 + \theta^2)$ uniformly in z for large θ .

We assume that $\alpha \neq 0$ so that zero does not lie on Γ and use the representation of Lemma 3 for $(zI - A(\theta^2))^{-1}$ for large θ^2 . $P(\theta^2)$ and $(zI - Q(\theta^2))^{-1}$ are bounded as θ^2 goes to infinity uniformly in z , while $1/|z - \lambda(\theta^2)|$ and the first row and column of $I - P(\theta^2)$ are bound by a constant divided by $1 + \theta^2$ as θ^2 goes to infinity by the construction of Γ and the properties of $\lambda(\theta^2)$ and $P(\theta^2)$ given in Proposition 2 and in (8) respectively. Since $(I - P(\theta^2))(zI - Q(\theta^2))^{-1}$ is equal to $(zI - Q(\theta^2))^{-1}(I - P(\theta^2))$ this gives the desired bound on entries in the first row and column of $(zI - A(\theta^2))^{-1}$.

If $\alpha = 0$ the proof can be applied to $(zI - (A(\theta^2) + \gamma I))^{-1}$ for z on the translate of Γ by γ for γ sufficiently small.

Proof of Lemma 4. For r sufficiently large the curve Γ_r surrounds all eigenvalues of B and $A(\theta^2)$ for fixed θ so that

$$(17) \quad F(\theta, t) = \frac{1}{2\pi i} \int_{\Gamma_r} e^{tz} \left[(zI - A(\theta^2))^{-1} - \begin{bmatrix} 0 & \cdots \\ \vdots & (zI - B)^{-1} \end{bmatrix} \right] dz.$$

Repeated applications of the mean value theorem and the dominated convergence theorem justify taking differentiation under the integral and give

$$(18) \quad \frac{\partial^{i+j}}{\partial t^i \partial \theta^j} F(\theta, t) = \frac{1}{2\pi i} \int_{\Gamma} z^i e^{t\alpha} \frac{\partial^j}{\partial \theta^j} \left[(zI - A(\theta^2))^{-1} - \begin{bmatrix} 0 & \cdots \\ \vdots & (zI - B)^{-1} \end{bmatrix} \right] dz.$$

Now letting r go to infinity we see that the lemma holds since the integral over the line segment from $(-\alpha - r, \beta + r)$ to $(-\alpha - r, -\beta - r)$ goes to zero because of the exponential term $e^{t\alpha}$.

We now continue with the proof of the proposition. From Lemma 4

$$(19) \quad \left\| \frac{\partial^{i+j}}{\partial t^i \partial \theta^j} F(\theta, t) \right\| \leq \frac{e^{-t\alpha}}{2\pi i} \int_{\Gamma} |z^i| |e^{t(s+\alpha)}| \cdot \left\| \frac{\partial^j}{\partial \theta^j} \left[(zI - A(\theta^2))^{-1} - \begin{bmatrix} 0 & \cdots \\ \vdots & (zI - B)^{-1} \end{bmatrix} \right] \right\| |dz|.$$

If $j = 0$, then by (8), (9), Lemma 2 and the construction of Γ that there is a constant M_0 such that

$$\left\| (zI - A(\theta^2))^{-1} - \begin{bmatrix} 0 & \cdots \\ \vdots & (zI - B)^{-1} \end{bmatrix} \right\| \leq \frac{M_0}{1 + \theta^2}.$$

For $j \geq 1$ we may use Lemma 3 so that

$$(20) \quad \left\| \frac{\partial^{i+j}}{\partial t^i \partial \theta^j} F(\theta, t) \right\| \leq \frac{e^{-t\alpha} M_i}{2\pi(1 + \theta^2)} \int_{\Gamma} |z^i| |e^{t(s+\alpha)}| |dz|.$$

Since $|e^{t(s+\alpha)}| \leq |e^{\delta(s+\alpha)}|$ for z on Γ and $t \geq \delta$, we have the bound given in the proposition with

$$(21) \quad M_{i,j}^{\delta} = \frac{M_i}{2\pi} \int_{\Gamma} |z^i| |e^{\delta(s+\alpha)}| |dz| \quad \text{for } i, j \geq 0.$$

Finally $e^{tA(\theta^2)} = e^{t\lambda(\theta^2)} P(\theta^2) + (I - P(\theta^2)) e^{tQ(\theta^2)}$ for θ^2 sufficiently large by Lemma 2 and the second representations of the exponential of a matrix given in Section 4 so that $\|e^{tA(\theta^2)}\|$ has a finite upper bound for $0 \leq t \leq 1$ and all θ as claimed.

Proof of Proposition 4. The proof is given by the two following lemmas. We recall the definition of the class of functions S from Section 4.

Lemma 5. *The representation of solutions to (3) given in Proposition 4 holds for solutions with initial conditions whose entries are in S .*

Lemma 6. *If the representation in Proposition 4 holds for solutions to (3) with initial conditions whose entries are in S , then the representation holds for all solutions of (3) with bounded initial conditions.*

Proof of Lemma 5. Let $\psi(x)$ be an $n + 1$ column vector with entries in S as functions of x . We set

$$(22) \quad W(x, t) = \int_{-\infty}^{\infty} \tilde{F}(x - y, t) \psi(y) dy + \begin{bmatrix} 0 & \cdots \\ \vdots & \\ \vdots & e^{tB} \end{bmatrix} \psi(x)$$

for $t > 0$. We note that by Proposition 3, $\tilde{F}(x, t)$ is defined and bounded in x for $t > 0$. We need to show that W is the solution to (3) with initial value ψ . We have

$$(23) \quad \begin{aligned} \int_{-\infty}^{\infty} \tilde{F}(x - y, t) \psi(y) dy &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\theta(x-y)} F(\theta, t) \psi(y) d\theta dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\theta x} e^{tA(\theta^2)} \left[\int_{-\infty}^{\infty} e^{-iy\theta} \psi(y) dy \right] d\theta \\ &\quad - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\theta x} \begin{bmatrix} 0 & \cdots \\ \vdots & \\ \vdots & e^{tB} \end{bmatrix} \left[\int_{-\infty}^{\infty} e^{-iy\theta} \psi(y) dy \right] d\theta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\theta x} e^{tA(\theta^2)} \hat{\psi}(\theta) d\theta - \begin{bmatrix} 0 & \cdots \\ \vdots & \\ \vdots & e^{tB} \end{bmatrix} \int_{-\infty}^{\infty} \frac{e^{i\theta x}}{2\pi} \hat{\psi}(\theta) d\theta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\theta x} e^{tA(\theta^2)} \hat{\psi}(\theta) d\theta - \begin{bmatrix} 0 & \cdots \\ \vdots & \\ \vdots & e^{tB} \end{bmatrix} \psi(x) \end{aligned}$$

where we have used Fubini's theorem to change the order of integration. This is possible since $|e^{i\theta(x-y)} F(\theta, t) \psi(y)|$ is bounded by

$$\frac{e^{-\alpha t} M_{00}^s \psi(y)}{1 + \theta^2} \quad \text{for } t > \delta > 0.$$

Together (22) and (23) give

$$(24) \quad W(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\theta x} e^{tA(\theta^2)} \hat{\psi}(\theta) d\theta.$$

So that W is the inverse Fourier transform of $e^{tA(\theta^2)} \hat{\psi}(\theta)$. From

$$e^{tA(\theta^2)} = F(\theta, t) + \begin{bmatrix} 0 & \cdots \\ \vdots & \\ \vdots & e^{tB} \end{bmatrix}$$

and Proposition 3 it follows that $e^{tA(\theta^2)} \hat{\psi}(\theta)$ has entries in S as functions of θ and using the mean value theorem and the dominated convergence theorem we see that

$$(25) \quad \begin{pmatrix} \frac{\partial^2 W^0}{\partial x^2}(x, t) \\ 0 \\ \vdots \end{pmatrix} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\theta x} \begin{pmatrix} -\theta^2 & 0 & \cdots \\ 0 & & \\ \vdots & & \end{pmatrix} e^{tA(\theta^2)} \hat{\psi}(\theta) d\theta.$$

Also differentiation of $W(x, t)$ with respect to t can be carried under the integral in (24) by using the mean value and dominated convergence theorems and Proposition 3. This gives

$$(26) \quad \frac{\partial W}{\partial t}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\theta x} A(\theta^2) e^{tA(\theta^2)} \hat{\psi}(\theta) d\theta.$$

Taken together (25) and (26) show that W is a solution to (3). To obtain the initial conditions we note that

$$\begin{aligned} \lim_{t \rightarrow 0} W(x, t) &= \lim_{t \rightarrow 0} \int_{-\infty}^{\infty} \frac{e^{i\theta x}}{2\pi} e^{tA(\theta^2)} \hat{\psi}(\theta) d\theta \\ &= \int_{-\infty}^{\infty} \frac{e^{i\theta x}}{2\pi} \hat{\psi}(\theta) d\theta = \psi(x) \end{aligned}$$

using the dominated convergence theorem. This is justified by the bound on $e^{tA(\theta^2)}$ for $0 \leq t \leq 1$ and all θ given in Proposition 3. This completes the proof of the lemma since W is bounded by Proposition 3 and (24).

Proof of Lemma 6. Let W be a solution of (3) with initial values $W(x, 0) = \phi(x)$ where $\|\phi(x)\| \leq M$ for all x . We define for $m = 1, 2, \dots$

$$(27) \quad \phi_m(x) = \int_{-m}^m \frac{m}{(2\pi)^{1/2}} e^{-m^2(x-y)^2/2} \phi(y) dy.$$

It can be verified directly that $\|\phi_m(x)\| \leq M$ for all x and ϕ_m has entries in S for $m = 1, 2, \dots$. Also on any bounded interval ϕ_m converges uniformly to ϕ so by the first paragraph of Section 4 if W_m is the solution to (3) with initial values given by ϕ_m , then $W_m(x, t)$ converges to $W(x, t)$ for $t > 0$ and all x . But in this case

$$\begin{aligned} (28) \quad W(x, t) &= \lim_{m \rightarrow \infty} W_m(x, t) = \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} \check{F}(x-y, t) \phi_m(y) dy + \begin{pmatrix} 0 & \cdots \\ \vdots & \\ \vdots & e^{tB} \end{pmatrix} \phi_m(x) \\ &= \int_{-\infty}^{\infty} \check{F}(x-y, t) \phi(y) dy + \begin{pmatrix} 0 & \cdots \\ \vdots & \\ \vdots & e^{tB} \end{pmatrix} \phi(x). \end{aligned}$$

by the dominated convergence theorem. We recall that the integrability of $\check{F}(x, t)$ in x follows from Proposition 3 as was shown in the proof of (i).

Proof of Theorem 2 and corollary. The proof of the theorem follows from two lemmas. We state the lemmas, give a proof of the theorem, prove the lemmas and finally give the standard properties of solutions to ordinary differential equations near critical points which give the corollary.

Lemma 7. *If the matrix in Theorem 2 has a purely imaginary eigenvalue for some $v \neq 0$, then the system (3) is not exponentially stable.*

Lemma 8. *If the eigenvalues $\alpha_0, \dots, \alpha_n$ of $A(0)$ all have negative real parts, then the condition on the eigenvalues of the matrix in Theorem 2 holds for v sufficiently large.*

Proof of the theorem. If the system (3) is exponentially stable, then by (ii) the real parts of $\alpha_0, \dots, \alpha_n$ are negative. By Lemma 8 the condition on the eigenvalues in Theorem 2 holds for v sufficiently large, and by Lemma 7 it must continue to hold for all $v > 0$ since no eigenvalue can lie on or cross the imaginary axis and the eigenvalues of a matrix depend continuously on the entries of the matrix. Since no distinct complex conjugate can exist, the eigenvalue with negative real part must be real for all $v > 0$.

Proof of Lemma 7. A complex number λ is an eigenvalue of the matrix in Theorem 2 if and only if the determinant of

$$\begin{bmatrix} -v - \lambda & -a & -r \\ 1 & -\lambda & 0 \dots \\ 0 & -\frac{1}{v}c & -\frac{1}{v}B - \lambda I \\ \vdots & & \end{bmatrix}$$

is zero. Adding $v + \lambda$ times the second row to the first row and then expanding by the first column, we see that this determinant is minus the determinant of

$$\begin{bmatrix} -a - \lambda(v + \lambda) & -r \\ -\frac{1}{v}c & -\frac{1}{v}B - \lambda I \end{bmatrix}$$

which is a nonzero multiple of

$$\left[\det \begin{bmatrix} a + \lambda v & r \\ c & B + \lambda v I \end{bmatrix} + \lambda^2 \det (B + \lambda v I) \right].$$

In setting $\mu = -\lambda v$ we have that λ is an eigenvalue if and only if

$$\begin{aligned} (29) \quad 0 &= \det (A(0) - \mu I) + \frac{\mu^2}{v^2} \det (B - \mu I) \\ &= (\alpha_0 - \mu) \dots (\alpha_n - \mu) + \frac{\mu^2}{v^2} (\beta_1 - \mu) \dots (\beta_n - \mu). \end{aligned}$$

If $i\tau$ with τ real is an eigenvalue of the matrix in Theorem 2, then setting $\mu = -i\tau$ we have

$$(30) \quad (\alpha_0 + i\tau) \cdots (\alpha_n + i\tau) - \tau^2(\beta_1 + i\tau) \cdots (\beta_n + i\tau) = 0$$

and by Proposition 1 and purely imaginary number $-i\tau$ is an eigenvalue of $A(\tau^2)$ so that by Theorem 1 the system (3) is not exponentially stable.

Proof of Lemma 8. By the methods used in the proof of Proposition 2 it is seen that as v tends to infinity there is a root $\mu(v^2)$ to (29) which is the inverse function of

$$g(\mu) = \frac{\mu^2(\mu - \beta_1) \cdots (\mu - \beta_n)}{(\mu - \alpha_0) \cdots (\mu - \alpha_n)} \quad \text{for large } \mu.$$

This root tends to infinity as v^2 tends to infinity. Also just as in the proof of Proposition 2 the winding number about the origin of

$$(\alpha_0 - \mu) \cdots (\alpha_n - \mu) + \frac{\mu^2}{v^2}(\beta_1 - \mu) \cdots (\beta_n - \mu)$$

for μ on a small circle about any of the α_i is equal to the multiplicity of the eigenvalue α_i for v^2 sufficiently large. Since μ is a root of (29) if and only if $\lambda = -\mu/v$ is an eigenvalue of the matrix in Theorem 3 and the polynomial in (29) is of degree $n + 2$, the proof is complete.

We remark here that converse of Theorem 2 is false. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ the least upper bound of the real parts of the eigenvalues of $A(\theta^2)$ is 0, but the condition on the eigenvalues of the matrix in Theorem 2 holds for all $v > 0$. In fact, we see from the above proof that for the condition on the eigenvalues of the matrix in Theorem 2 to hold it is sufficient that the real parts of the eigenvalues of $A(\theta^2)$ be negative for all real θ .

Proof of the corollary. For $v > 0$ fixed if there is one negative eigenvalue $-\lambda$ to the matrix in (4) and all other eigenvalues have positive real part, then by the theory of constant coefficient ordinary differential equations any real solution of (4) tending to zero as y goes to infinity is of the form

$$+e^{-\lambda(y-y_0)}Z \quad \text{or} \quad -e^{-\lambda(y-y_0)}Z$$

for some y_0 and for Z a fixed eigenvector of the matrix in (4) with eigenvalue $-\lambda$.

Under the assumption that the first two partial derivatives of $g^0, \dots, g^n$ exist and are continuous, standard results on the conditional stability of non-linear ordinary differential equations as found, for example, in Section 4 of Chapter 13 of [1] assert that there exists $\delta > 0$ such that for any fixed choice of sign and of y_0 resulting in a solution say

$$e^{-\lambda(y-y_0)}Z$$

to (4) there is a unique solution

$$\begin{pmatrix} \frac{d\phi^0}{dy}(y) \\ \phi(y) \end{pmatrix}$$

to (2) with

$$\left\| \begin{pmatrix} \frac{d\phi^0}{dy} \\ \phi(y) \end{pmatrix} - e^{-\lambda(v-v_0)} Z \right\| \leq K e^{(-\lambda-\delta)v}$$

for sufficiently large y and for some constant K with a similar statement for a solution $-e^{-\lambda(v-v_0)} Z$. Moreover, all solutions of (2) tending to zero arise in this manner. Since translates in y of a solution tending to zero are still solutions tending to zero, the corollary holds.

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