Singular Integrals whose Kernels Involve Certain Sturm-Liouville Functions, II

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1. Introduction. Here, as in [12], we study the behaviour for large positive ν of the singular integral

$$\int_b^\infty f(t)w(\nu, t-x) dt,$$

where $b < x \leq b-a$ for some $a < 0$, a and b are finite, and, for each $\nu > 0$, $w(\nu, t)$ is the solution of

$$d^2w/dt^2 = [\nu^2t + q(t)]w,$$

on the interval $[a, \infty)$, which satisfies

$$(1.1) \quad \lim_{t \rightarrow \infty} w(\nu, t)t^{1/4} \exp\left(\frac{2}{3}\nu t^{3/2}\right) = \frac{1}{2}\pi^{-1/2}\nu^{1/2}.$$

(The assumption in [12, §4] that $a \leq b$ is unnecessary and is not used in any of the proofs.) To ensure that, for each $\nu > 0$, the condition (1.1) determines $w(\nu, t)$ uniquely, we suppose that q is continuous on $[a, \infty)$ and that

$$\int_0^\infty t^{-1/2} |q(t)| dt$$

exists.

In [12], we considered the convergence, as $\nu \rightarrow \infty$, of the foregoing singular integral; here, we consider its Cesàro summability.

The principal result in [12] is that, under appropriate hypotheses on f ,

$$(1.2) \quad \lim_{\nu \rightarrow \infty} \int_b^\infty f(t)w(\nu, t-x) dt = \frac{2}{3}f(x-) + \frac{1}{3}f(x+).$$

In particular, we have the following result, as a special case of [12, p. 865, Theorem 4.1].

Theorem 1.1. *Let f be of bounded variation on $[b, x]$. Then*

$$(1.3) \quad \lim_{\nu \rightarrow \infty} \int_b^x f(t) w(\nu, t - x) dt = \frac{2}{3} f(x-).$$

In the present paper we prove (Theorem 5.1) that the hypothesis that f be of bounded variation may be weakened to the requirement that f be integrable on $[b, x]$ and that $f(x-)$ exist, provided that the limit relation in (1.3) is interpreted in the sense of Cesàro (C, k) summability, when $k > \frac{1}{2}$, but not necessarily when $0 \leq k \leq \frac{1}{2}$. We show (Theorem 5.2) that if, in addition, f satisfies a local Lipschitz condition to the left of the point x , then (1.3) holds for Cesàro (C, k) summability when $k \geq \frac{1}{2}$, but still not necessarily when $0 \leq k < \frac{1}{2}$.

The methods used are those of the theory of singular integrals; see [8; 7, ch. 7; 13, ch. 10]. In particular, we use the fact that the validity of (1.3) (or a summability analogue of (1.3)) for all integrable functions f for which $f(x-)$ exists depends on the boundedness, in ν , of a set of numbers called the Lebesgue constants. To discuss the order of magnitude of these constants we use an approximation

$$w(\nu, t) = \nu^{2/3} Ai(\nu^{2/3} t) + \epsilon(\nu, t),$$

with the usual notation for the Airy function $Ai(x)$ [11, p. 75]. We use a form of this approximation, including bounds for $\epsilon(\nu, t)$, which is due to F. W. J. Olver [14], and which has been discussed in [12, p. 865].

This study was motivated by work of L. Lorch and P. Szego [10] who proved a result, analogous to Theorem 5.1, on the Cesàro summability as $\nu \rightarrow \infty$, of

$$(1.4) \quad \int_0^A f(t) \nu J_\nu(\nu t) dt, \quad A > 1,$$

where J_ν denotes the Bessel function of the first kind. Their earlier theorem [9, Theorem 1] on the ordinary convergence of (1.4), as $\nu \rightarrow \infty$, was shown by the present writer [12, p. 872] to follow from a result of the type (1.2) by a special choice of $q(t)$, and hence of $w(\nu, t)$. The methods of the present paper follow closely those used by Lorch and Szego in [10].

In §6 we give an application of our general results to a singular integral involving the Whittaker confluent hypergeometric function $W_{\nu, m}(x)$.

Certain aspects of our theorem on a singular integral involving an Airy function [12, Theorem 3.1] have been generalized by C. Comstock [2]. He considers a kernel involving Bessel functions of order $1/m$, where m is odd and $m \geq 3$. We mention also that results similar to those of [12, Theorem 3.1] have been found useful by E. L. Roetman [15] in discussing an initial value problem for a third order linear partial differential equation.

As in the earlier paper, all of the quantities discussed here are assumed to be real.

2. Cesàro summability. When $k \geq 0$, we define $C_{k,\nu}\{F(\nu)\}$, the (C, k) mean of $F(\nu)$, by:

$$(2.1) \quad \begin{aligned} C_{0,\nu}\{F(\nu)\} &= F(\nu); \\ C_{k,\nu}\{F(\nu)\} &= k\nu^{-k} \int_1^\nu (\nu - p)^{k-1} F(p) dp, \quad k > 0, \end{aligned}$$

in case the integral exists. We say that $F(\nu)$ is (C, k) summable to B , as $\nu \rightarrow \infty$, and write

$$(C, k) - \lim_{\nu \rightarrow \infty} F(\nu) = B,$$

if and only if

$$(2.2) \quad \lim_{\nu \rightarrow \infty} C_{k,\nu}\{F(\nu)\} = B.$$

Remark. In the usual definition (see, e.g., [3, pp. 311–312] or [6, pp. 110–111]), the lower limit of integration in (2.1) is taken to be 0. However, it is clear that as long as the integrals involved exist, the lower limit of integration in (2.1) has no effect on the validity of (2.2). We choose 1 as limit of integration so as to be able to handle functions whose behaviour in a neighbourhood of 0 is unknown or difficult to determine.

The following fundamental result is proved in [3, p. 314]. That it remains valid with our definition of (C, k) summability follows from the preceding remark.

Lemma 2.1. *If, for some $k \geq 0$,*

$$(C, k) - \lim_{\nu \rightarrow \infty} F(\nu) = B,$$

then, for all $\alpha > 0$,

$$(C, k + \alpha) - \lim_{\nu \rightarrow \infty} F(\nu) = B.$$

3. The Lebesgue constants. The following theorem is due to H. Lebesgue [8, p. 70]; see also [7, pp. 447–448; 13, p. 15].

Theorem 3.1. *Let $\varphi(\nu, t)$ be a measurable function of t for $0 \leq t \leq c$, for each fixed $\nu > 0$. Suppose that*

$$\lim_{\nu \rightarrow \infty} \int_0^\beta \varphi(\nu, t) dt = \ell$$

for each β satisfying $0 < \beta \leq c$. Suppose also that

$$|\varphi(\nu, t)| \leq K(\delta), \quad \delta \leq t \leq c,$$

for each δ satisfying $0 < \delta \leq c$ and that

$$\int_0^c |\varphi(\nu, t)| dt \leq K$$

for all $\nu > 0$. Let a function f defined on $[0, c]$ be such that $f \in L[0, c]$ and $f(0+)$ exists. Then

$$(3.1) \quad \lim_{\nu \rightarrow \infty} \int_0^c f(t) \varphi(\nu, t) dt = \ell f(0+).$$

In the hypotheses of Theorem 3.1 it is assumed that the numbers

$$\Lambda(\nu) = \int_0^c |\varphi(\nu, t)| dt,$$

known as the Lebesgue constants, form a bounded set. Lebesgue also showed that the boundedness of these quantities is a necessary condition for the validity of (3.1). He proved essentially the following result [8, p. 70]; see also [7, pp. 456–457].

Theorem 3.2. *Let $\varphi(\nu, t)$ be a measurable function of t for $0 \leq t \leq c$, for each $\nu > 0$. Let*

$$\int_0^c |\varphi(\nu, t)| dt$$

be an unbounded function of ν , as $\nu \rightarrow \infty$. Then there exists a function f , continuous on $[0, c]$, for which

$$\int_0^c f(t) \varphi(\nu, t) dt$$

is unbounded, as $\nu \rightarrow \infty$.

Theorem 3.2 may also be obtained by applying the principle of uniform boundedness to the Banach space of continuous functions on $[0, c]$ with the sup norm; see [1, p. 80, Théorème 5; 5, p. 76].

In references [1], [5], [8], and [13] the variable which we designate here as ν is supposed to approach infinity through integer values. It is clear, however, that the results which we have given in this section are still valid when ν approaches infinity continuously through real values; Hobson [7] formulates his results so as to allow for this possibility.

4. Estimates for the Lebesgue constants. In this section, we give asymptotic expressions (Theorem 4.1) for the quantities that correspond to $\Lambda(\nu)$ when the theorems of §3 are applied to integrals involving $\nu^{2/3} Ai(\nu^{2/3} t)$. First, we have a lemma.

Lemma 4.1. *Let k and c be fixed, $0 \leq k < 1$, $0 < c < \infty$. Then, as $\nu \rightarrow \infty$*

$$(4.1) \quad C_{k,\nu} \{ \nu^{2/3} Ai(-\nu^{2/3} t) \} = a_k t^{-1/4-3k/2} \nu^{-k+1/2} \cos \left(\frac{2}{3} \nu t^{3/2} - \frac{1}{2} k\pi - \frac{\pi}{4} \right) \\ + t^{-7/4} O(\nu^{-1/2}), \quad 0 < t \leq c,$$

the O -term holding uniformly in t , $0 < t \leq c$, where

$$(4.2) \quad a_k = \pi^{-1/2} \Gamma(k+1) (3/2)^k.$$

Proof. Throughout the proof, the O -terms hold uniformly in t , $0 < t \leq c$, as $\nu \rightarrow \infty$.

We see, using [16, p. 199, ff.], that

$$(4.3) \quad \nu^{2/3} Ai(-\nu^{2/3} t) = \pi^{-1/2} \nu^{1/2} t^{-1/4} \cos\left(\frac{2}{3} \nu t^{3/2} - \frac{\pi}{4}\right) + R(\nu, t),$$

where

$$(4.4) \quad |R(\nu, t)| \leq C \nu^{-1/2} t^{-7/4} \quad (\nu > 0, t > 0),$$

for some constant $C > 0$. Thus we see that (4.1) holds in case $k = 0$. On account of (4.4), we have, for $k > 0$,

$$|C_{k,\nu} \{R(\nu, t)\}| \leq C k B(k, \frac{1}{2}) \nu^{-1/2} t^{-7/4} \quad (\nu > 0, t > 0),$$

and so we get, from (4.3),

$$(4.5) \quad \begin{aligned} & C_{k,\nu} \{\nu^{2/3} Ai(-\nu^{2/3} t)\} \\ &= k \pi^{-1/2} t^{-1/4} \nu^{-k} \int_1^\nu (\nu - p)^{k-1} p^{1/2} \cos\left(\frac{2}{3} p t^{3/2} - \frac{\pi}{4}\right) dp + t^{-7/4} O(\nu^{-1/2}). \end{aligned}$$

The error made in replacing $p^{1/2}$ by $\nu^{1/2}$ in the integral in (4.5) is

$$(4.6) \quad k \pi^{-1/2} t^{-1/4} \nu^{-k} \int_1^\nu (\nu^{1/2} - p^{1/2})(\nu - p)^{k-1} \cos\left(\frac{2}{3} p t^{3/2} - \frac{\pi}{4}\right) dp.$$

Now

$$(4.7) \quad (\nu^{1/2} - p^{1/2})(\nu - p)^{k-1} = (\nu^{1/2} - p^{1/2})^k (\nu^{1/2} + p^{1/2})^{k-1}.$$

Since $0 \leq k < 1$, both factors on the right-hand side of (4.7) are nonincreasing functions of p , when $1 \leq p \leq \nu$. Thus we can apply the second mean value theorem to the integral in (4.6) to see that the error referred to is bounded, in absolute value, by $K t^{-7/4} \nu^{-1/2}$, where K is some constant which depends only on k . Hence, (4.5) becomes

$$(4.8) \quad \begin{aligned} & C_{k,\nu} \{\nu^{2/3} Ai(-\nu^{2/3} t)\} \\ &= k \pi^{-1/2} t^{-1/4} \nu^{-k+1/2} \int_0^\nu (\nu - p)^{k-1} \cos\left(\frac{2}{3} p t^{3/2} - \frac{\pi}{4}\right) dp + t^{-7/4} O(\nu^{-1/2}), \end{aligned}$$

the effect of the change of interval of integration from $(1, \nu)$ to $(0, \nu)$ being

absorbed in the term $t^{-7/4}O(\nu^{-1/2})$. Now

$$\begin{aligned}
 \int_0^\nu (\nu - p)^{k-1} \cos\left(\frac{2}{3}pt^{3/2} - \frac{\pi}{4}\right) dp \\
 &= (3/2)^k t^{-3k/2} \int_0^{(2/3)\nu t^{3/2}} x^{k-1} \cos\left(x - \frac{2}{3}\nu t^{3/2} + \frac{\pi}{4}\right) dx \\
 &= (3/2)^k t^{-3k/2} \int_0^\infty x^{k-1} \cos\left(x - \frac{2}{3}\nu t^{3/2} + \frac{\pi}{4}\right) dx + t^{-3/2}\nu^{k-1}O(1) \\
 &= (3/2)^k t^{-3k/2} \Gamma(k) \cos\left(\frac{2}{3}\nu t^{3/2} - \frac{k\pi}{2} - \frac{\pi}{4}\right) + t^{-3/2}\nu^{k-1}O(1);
 \end{aligned}$$

the second equality follows from an application of the second mean value theorem, while the third follows on using equations (37) and (38) in [4, p. 13]. If we substitute this result in (4.8), we get (4.1), and the proof is complete.

The desired estimates for the Lebesgue constants are given by the following theorem. (This is motivated by [10, Theorem 4.2]; however, the α appearing there has a different significance.)

Theorem 4.1. *Let $0 < c < \infty$ and suppose that*

$$\Lambda_k(\nu, \alpha) = \int_0^c t^\alpha |C_{k,\nu}\{\nu^{2/3} Ai(-\nu^{2/3}t)\}| dt,$$

where $\nu > 0$, $\alpha \geq 0$ and $0 \leq k < 1$. Then, as $\nu \rightarrow \infty$,

$$(4.9) \quad \Lambda_k(\nu, \alpha) = c_k(\alpha)\nu^{1/2-k} + O(1), \quad 0 \leq k < 1/2,$$

where

$$(4.10) \quad c_k(\alpha) = 2\pi^{-3/2}(3/2)^{k-1}\Gamma(k+1)c^{\alpha+3/4-3k/2}/(\frac{2}{3}\alpha + \frac{1}{2} - k);$$

$$(4.11) \quad \Lambda_{1/2}(\nu, 0) = \pi^{-1}(2/3)^{1/2} \log \nu + O(1);$$

$$(4.12) \quad \Lambda_{1/2}(\nu, \alpha) = O(1), \quad \alpha > 0;$$

and

$$(4.13) \quad \Lambda_k(\nu, \alpha) = O(1), \quad 1/2 < k < 1.$$

Proof. Throughout the proof all O -terms will refer to $\nu \rightarrow \infty$. For all α and k considered,

$$(4.14) \quad \Lambda_k(\nu, \alpha) = \int_{\nu^{-2/3}}^c t^\alpha |C_{k,\nu}\{\nu^{2/3} Ai(-\nu^{2/3}t)\}| dt + O(1),$$

because, since $Ai(t)$ is bounded,

$$|C_{k,\nu}\{\nu^{2/3} Ai(-\nu^{2/3}t)\}| \leq Kk\nu^{-k} \int_0^\nu (\nu - p)^{k-1} p^{2/3} dp \leq K_1\nu^{2/3},$$

where K and K_1 are constants. We now use the result of Lemma 4.1. The contribution from the remainder term in (4.1) to the integral in (4.14) is $O(1)$. Thus (4.14) becomes

$$\Lambda_k(\nu, \alpha) = a_k \nu^{1/2-k} \int_{\nu^{-1/s}}^c t^{\alpha-1/4-3k/2} \left| \cos \left(\frac{2}{3} \nu t^{3/2} - \frac{1}{2} k\pi - \frac{\pi}{4} \right) \right| dt + O(1),$$

a_k being given by (4.2). Making the change of variable

$$(2/3)t^{3/2} = \theta$$

and writing $\beta = 2\alpha/3 - 1/2 - k$, we find

$$(4.15) \quad \Lambda_k(\nu, \alpha) = (3/2)^\beta a_k \nu^{-k+1/2} \cdot \int_{(2/3)\nu^{-1}}^{(2/3)c^{3/2}} \theta^\beta |\cos(\nu\theta - k\pi/2 - \pi/4)| d\theta + O(1).$$

The error made in replacing $|\cos(\nu\theta - k\pi/2 - \pi/4)|$ by its mean value $2/\pi$ in the integral in (4.15) is

$$O(1)\nu^{-2\alpha/3} \int_{2/3}^{(2/3)\nu c^{3/2}} x^\beta \{ |\cos(x - k\pi/2 - \pi/4)| - 2/\pi \} dx,$$

and this is $O(1)$ by the second mean value theorem. From (4.15), we get

$$\Lambda_k(\nu, \alpha) = (2/\pi)(3/2)^\beta a_k \nu^{1/2-k} \int_{(2/3)\nu^{-1}}^{(2/3)c^{3/2}} \theta^\beta d\theta + O(1).$$

Thus

$$(4.16) \quad \begin{aligned} \Lambda_k(\nu, \alpha) &= 4a_k[3\pi(\beta+1)]^{-1}[c^{3(\beta+1)/2} - \nu^{-\beta-1}]\nu^{1/2-k} + O(1), \quad \beta \neq -1; \\ \Lambda_k(\nu, \alpha) &= 4a_k(3\pi)^{-1}[(3/2)\log c + \log \nu]\nu^{1/2-k} + O(1), \quad \beta = -1. \end{aligned}$$

The assertions of the theorem follow readily from (4.16).

5. The principal theorems. We are now in a position to prove our results on the Cesàro summability of a singular integral involving the solution $w(\nu, t)$ of

$$d^2w/dt^2 = [\nu^2t + q(t)]w,$$

on $[a, \infty)$, that satisfies (1.1). It is supposed that $-\infty < a < 0$ and that q satisfies the conditions given in the Introduction.

Theorem 5.1. *Let a function f be defined and Lebesgue integrable on $[b, x]$ where $b < x \leq b - a$, and let $f(x-)$ exist. Then, if $k > 1/2$,*

$$(5.1) \quad (C, k) - \lim_{\nu \rightarrow \infty} \int_b^x f(t)w(\nu, t-x) dt = \frac{2}{3}f(x-).$$

There exists a continuous function f on $[b, x]$ for which

$$C_{1/2, \nu} \left\{ \int_b^x f(t) w(\nu, t - x) dt \right\}$$

is an unbounded function of ν , as $\nu \rightarrow \infty$.

Proof. In view of the equiconvergence theorem [12, Theorem 4.2], and Lemma 2.1, it will be sufficient to prove (5.1) in the case where $w(\nu, t) = \nu^{2/3} Ai(\nu^{2/3} t)$ and $1/2 < k < 1$. However, it is necessary to verify that the (C, k) mean of

$$\int_b^x f(t) w(\nu, t - x) dt$$

has a meaning; that is to say, we must show that the repeated integral

$$k\nu^{-k} \int_1^\nu (\nu - p)^{k-1} \left\{ \int_b^x f(t) w(p, t - x) dt \right\} dp,$$

exists or, equivalently, that

$$k\nu^{-k} \int_1^\nu (\nu - p)^{k-1} \left\{ \int_{b-x}^0 f(\theta + x) w(p, \theta) d\theta \right\} dp,$$

exists. Its existence is a consequence of the continuity of $w(\nu, t)$, when $1 \leq \nu < \infty$ and $a \leq t \leq 0$, which follows, in turn, from the fact that $w(\nu, t) = \nu^{2/3} u(\nu^{2/3} t)$, where $u(x)$ satisfies

$$d^2 u / dx^2 = \{x + \nu^{-4/3} q(\nu^{-2/3} x)\} u$$

on the interval $\nu^{2/3} a \leq x < \infty$.

Now, when $1/2 < k < 1$,

$$\begin{aligned} C_{k, \nu} \left\{ \int_b^x f(t) \nu^{2/3} Ai(\nu^{2/3}(t - x)) dt \right\} \\ &= k\nu^{-k} \int_1^\nu (\nu - p)^{k-1} \left\{ \int_b^x f(t) p^{2/3} Ai(p^{2/3}(t - x)) dt \right\} dp \\ &= k\nu^{-k} \int_1^\nu (\nu - p)^{k-1} \left\{ \int_{b-x}^0 f(\theta + x) p^{2/3} Ai(p^{2/3} \theta) d\theta \right\} dp \\ &= \int_0^{x-b} f(x - \varphi) \left\{ k\nu^{-k} \int_1^\nu (\nu - p)^{k-1} p^{2/3} Ai(-p^{2/3} \varphi) dp \right\} d\varphi \\ &= \int_0^{x-b} f(x - \varphi) [C_{k, \nu} \{ \nu^{2/3} Ai(-\nu^{2/3} \varphi) \}] d\varphi \end{aligned}$$

the interchange of orders of integration being justified by Fubini's theorem.

The result (5.1) will now follow from Theorem 3.1, provided that, when

$1/2 < k < 1$, the following three conditions hold:

$$(i) \quad \lim_{\nu \rightarrow \infty} \int_0^\beta C_{k,\nu} \{ \nu^{2/3} Ai(-\nu^{2/3}t) \} dt = 2/3,$$

for each β satisfying $0 < \beta \leq x - b$;

$$(ii) \quad |C_{k,\nu} \{ \nu^{2/3} Ai(-\nu^{2/3}t) \}| \leq K(\delta, k), \quad (\delta \leq t \leq x - b, \nu \geq \nu_0),$$

for some ν_0 and for each δ satisfying $0 < \delta < x - b$, and

$$(iii) \quad \int_0^{x-b} |C_{k,\nu} \{ \nu^{2/3} Ai(-\nu^{2/3}t) \}| dt \leq K(k), \quad (\nu \geq \nu_0).$$

For each β , (i) follows from [12, p. 861, (3.5)] and Lemma 2.1. Lemma 4.1 shows that (ii) holds, while (iii) follows directly from the result (4.13) of Theorem 4.1. Thus (5.1) is proved.

The final assertion of Theorem 5.1 will follow from Theorem 3.2 and the equiconvergence theorem [12, p. 867, Theorem 4.2], provided that

$$\int_0^{x-b} |C_{1/2,\nu} \{ \nu^{2/3} Ai(-\nu^{2/3}t) \}| dt$$

is an unbounded function of ν . It is clear from the result (4.11) of Theorem 4.1 that this is indeed the case. This completes the proof of Theorem 5.1.

The (C, k) process becomes effective also when $k = 1/2$, provided that an extra assumption is made concerning $f(t)$. The precise result is as follows.

Theorem 5.2. *Let a function f be defined and Lebesgue integrable on $[b, x]$, where $b < x \leq b - a$. Let $f(x-)$ exist and let*

$$(5.2) \quad f(t) = f(x-) + O([x - t]^\alpha), \quad t \rightarrow x-,$$

where α is some positive number. Then, if $k \geq 1/2$,

$$(5.3) \quad (C, k) - \lim_{\nu \rightarrow \infty} \int_b^x f(t)w(\nu, t - x) dt = \frac{2}{3}f(x-).$$

For each $\alpha > 0$ and for each k , $0 \leq k < 1/2$, there exists a continuous function f on $[b, x]$, satisfying (5.2), for which

$$C_{k,\nu} \left\{ \int_b^x f(t)w(\nu, t - x) dt \right\}$$

is an unbounded function of ν , as $\nu \rightarrow \infty$.

Proof. It will be sufficient to prove (5.3) in the case where $k = 1/2$ and $w(\nu, t) = \nu^{2/3} Ai(\nu^{2/3}t)$. The existence of the (C, k) mean of

$$\int_b^x f(t)w(\nu, t - x) dt$$

is established as in the proof of Theorem 5.1.

We may assume, without loss of generality, that $\alpha < 1$. Then

$$\begin{aligned}
 (5.4) \quad C_{1/2, \nu} & \left\{ \int_b^x f(t) \nu^{2/3} Ai(\nu^{2/3}(t-x)) dt \right\} \\
 &= C_{1/2, \nu} \left\{ \int_0^{x-b} f(x-\theta) \nu^{2/3} Ai(-\nu^{2/3}\theta) d\theta \right\} \\
 &= f(x-) C_{1/2, \nu} \left\{ \int_0^{x-b} \nu^{2/3} Ai(-\nu^{2/3}\theta) d\theta \right\} \\
 &\quad + \int_0^{x-b} \theta^{-\alpha} [f(x-\theta) - f(x-)] C_{1/2, \nu} \{ \theta^\alpha \nu^{2/3} Ai(-\nu^{2/3}\theta) \} d\theta
 \end{aligned}$$

where we have used the fact that

$$\begin{aligned}
 C_{1/2, \nu} & \left\{ \int_0^{x-b} [\theta^{-\alpha} (f(x-\theta) - f(x-))] \theta^\alpha \nu^{2/3} Ai(-\nu^{2/3}\theta) d\theta \right\} \\
 &= \int_0^{x-b} \theta^{-\alpha} [f(x-\theta) - f(x-)] C_{1/2, \nu} \{ \theta^\alpha \nu^{2/3} Ai(-\nu^{2/3}\theta) \} d\theta,
 \end{aligned}$$

an easy consequence of Fubini's theorem. The first expression on the right-hand side of (5.4) approaches $(2/3)f(x-)$, as $\nu \rightarrow \infty$, by Lemma 2.1. It will follow from Theorem 3.1 that the second expression on the right-hand side of (5.4) approaches 0 as $\nu \rightarrow \infty$, provided that the following three conditions are satisfied:

$$(i) \quad \lim_{\nu \rightarrow \infty} \int_0^\beta t^\alpha C_{1/2, \nu} \{ \nu^{2/3} Ai(-\nu^{2/3}t) \} dt = 0,$$

for each β satisfying $0 < \beta \leq x - b$;

$$(ii) \quad |t^\alpha C_{1/2, \nu} \{ \nu^{2/3} Ai(-\nu^{2/3}t) \}| \leq K(\delta) \quad (\delta \leq t \leq x - b, \nu \geq \nu_0),$$

for each δ satisfying $0 < \delta \leq x - b$ and some ν_0 ; and

$$(iii) \quad \int_0^{x-b} t^\alpha |C_{1/2, \nu} \{ \nu^{2/3} Ai(-\nu^{2/3}t) \}| dt \leq K \quad (\nu \geq \nu_0).$$

It is clear from Lemma 2.1 and [12, p. 861, Theorem 3.1] that (i) holds for each β . Lemma 4.1 shows that (ii) is satisfied, while (iii) follows from the result (4.12) of Theorem 4.1. This completes the proof of (5.3).

The unboundedness in ν of

$$\int_0^{x-b} t^\alpha |C_{k, \nu} \{ \nu^{2/3} Ai(-\nu^{2/3}t) \}| dt$$

when $0 \leq k < 1/2$ (Theorem 4.1), shows, on applying Theorem 3.2, that there exists a continuous function g on $[0, x - b]$ for which

$$\int_0^{x-b} g(t) t^\alpha C_{k, \nu} \{ \nu^{2/3} Ai(-\nu^{2/3}t) \} dt$$

is an unbounded function of ν . If $f(t) = (x - t)^a g(x - t)$, then f clearly satisfies all the required conditions and the equiconvergence theorem [12, Theorem 4.2] gives the last part of Theorem 5.2.

6. Applications. The results of the present paper, like those of [12], may be applied to integrals involving various special functions. We indicate first the form which our results assume when applied to the solution $W(\nu, x)$, defined in [12, p. 868] of the equation

$$d^2y/dx^2 = \{\nu^2 p(x) + r(x)\}y;$$

$p(x)$ and $r(x)$ are supposed to satisfy the conditions imposed in [12].

Using Theorem 5.1 and the changes of variable (5.7) and (5.8) in [12, p. 868], and with the notation of [12, §5], we get the following result.

Theorem 6.1. *Suppose that the function $g(x)$, defined on $[A, x_0]$, is such that $g(x) \in L[A, x_0]$ and that $g(x_0-)$ exists. Then, if $k > 1/2$,*

$$(C, k) - \lim_{\nu \rightarrow \infty} \int_A^{x_0} g(x) W(\nu, x) dx = \frac{2}{3} g(x_0-).$$

There exists a continuous function $g(x)$ on $[A, x_0]$ for which

$$C_{1/2, \nu} \left\{ \int_A^{x_0} g(x) W(\nu, x) dx \right\}$$

is an unbounded function of ν , as $\nu \rightarrow \infty$.

A similar generalization of Theorem 5.2 can be formulated. These generalizations and the results of [12, §6] may be applied to the functions considered in [12, §6]. We give here only the application to the Whittaker function $W_{\nu, m}(x)$, as defined, for example, in [11, Ch. 7].

Theorem 6.2. *Let $0 < A < 1$, let g be defined on $[A, \infty)$ and let it satisfy: (i) $g(x) \in L(A, 1)$; (ii) $g(1-)$ exists; (iii) $g(1+)$ exists; (iv) $g(x) \exp(-\lambda x) \in L(1, \infty)$ for some real λ . Then, if m is real, and $k > 1/2$*

$$(6.1) \quad (C, k) - \lim_{\nu \rightarrow \infty} [\Gamma(\nu)]^{-1} \int_A^{\infty} g(x) W_{\nu, m}(4\nu x) dx = \frac{2}{3} g(1-) + \frac{1}{3} g(1+).$$

If, in addition,

$$(6.2) \quad g(x) = g(1-) + O[(1-x)^\alpha], \quad x \rightarrow 1-,$$

for some $\alpha > 0$, then (6.1) is valid for $k = 1/2$. There exists a continuous function g on $[A, 1]$ such that

$$C_{1/2, \nu} \left\{ [\Gamma(\nu)]^{-1} \int_A^1 g(x) W_{\nu, m}(4\nu x) dx \right\}$$

is an unbounded function of ν , as $\nu \rightarrow \infty$; for each k satisfying $0 \leq k < 1/2$ and for each $\alpha > 0$ there exists a continuous function g on $[A, 1]$, satisfying (6.2),

such that

$$C_{k,\nu} \left\{ [\Gamma(\nu)]^{-1} \int_A^1 g(x) W_{\nu,m}(4\nu x) dx \right\}$$

is an unbounded function of ν , as $\nu \rightarrow \infty$.

Remarks on the proof. Theorem 6.2 follows on applying Theorem 6.1 and the corresponding analogue of Theorem 5.2 to the case where

$$W(\nu, x) = A_\nu W_{\nu,m}(4\nu x)$$

and

$$A_\nu = (2\pi)^{-1/2} e^\nu \nu^{1/2-\nu}$$

(cf [12, p. 870]). This gives the conclusion of Theorem 6.2 with the factor $[\Gamma(\nu)]^{-1}$ replaced by A_ν . Stirling's formula shows that

$$[\Gamma(\nu)]^{-1} = A_\nu [1 + O(\nu^{-1})], \quad \nu \rightarrow \infty.$$

Thus, (6.1) follows from the corresponding result with $[\Gamma(\nu)]^{-1}$ replaced by A_ν since

$$(6.3) \quad \lim_{\nu \rightarrow \infty} \nu^{-1} A_\nu \int_1^\infty g(x) W_{\nu,m}(4\nu x) dx = 0$$

and

$$(6.4) \quad \lim_{\nu \rightarrow \infty} \nu^{-1} A_\nu \int_A^1 g(x) W_{\nu,m}(4\nu x) dx = 0.$$

The assertion (6.3) follows from [12, Theorem 6.1] and (6.4) follows from the fact that under the hypotheses of Theorem 5.1, using the boundedness of $\nu^{-2/3} w(\nu, t)$, we have

$$\lim_{\nu \rightarrow \infty} \nu^{-1} \int_a^x f(t) w(\nu, t - x) dt = 0.$$

Lemma 2.1 shows that the following corollary holds; cf. [10, p. 205, Corollary].

Corollary 6.1. *If $g(x)$ satisfies the hypotheses (i), (ii), (iii) and (iv) of Theorem 6.2 and if*

$$\lim_{\nu \rightarrow \infty} [\Gamma(\nu)]^{-1} \int_A^\infty g(x) W_{\nu,m}(4\nu x) dx$$

exists, it is equal to $\frac{2}{3}g(1-) + \frac{1}{3}g(1+)$.

Finally, we remark that using an appropriate differential equation and changes of variable (cf. [12, p. 872]) Theorem 5.1 can be used to rederive the result of Lorch and Szego [10, Theorem 4.1] on the Cesàro summability of a singular integral involving the Bessel function $J_\nu(t)$.

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