A Variant of the Lusternik-Schnirelman Theory

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1. Introduction. In this paper we consider the existence of critical points of an even, real valued function on a Banach space. The general approach is by a variant of the Lusternik-Schnirelman theory. An application is made to nonlinear eigenvalue problems of the form

$$(1) \quad \lambda g' = h',$$

where g' and h' are gradient operators and where the value of λ is specified. This contrasts with other treatments of (1) by the Lusternik-Schnirelman theory, where (1) is dealt with as an isoperimetric variational problem and consequently the value of λ cannot be specified. Also, it is to be noted that no definiteness assumptions are imposed upon g' , which contrasts with all other published results concerning (1), except for some results of Amann [13].

Another feature of isoperimetric variational problems is that the parameter appears linearly. It seems likely that certain eigenvalue problems in which the parameter appears nonlinearly could be considered in terms of finding the critical points of a function on a Banach space, although the author has not yet considered any such problems.

We obtain the main theorem of the Lusternik-Schnirelman theory specialized to the case of an even function on a Banach space (Theorem 8). The proof and the preliminary results are largely modifications of results of Palais [2, 5] and Schwartz [1]. However, Theorem 8 is not a specialization of their results. Also, it should be noted that the specialization to a Banach space permits great simplification in the technical details of the proof.

After deriving Theorem 8 we consider corollaries which require special types of nonlinearity in the gradient function, which are related to superlinearity and to asymptotic linearity [12].

Finally, we consider eigenvalue problems of the form (1), with λ specified. Other authors who have, in effect, considered eigenvalue problems with the parameter fixed are Hempel [10], who treats elliptic problems in differential equations, and Coffman [11], who treats Hammerstein integral equations.

2. Preliminary results. Let E be a real Banach space, f a real valued C^1 function on E , E^* the set of regular (non-critical) points of f , and v a vector field on E^* .

A C^1 function ϕ from an interval of the reals into E such that $\phi'(t) = v(\phi(t))$ is called a solution curve of v . If 0 is in the domain of ϕ , then $\phi(0)$ is called the initial value of the solution curve. We shall say that v is C^{1-} (locally Lipschitz continuous) if for every $x \in E^*$ there exists a neighborhood $U \subset E^*$ of x and $K > 0$ such that $\|v(y) - v(z)\| \leq K \|y - z\|$ for $y, z \in U$.

Lemma 1 is the relevant part of results of Palais [5], specialized to the present case, and Definition 2 is also taken from [5].

1. Lemma. *Let v be a C^{1-} vector field on E^* . Then*

(a) *For each $x \in E^*$ there is a solution curve ϕ_x of v in E^* with initial value x such that every solution curve in E^* with initial value x is a restriction of ϕ_x . The domain of ϕ_x is an open interval $(t^-(x), t^+(x))$.*

(b) *Either $t^-(x) = -\infty$ ($t^+(x) = \infty$) or $\phi_x(t)$ has no limit point in E^* as $t \rightarrow t^-(x)$ ($t \rightarrow t^+(x)$).*

(c) *The set $D = \{(x, t) \mid t^-(x) < t < t^+(x)\}$ is open in $E^* \times \mathbb{R}$. Let $\sigma_t(x) = \phi_x(t)$. The map $x \rightarrow \sigma_t(x)$ is continuous. The family of maps σ_t is said to be the flow defined by v .*

We let (u, x) denote the pairing between an element u of the space dual to E and an element x of E . Also, we let $\|\cdot\|$ denote the norm in both E and its dual space.

2. Definition. $v \in E$ is called a pseudo-gradient vector for f at x if

(a) $\|v\| \leq 2 \|f'(x)\|$

(b) $\langle f'(x), v \rangle \geq \|f'(x)\|^2$.

If $v(x)$ is a C^{1-} vector field on E^* such that $v(x)$ is a pseudo-gradient vector for f at each $x \in E^*$, then $v(x)$ is called a pseudo-gradient vector field for f in E^* .

3. Lemma. *Let f be even. Then there exists an odd C^{1-} pseudo-gradient vector field for f in E^* .*

The existence of a C^{1-} pseudo-gradient vector field $v(x)$ for f in E^* follows from a result of Palais [5]. It is easily seen that $\bar{v}(x) = \frac{1}{2}(v(x) - v(-x))$ is a C^{1-} pseudo-gradient vector field for f in E^* which is odd.

We shall assume that $f(0) = 0$ and that f satisfies the following Palais-Smale type condition:

(A) If a sequence $\{x_n\} \subset E$ is such that $f(x_n) < 0$, $f(x_n)$ is bounded below, and $f'(x_n) \rightarrow 0$, then $\{x_n\}$ contains a convergent subsequence.

For verification of Condition (A) in particular cases it will be convenient to verify the following two conditions, which taken together are equivalent to Condition (A):

(B) For every $k < 0$ there exists $R > 0$ and $a > 0$ such that $\|f'(x)\| \geq a$ for x such that $k < f(x) < 0$ and $\|x\| > R$.

(C) If a bounded sequence $\{x_n\} \subset E$ is such that $f(x_n) < 0$, $f(x_n)$ is bounded below, and $f'(x_n) \rightarrow 0$, then $\{x_n\}$ contains a convergent subsequence.

We let $K = \{x \in E \mid f'(x) = 0\}$, $K_c = K \cap f^{-1}(c)$, and $f_c = f^{-1}((-\infty, c])$.

The following theorem and its proof are analogous to theorems and proofs from a number of other sources [1-7, 13, 14], but it is not a specialization of any of these results.

4. Theorem. *Let f be an even C^1 function satisfying Condition (A). For any $c < 0$ let U be a neighborhood of K_c . Then there exists $d > 0$ and a flow $\eta_t(x)$, defined for $(t, x) \in [0, 1] \times (f_{c+d} - U)$, and odd in x , such that $\eta_1(f_{c+d} - U) \subset f_{c-d}$. In particular, if c is a regular value of f , then $\eta_1(f_{c+d}) \subset f_{c-d}$.*

Proof. Consider first the case where $K_c \neq \emptyset$. Then Condition (A) implies that K_c is compact; hence U contains a neighborhood of K_c of the form

$$(2) \quad U_\epsilon = \{x \in E \mid d(x, K_c) < \epsilon\}, \quad \epsilon > 0.$$

It will suffice to prove that for any such set U_ϵ there exists $d > 0$ and a flow $\eta_t(x)$, defined for $(t, x) \in [0, 1] \times (f_{c+d} - U_\epsilon)$, and odd in x , such that $\eta_1(f_{c+d} - U_\epsilon) \subset f_{c-d}$.

It follows from Condition (A) that for any $\epsilon > 0$ there exist $b > 0$ and $d_1 > 0$ such that

$$(3) \quad \|f'(x)\| \geq b \quad \text{if} \quad x \in f_{c+d_1} - f_{c-d_1} - U_{\epsilon/2}.$$

Given any such ϵ , b , d_1 , choose d_1 smaller, if necessary, and choose d so that

$$(4) \quad 0 < d < d_1 \leq \min(b/4, b^2/4, b\epsilon/12).$$

Let $g(x) = \text{dist}(x, A) / (\text{dist}(x, A) + \text{dist}(x, B))$, where

$$A = (E - f_{c+d_1}) \cup f_{c-d_1}, \quad B = f_{c+d} - f_{c-d}.$$

It is easily seen that g is even C^{1-} and

$$(5) \quad g(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}, \quad 0 \leq g(x) \leq 1 \quad \text{otherwise.}$$

By Lemma 3 there is an odd pseudo-gradient vector field $v(x)$ for f on E^* . Let

$$h(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq 1, \\ 1/s & \text{if } s \geq 1, \end{cases}$$

$$V(x) = -g(x)h(\|v(x)\|)v(x).$$

V is an odd, bounded, C^{1-} vector field on E^* . Let $\eta_t(x)$ be the flow generated by $V(x)$. Clearly $\eta_t(x)$ is odd in x .

For fixed x let

$$(6) \quad \alpha(t) = f(\eta_t(x)), \quad \beta(t) = \|\eta_t(x) - x\|.$$

It will suffice to show that

$$(7) \quad t^+(x) > 1, \quad \alpha(1) < c - d$$

for $x \in f_{c+d} - U_\epsilon$. Since $V \equiv 0$ on f_{c-d} , (7) clearly holds for $x \in f_{c-d}$. It remains to show that (7) holds for $x \in D = f_{c+d} - f_{c-d} - U_\epsilon$.

Let $x \in D$. $\alpha(t)$ is a decreasing differentiable function, in fact, using Definition 2,

$$(8) \quad \begin{aligned} \alpha'(t) &= (f'(\eta_t(x)), V(\eta_t(x))) \\ &\leq -\frac{1}{2} \|f'(\eta_t(x))\| \|V(\eta_t(x))\|. \end{aligned}$$

Since $V \equiv 0$ on f_{c-d} , for $0 \leq t < t^+(x)$,

$$(9) \quad 0 \leq \alpha(0) - \alpha(t) < d + d_1 < 2d_1.$$

Let $T = \min [t^+(x), \inf \{t \in [0, t^+(x)] \mid \eta_t(x) \in U_{\epsilon/2}\}]$. Then for $0 \leq t < T$, by (3), (6), and (8),

$$\begin{aligned} \alpha(0) - \alpha(t) &\geq \frac{1}{2} \int_0^t \|f'(\eta_s(x))\| \|V(\eta_s(x))\| ds \\ &\geq \frac{1}{2} b \left\| \int_0^t V(\eta_s(x)) ds \right\| \\ &\geq \frac{1}{2} b \beta(t). \end{aligned}$$

Hence, by (4) and (9), $\beta(t) < \epsilon/3$ for $0 \leq t < T$. Hence, by the continuity in t of $\eta_t(x)$, $T = t^+(x)$ and $\eta_t(x) \notin U_{\epsilon/2}$ for $x \in D$ and $0 \leq t < t^+(x)$.

Now we show that $t^+(x) = \infty$ for $x \in D$. Suppose the contrary, and let $x_0 \in D$ and $\{t_n\}$ be such that $t^+(x_0) < \infty$ and $t_n \rightarrow t^+(x_0) -$. Since V is bounded, $\{\eta_{t_n}(x_0)\}$ is a Cauchy sequence. The limit of $\eta_{t_n}(x_0)$ lies in E^* since $\beta(t_n) < \epsilon/3$, which contradicts Lemma 1(b). Hence $t^+(x) = \infty$ for $x \in D$, and this verifies the first inequality of (7) for $x \in D$.

To verify the second inequality of (7) for $x \in D$, (3), (5), and (8) imply that $\alpha'(t) \leq -\frac{1}{2} \min(b^2, b)$ for $x \in D$ and $\eta_t(x) \notin f_{c-d}$. Hence, by (4), $\alpha(1) < c - d$ for $x \in D$. This completes the proof of the theorem for the case $K_\epsilon \neq \emptyset$.

The proof for the case $K_\epsilon = \emptyset$ is simpler. The sets U_ϵ and $U_{\epsilon/2}$ are replaced by $\emptyset$, and the other details are obvious.

Next we introduce the concept of genus and list some relevant properties. The class of subsets of $E - \{0\}$ closed and symmetric with respect to the origin will be denoted by $S(E)$.

5. Definition. If $A \in S(E)$, the *genus* of A , denoted by $\gamma(A)$, is the least integer n such that there exists an odd continuous map ψ from A into $\mathbb{R}^n - \{0\}$. We set the genus of the empty set to zero.

6. Lemma. *The genus has the following properties:*

- (a) *If $A_1, A_2 \in S(E)$ and if there exists an odd continuous map $f: A_1 \rightarrow A_2$, then $\gamma(A_1) \leq \gamma(A_2)$; in particular, if $A_1 \subset A_2$ then $\gamma(A_1) \leq \gamma(A_2)$.*
- (b) *For $A_1, A_2 \in S(E)$, $\gamma(A_1 \cup A_2) \leq \gamma(A_1) + \gamma(A_2)$.*
- (c) *If $A \in S(E)$ and A is compact, then $\gamma(A) < \infty$ and A has an open neighborhood U such that $\bar{U} \in S(E)$ and $\gamma(\bar{U}) = \gamma(A)$.*
- (d) *If $A \in S(E)$ and if there exists an odd homeomorphism of the n -sphere onto A , then $\gamma(A) = n + 1$.*
- (e) *If E is a Hilbert space, and F is an n -dimensional subspace of E , and $A \in S(E)$ is such that $A \cap F^\perp = \emptyset$, then $\gamma(A) \leq n$.*

Properties (a) to (d) are taken from [11, p. 407], and the proofs are given there. For the proof of property (e) we project A orthogonally into $F - \{0\}$, which is an odd continuous map, and then project along rays from the origin into $F \cap \{x \in E \mid \|x\| = 1\}$, which is an odd continuous map into a set homeomorphic to the $n - 1$ sphere. The result now follows from properties (a) and (d).

7. Definition. For $m \leq \dim E$ let

$$(10) \quad c_m(f) = \inf_{\gamma(A) \geq m} \sup_{x \in A} f(x).$$

Clearly $-\infty \leq c_1(f) \leq c_2(f) \leq c_3(f) \leq \dots \leq 0$, since $f(0) = 0$.

3. Main theorem and corollaries. Now we come to the analog of the principal result of the Lusternik–Schnirelman theory on manifolds.

8. Theorem. *Let f be even, C^1 , satisfy Condition (A), and let $f(0) = 0$. Then if $-\infty < c_m(f) < 0$, K_{c_m} is non-empty and compact. Moreover, if $-\infty < c_m(f) = c_n(f) = c < 0$, then $\gamma(K_c) \geq n - m + 1$.*

Proof. The assertion that each K_{c_m} is compact follows immediately from Condition (A). The assertion that K_{c_m} is non-empty is a consequence of the latter assertion of the theorem. Hence it remains to prove the latter assertion of the theorem. Suppose that $n \geq m$. Let U be an open symmetric neighborhood of K_c such that $\bar{U} \in S(E)$ and $\gamma(\bar{U}) = \gamma(K_c)$ as in Lemma 6(c). Choose $d > 0$ and a flow η_t such that $c + d < 0$ and $\eta_1(f_{c+d} - U) \subset f_{c-d}$ as in Theorem 4. By Definition 7 there exists $A \in S(E)$ such that $A \subset f_{c+d}$ and $\gamma(A) \geq n$. Let $B = A - U$. Then by Lemma 6(a, b)

$$(11) \quad \gamma(K_c) \geq \gamma(A) - \gamma(B).$$

Also, it is clear that $B \in S(E)$, $\eta_1(B) \in S(E)$, and $\eta_1(B) \subset f_{c-d}$. By Lemma 6(a), $\gamma(\eta_1(B)) \geq \gamma(B)$. If $\gamma(B) \geq m$, then $\gamma(\eta_1(B)) \geq m$ and we arrive at the contradiction that $c_m(f) \leq c - d$. Hence $\gamma(B) \leq m - 1$ and from (11) we obtain that $\gamma(K_c) \geq n - m + 1$.

Using Theorem 8 we shall express lower bounds for the number of antipodal pairs of critical points of a function f in terms of the quantities

$$(12) \quad \begin{aligned} i_1(f) &= \lim_{a \rightarrow 0-} \gamma(f_a), \\ i_2(f) &= \lim_{a \rightarrow -\infty} \gamma(f_a). \end{aligned}$$

It is easily seen that $c_m(f) < 0$ if and only if $m \leq i_1(f)$ and $c_m(f) > -\infty$ if and only if $m > i_2(f)$. To prove the former assertion, first suppose that $m \leq i_1(f)$. Then there exists $a < 0$ such that $\gamma(f_a) \geq m$. Clearly $f_a \in S(E)$ and $\sup_{x \in f_a} f(x) = a$; hence $c_m(f) \leq a < 0$. Now suppose that $c_m(f) < 0$. Then there exists $A \in S(E)$ such that $\gamma(A) \geq m$ and $\sup_{x \in A} f(x) = b < 0$. Since $f_b \supset A$, $\gamma(f_b) \geq m$; hence $\lim_{a \rightarrow 0-} \gamma(f_a) \geq m$. The other assertion is also easy.

From Theorem 8 and the preceding considerations we immediately have:

9. Theorem. *Let f be even, C^1 , satisfy Condition (A), and let $f(0) = 0$. If $i_1(f) > i_2(f)$, then for each integer m such that $i_2(f) < m \leq i_1(f)$ there is at least one antipodal pair $(x_m, -x_m)$ of critical points of f such that $f(x_m) = c_m(f)$.*

We shall consider two special cases of Theorem 9. First let us recall some well known facts about quadratic forms. A function $q: E \rightarrow R$ is called a quadratic form if there exists a symmetric bilinear form $\beta: E \times E \rightarrow R$ such that $q(x) = \beta(x, x)$. A quadratic form q is continuous if and only if there is a constant K such that $|q(x)| \leq K \|x\|^2$. In this case there is a continuous symmetric linear operator L from E to the space dual to E such that $q(x) = \frac{1}{2}(Lx, x)$, and q is differentiable to all orders, the first and second derivatives being given by $q'(x) = Lx$, $q''(x) = (L \cdot, \cdot)$, and higher derivatives vanishing identically. We shall say that q is regular if $\|Lx\| \geq \alpha \|x\|$ for some $\alpha > 0$. The index $i(q)$ is defined to be the maximal dimension of a subspace of E on which q is negative definite.

The following lemma is similar to a result of Lax and Phillips [15, p. 223].

10. Lemma. *Let $q(x)$ be a continuous regular quadratic form on a Hilbert space H , such that $i(q) = k < \infty$. Then there exists a k -dimensional subspace F and $c > 0$ such that $q(x) \geq c \|x\|^2$ for $x \in F^\perp$.*

Proof. Let L be the continuous symmetric linear operator such that $q(x) = \frac{1}{2}(Lx, x)$. Then from the spectral representation theorem for continuous symmetric linear operators it is clear that L possesses k linearly independent eigenvectors with corresponding negative eigenvalues. Let F be the span of these eigenvectors. Then $F^\perp$ is an invariant subspace under L . Let $\ell = \inf_{x \in S} q(x)$, where S is the unit sphere in $F^\perp$. Given any $\epsilon > 0$ choose $x_0 \in S$ so that $\ell \leq q(x_0) < \ell + \epsilon$. We have $Lx_0 = 2q(x_0)x_0 + v$, where $v \perp x_0$ and $v \perp F$. Also, for real t ,

$$(13) \quad q(x_0 + tv) = q(x_0) + t \|v\|^2 + t^2 q(v) \geq \ell.$$

It follows from the inequality in (13) that $\|v\| \leq (2\epsilon \|L\|)^{1/2}$. Since also $2q(x_0) \geq \|Lx_0\| - \|v\| \geq \alpha - \|v\|$, it follows for ϵ sufficiently small that $\ell \geq \alpha/2 > 0$, and hence the lemma follows.

11. Theorem. Let E be a real Banach space and f be an even, real valued, C^1 function on E . Suppose further that f satisfies Condition (C), $f(0) = 0$, f is bounded below, $f(x) \geq 0$ for large $\|x\|$, and $f(x) = q(x) + o(\|x\|^2)$ as $\|x\| \rightarrow 0$, where q is a quadratic form of index k . Then for each integer $m = 1, 2, \dots, k$, there is at least one antipodal pair $(x_m, -x_m)$ of critical points of f such that $f(x_m) = c_m(f)$.

Proof. Since $f(x) \geq 0$ for large $\|x\|$, Condition (B) holds. Since also f satisfies Condition (C), Condition (A) holds. Clearly, for $a < 0$ and sufficiently close to zero, f_a contains a set odd homeomorphic to the k -sphere; hence by (12) and Lemma 6(ad), $i_1(f) \geq k$. Since f is bounded below, $i_2(f) = 0$. The theorem now follows from Theorem 9.

12. Theorem. Let H be a real Hilbert space and f be an even, real valued, C^1 function on H . Suppose further that f satisfies Condition (C), $f(0) = 0$, f is bounded below on bounded sets, and that $f(x) = q_1(x) + o(\|x\|^2)$ as $\|x\| \rightarrow 0$, where q_1 is a quadratic form of index k , and $f(x) = q_2(x) + r_2(x)$, where q_2 is a continuous regular quadratic form of finite index $\ell < k$ and where $r_2(x) = o(\|x\|^2)$ and $\|r'_2(x)\| = o(\|x\|)$ as $\|x\| \rightarrow \infty$. Then for each integer m such that $\ell < m \leq k$, there is at least one antipodal pair $(x_m, -x_m)$ of critical points of f such that $f(x_m) = c_m(f)$.

Proof. Since q_2 is a regular quadratic form and $\|r'_2(x)\| = o(\|x\|)$ as $\|x\| \rightarrow \infty$, Condition (B) holds. Since also Condition (C) holds, Condition (A) holds. As in the proof of Theorem 11, $i_1(f) \geq k$. By Lemma 10 there exists an ℓ dimensional subspace F such that $q(x) \geq c\|x\|^2$ for $x \in F^\perp$, some $c > 0$. It follows that f is bounded below on $F^\perp$ and hence $f_a \cap F^\perp = \emptyset$ for some $a < 0$. Therefore, by Lemma 6(a), $\gamma(f_a) \leq \ell$ for some $a < 0$, and so $i_2(f) \leq \ell$. The theorem now follows from Theorem 9.

4. Nonlinear eigenvalue problems. The results of Section 3 may be applied to nonlinear eigenvalue problems of the form

$$(14) \quad \lambda g'(x) = h'(x)$$

in which the value of λ is specified. We shall assume that g and h are real valued functions on a separable, real Hilbert space H , and that:

(D) g is even, C^1 , bounded (on bounded sets), $g(0) = 0$, and g' is compact, $g'(x) = L_1x + o(\|x\|)$ as $\|x\| \rightarrow 0$, where L_1 is a continuous, symmetric, linear operator. It follows that L_1 is compact [9, p. 135].

h is even, C^1 , bounded, $h(0) = 0$, and h' is proper, i.e., the inverse image under h' of a compact set in H is compact in H , $h'(x) = M_1x + o(\|x\|)$ as $\|x\| \rightarrow 0$, where M_1 is a symmetric linear operator such that $c_1\|x\| \leq (M_1x, x) \leq c_2\|x\|^2$, some $c_2 \geq c_1 > 0$.

13. Lemma. Let $f = h - \lambda g$, where g and h satisfy Conditions (D). Then f satisfies Condition (C).

Proof. In fact we shall see that f satisfies a stronger condition. Let $\{x_n\}$

be a bounded sequence in H such that $f'(x_n) \rightarrow 0$. There is a weakly convergent subsequence, which we may suppose to be $\{x_n\}$. Since g' is compact, a subsequence of $g'(x_n)$ converges, which we may suppose to be $g'(x_n)$. Then $h'(x_n) = \lambda g'(x_n) + f'(x_n)$ converges, and so the set $\{h'(x_n)\}$ is relatively compact. Since h' is proper, $\{x_n\}$ is relatively compact and therefore contains a convergent subsequence.

Next consider the eigenvalue problem

$$(15) \quad \lambda L_1 x = M_1 x$$

where L_1 and M_1 are the same as in Conditions (D). By well known results, (15) possesses a set of solutions (λ_i^1, x_i^1) , where

$$(16) \quad \cdots \leq \lambda_{-3}^1 \leq \lambda_{-2}^1 \leq \lambda_{-1}^1 < 0 < \lambda_1^1 \leq \lambda_2^1 < \lambda_3^1 \leq \cdots,$$

either series $\lambda_i^1, i > 0$ or $\lambda_i^1, i < 0$ may be finite, and where $\{x_i^1\}$ forms an orthonormal set. Moreover we may write $H = P^1 \oplus N^1 \oplus Q^1$, where $P^1 = \text{span } \{x_i^1 \mid i < 0\}$, N^1 is the null space of L_1 , and $Q^1 = \text{span } \{x_i^1 \mid i > 0\}$. Also, we let

$$\beta_1(x, y) = (M_1 x, y) - \lambda(L_1 x, y),$$

$$q_1(x) = \frac{1}{2}\beta_1(x, x),$$

$$f(x) = h(x) - \lambda g(x).$$

From (D) it follows that $f(x) = q_1(x) + o(\|x\|^2)$ as $\|x\| \rightarrow 0$. It is easily seen that $\beta_1(x_i^1, x_j^1) = 0$ if $i \neq j$. Hence q_1 is negative definite on the span of those x_i^1 for which $q(x_i^1) < 0$ and q_1 is non-negative definite on the orthogonal complement of this subspace. Also, $q_1(x_i^1) = \frac{1}{2}(1 - \lambda/\lambda_i^1)(M_1 x_i^1, x_i^1)$; hence $q(x_i^1) < 0$ if and only if $\lambda/\lambda_i^1 > 1$. It follows that $i(q_1) \geq k$ if and only if $\lambda > \lambda_k^1$ or $\lambda < \lambda_{-k}^1$, where $k > 0$.

From Lemma 13 and Theorem 11 we conclude:

14. Theorem. *Suppose g and h satisfy Conditions (D), that $\lambda > \lambda_k$ or $\lambda < \lambda_{-k}$ for some $k > 0$, and that $h(x) - \lambda g(x) \geq 0$ for large $\|x\|$. Then for each integer $m = 1, 2, \dots, k$, (14) has at least one antipodal pair $(x_m, -x_m)$ of solutions such that $h(x_m) - \lambda g(x_m) = c_m(h - \lambda g)$.*

Next we apply Theorem 12 to (14). First we introduce the following conditions:

(E) g and h satisfy Conditions (D) and in addition:

$$g'(x) = L_2 x + o(\|x\|) \quad \text{as } \|x\| \rightarrow \infty.$$

By [9, p. 207], L_2 is compact. Also

$$h'(x) = M_2 x + o(\|x\|) \quad \text{as } \|x\| \rightarrow \infty,$$

where $a\|x\|^2 \leq (M_2 x, x) \leq b\|x\|^2$ for some $b \geq a > 0$.

Analogously to (15) the eigenvalue problem

$$(17) \quad \lambda L_2 x = M_2 x$$

has a set of eigensolutions (λ_i^2, x_i^2) , where

$$\cdots \leq \lambda_{-3}^2 \leq \lambda_{-2}^2 \leq \lambda_{-1}^2 < 0 < \lambda_1^2 < \lambda_2^2 \leq \lambda_3^2 \leq \cdots$$

and $\{x_i^2\}$ is a complete orthonormal set in the orthogonal complement of the null space of L_2 .

15. Theorem. Suppose that g and h satisfy Conditions (E), that either $\lambda_k^1 < \lambda < \lambda_{i+1}^2$ or $\lambda_{-i-1}^2 < \lambda < \lambda_{-k}^1$, where $k > l \geq 0$, and $\lambda \neq \lambda_i^2$ for any i . Then for each integer m such that $l < m \leq k$, (14) has at least one antipodal pair $(x_m, -x_m)$ of solutions such that $h(x_m) - \lambda g(x_m) = c_m(h - \lambda g)$.

Proof. Most of the hypotheses of Theorem 12 for the function $f = h - \lambda g$ are immediate and need not be discussed. Starting on the other hypotheses, as in the proof of Theorem 14, we find that $i_1(f) \geq k$. Next we introduce the forms

$$\beta_2(x, y) = (M_2 x, y) - \lambda(L_2 x, x), \quad q_2(x) = \frac{1}{2}\beta_2(x, x).$$

It is easily shown that

$$(18) \quad \beta_2(x_i^2, x_j^2) = 0 \quad \text{if} \quad i \neq j.$$

From (E) it is clear that $f = q_2 + r$ where $r(x) = o(\|x\|^2)$ and $\|r'(x)\| = o(\|x\|)$ as $\|x\| \rightarrow \infty$. Also it can be shown that $i(q_2) \leq l$ in a manner similar to the verification that $i(q_1) \geq k$ in the proof of Theorem 14.

We claim that $q_2(x)$ is regular. Consider the case where $\lambda_k^1 < \lambda < \lambda_{i+1}^2$. Without loss of generality we may assume that either $\lambda_i^2 < \lambda$ or $0 < \lambda < \lambda_i^2$. Consider the former case.

Let $F = \text{span}(x_i^2, \dots, x_l^2)$.

For any $x \in H$ write $x = u + v$, where $u \in F$, $v \in F^\perp$. Either $\|x\| \leq \frac{1}{2}\|u\|$ or $\|x\| \leq \frac{1}{2}\|v\|$. Suppose first that $\|x\| \leq \frac{1}{2}\|u\|$. Then

$$\begin{aligned} \|q'_2(x)\| &\geq |(q'_2(x), u)|/\|u\| \\ &= |\beta_2(x, u)|/\|u\| = |q_2(u)|/\|u\|, \end{aligned}$$

using (18) for the latter equality. For $i = 1, \dots, l$,

$$|q_2(x_i^2)| \geq (\lambda/\lambda_i^2 - 1)(M_2 x_i^2, x_i^2) > 0.$$

Hence, by (18) $|q_2(u)| \geq d\|u\|^2$ for some $d > 0$. Hence $\|q'_2(x)\| \geq 2d\|x\|$; in other words, $q_2(x)$ is a regular quadratic form. Arguments for each of the other cases show that in all cases $q_2(x)$ is regular.

The conclusion of the theorem now follows from Theorem 12 applied to $f = h - \lambda g$.

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