A Note on a Nonlinear Elliptic Equation

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This note concerns the elliptic boundary value problem

$$(1) \quad \begin{cases} \mathcal{L}u \equiv \sum_{|\sigma| \leq 2m} a_\sigma(x) D^\sigma u = g(x, u), & x \in \Omega, \\ \mathcal{B}_i u \equiv \sum_{|\sigma| \leq m_i} b_{i\sigma}(x) D^\sigma u = 0, & x \in \partial\Omega, \end{cases}$$

where $1 \leq i \leq m$, $m_i < 2m$, $x = (x_1, \dots, x_n) \in \Omega$ which is a smooth bounded domain in $\mathbb{R}^n$, $\mathcal{L}$ is uniformly elliptic [1] with smooth coefficients, g is continuous on $\bar{\Omega} \times \mathbb{R}$, the $\mathcal{B}_i$ have smooth coefficients, and the usual multi-index notation is being employed. Moreover the boundary conditions are complementing in the sense of [1]. Under the above conditions on $\mathcal{L}$ and the $\mathcal{B}_i$, $\mathcal{L}$ is a Fredholm map on the space of functions satisfying the boundary conditions (in various topologies), i.e., $\mathcal{L}$ has a finite dimensional kernel, a finite codimensional range in an appropriate space and the range is closed. In particular the Fredholm index of $\mathcal{L}$, $\chi = \dim \ker \mathcal{L} - \text{codim range } \mathcal{L}$ is finite.

We will obtain some existence results for (1) when $\chi > 0$ provided that g is odd in u . The oddness of g implies that $u = 0$ satisfies (1). In addition to this trivial solution (1) possesses infinitely many solutions when $\chi > 0$, in fact an unbounded component of solutions containing $u = 0$. Moreover a sharper estimate for the number of solutions of (1) can be given in terms of χ . This last statement follows from a general result (Theorem 14) from nonlinear functional analysis which is of independent interest. Precise existence statements for (1) are given in Theorems 9 and 11 below.

Our study of (1) was motivated by a recent paper of L. Nirenberg [2]. When certain technical conditions on $\mathcal{L}$, $\mathcal{B}_i$, and g were satisfied and $\chi \geq 0$, Nirenberg obtained a sufficient condition for the existence of solutions of (1). His criterion was in terms of the topological degree of a mapping when $\chi = 0$ and in terms of a generalized version of topological degree when $\chi > 0$. The $\chi = 0$ case contained an earlier result of Landesman and Lazer [3].

The various technical conditions of [2] are not needed here. Moreover our main tool is the usual Leray–Schauder degree theory together with the notion

of genus rather than Nirenberg's generalized degree. The results we obtain are disjoint from those of [2] since when g is odd in u , the mappings required in [2] cannot be constructed.

Essentially following Nirenberg, (1) will be converted to an equivalent operator equation. However to minimize regularity technicalities we prefer to use Hölder norms rather than work with Sobolev spaces as in [2]. Accordingly assume g is a continuously differentiable function on $\bar{\Omega} \times \mathbf{R}$. Let $\alpha \in (0, 1)$ and let $C^{j+\alpha}(\Omega)$ denote the set of j times continuous differentiable functions on $\bar{\Omega}$ whose j th derivatives are Hölder continuous in Ω with exponent α . Then $C^{j+\alpha}(\Omega)$ is a Banach space under the norm

$$\|u\|_{j+\alpha} = \sum_{|\sigma| \leq j} \max_{x \in \Omega} |D^\sigma u(x)| + \sum_{|\sigma|=j} \max_{\substack{x, y \in \Omega \\ x \neq y}} \frac{|D^\sigma u(x) - D^\sigma u(y)|}{|x - y|^\alpha}.$$

It follows from [1] that $\mathcal{L}$ is a Fredholm map from the set of $C^{2m+\alpha}(\Omega)$ functions satisfying the boundary conditions to $C^\alpha(\Omega)$. Let $V_k \equiv \text{span} \{v_1, \dots, v_k\}$ denote the kernel of $\mathcal{L}$ and $W_i \equiv \text{span} \{w_1, \dots, w_i\}$ denote the cokernel of $\mathcal{L}$. Set $E = C^\alpha(\Omega)$. Since V_k, W_i are isomorphic to $\mathbf{R}^k, \mathbf{R}^i$ respectively, we can write $E = \mathbf{R}^k \times \hat{E} = \mathbf{R}^i \times \tilde{E}$ where $\hat{E}$ and $\tilde{E}$ are respectively the subspaces of E orthogonal in an L^2 sense to V_k, W_i , i.e., $\varphi \in \hat{E}$ means

$$\int_{\Omega} \varphi v_i \, dx = 0, \quad 1 \leq i \leq k,$$

etc. Thus $u \in E$ implies $u = \sum_1^k \hat{a}_i v_i + \hat{u} = \sum_1^i \tilde{a}_i w_i + \tilde{u}$ or $u = (\hat{a}, \hat{u}) = (\tilde{a}, \tilde{u})$. An equivalent norm for E is $|u|_\alpha = \max(|\hat{a}|, \|\hat{u}\|_\alpha)$.

Let $\tilde{P}u = \tilde{u}$, i.e., $\tilde{P}$ is the L^2 orthogonal projector of E to $\tilde{E}$. Consider the inhomogeneous linear problem

$$(2) \quad \begin{cases} \mathcal{L}\hat{U} = \tilde{P}f, & x \in \Omega, \\ \mathcal{B}_i \hat{U} = 0, & x \in \partial\Omega, \quad 1 \leq i \leq m, \end{cases}$$

where $f \in E$. According to the existence theory of [1], there exists a unique $\hat{U} \in C^{2m+\alpha}(\Omega) \cap \hat{E}$ satisfying (2).

With these preliminaries in hand, (1) can be converted to an operator equation in E . Let $u = (\hat{a}, \hat{u}) \in E$ and let $G(u) = (\hat{A}, \hat{U}) \equiv (A(u), T(u))$ where (for $\chi \geq 0$)

$$(3) \quad \begin{cases} \hat{A}_i = \hat{a}_i - \int_{\Omega} g(x, u) w_i \, dx, & 1 \leq i \leq j, \\ = \hat{a}_i, & j+1 \leq i \leq k \quad (\text{if } k > j), \end{cases}$$

and

$$(4) \quad \begin{cases} \mathcal{L}\hat{U} = \tilde{P}g(x, u), & x \in \Omega, \\ \mathcal{B}_i \hat{U} = 0, & x \in \partial\Omega, \quad 1 \leq i \leq m. \end{cases}$$

Here $\hat{U} \in C^{2m+\alpha}(\Omega)$ is the unique solution of (4) in $\hat{E}$. The Schauder estimates [1] imply

$$(5) \quad \|\hat{U}\|_{2m+\alpha} \leq c \|\tilde{P}g(x, u)\|_{\alpha}.$$

Hence it easily follows from (3)–(5) that the map $G: E \rightarrow E$ is compact, i.e., is continuous and maps bounded sets to relatively compact sets. Moreover u is a solution of (1) if and only if $G(u) = u$.

Let $\Phi(u) = u - G(u)$. Then $\Phi: E \rightarrow E$. Moreover if $\chi > 0$, (3)–(4) show the range of Φ is a proper subspace of E . Henceforth we assume g is an odd function of u . Then G is an odd function of $u = (\hat{a}, \hat{u})$, i.e., $G(-(\hat{a}, \hat{u})) = -G((\hat{a}, \hat{u}))$. Formalizing what we have shown:

Lemma 6. *If g is an odd continuously differentiable function of u , G is a compact odd map of $E \rightarrow E$. Moreover if $\chi > 0$, then $\Phi(u) = u - G(u)$ maps E to a proper subspace of E .*

Next some preliminaries concerning Leray–Schauder degree are needed. Let $\mathcal{E}$ be a real Banach space and $\mathcal{Q}$ a bounded open subset of $\mathcal{E}$. Suppose $\Psi(u) = u - K(u)$ where K is a compact operator on $\mathcal{E}$. Let $b \in \mathcal{E}$, $b \notin \Psi(\partial\mathcal{Q})$. Then the Leray–Schauder degree of Ψ with respect to $\mathcal{Q}$ and b is well defined and will be denoted by $d(\Psi, \mathcal{Q}, b)$. For the properties of Leray–Schauder degree, see e.g., [4]. A set $\mathcal{Q} \subset \mathcal{E}$ will be called symmetric if $u \in \mathcal{Q}$ implies $-u \in \mathcal{Q}$. One of the important properties of degree is the Borsuk Antipodensatz:

Lemma 7. *Let $\mathcal{Q}$ be a symmetric bounded open neighborhood of 0 in $\mathcal{E}$. Suppose $\Psi(u) = u - K(u)$ where K is compact on $\mathcal{E}$, $\Psi \neq 0$ on $\partial\mathcal{Q}$, and Ψ is odd on $\partial\mathcal{Q}$. Then $d(\Psi, \mathcal{Q}, 0)$ is an odd integer.*

Under the hypotheses of Lemma 7 the continuity of $d(\Psi, \mathcal{Q}, b)$ in b implies $\Psi(\mathcal{Q})$ contains a neighborhood of 0 in $\mathcal{E}$. Thus an easy and well known corollary of Lemma 7 is:

Corollary 8. *Let $\mathcal{Q}$ be as in Lemma 7 and $\Psi(u) = u - K(u)$ where K is compact. If Ψ is odd on $\partial\mathcal{Q}$ and $\Psi(\mathcal{Q})$ is a proper subspace of $\mathcal{E}$, then there exists $\xi \in \partial\mathcal{Q}$ such that $\Psi(\xi) = 0 = \Psi(-\xi)$.*

Proof. If not, by Lemma 7 $d(\Psi, \mathcal{Q}, 0) \neq 0$ and hence by our above remark $\Psi(\mathcal{Q})$ contains a neighborhood of 0. But this is contrary to the fact that $\Psi(\mathcal{Q})$ is contained in a proper subspace of $\mathcal{E}$.

Combining Lemma 6 and Corollary 8 immediately gives:

Theorem 9. *Let $\mathcal{Q}$ be any symmetric bounded open neighborhood of 0 in E . Then if $\chi > 0$, there exists $u \in \partial\mathcal{Q}$ such that $\Phi(u) = 0 = \Phi(-u)$.*

Since zeros of Φ are solutions of (1), Theorem 9 implies (1) possesses solutions on the boundary of every symmetric bounded open neighborhood of 0 in E . A standard consequence of such a statement (see e.g., [5]) is:

Corollary 10. *The set of zeros of Φ possesses a symmetric unbounded component $\mathcal{C}$ containing $u = 0$.*

Proof. Note that components of zeros of Φ occur in antipodal pairs because of the oddness of Φ . Suppose the statement of Corollary 10 is not true. Then $u = 0$ belongs to a symmetric bounded component $\mathcal{C}$ of zeros of Φ . Let $\mathcal{O}$ be a δ neighborhood of $\mathcal{C}$ (i.e., all points within δ of $\mathcal{C}$) and let $\mathcal{S}$ denote the set of zeros of Φ . The compactness of G and boundedness of $\bar{\mathcal{O}}$ implies that $\mathcal{S} \cap \bar{\mathcal{O}}$ is a compact metric space under the induced topology from $\mathcal{E}$. Let $A = \partial\mathcal{O} \cap \mathcal{S}$ and $B = \mathcal{C}$. A lemma from point set topology [6] implies there exist disjoint compact sets $\hat{A}, \hat{B} \subset \bar{\mathcal{O}}$ such that $\hat{A} \supset A, \hat{B} \supset B$ and $\mathcal{S} \cap \bar{\mathcal{O}} = \hat{A} \cup \hat{B}$. Let $\mathcal{Q}$ be an ϵ neighborhood of $\hat{B}$ where ϵ is less than the distance from $\hat{A}$ to $\hat{B}$. Then $\mathcal{C} \subset \mathcal{Q}$ and $\partial\mathcal{Q} \cap \mathcal{S} = \emptyset$. It is clear that $\mathcal{Q}$ is a symmetric neighborhood of 0 but $\partial\mathcal{Q}$ contains no zeros of Φ , contrary to Theorem 9.

The Schauder estimates imply the map $\mathcal{C} \rightarrow C^{2m+\alpha}(\Omega)$, $u \rightarrow u$ is continuous. Hence $\mathcal{C}$ considered as a subset of $C^{2m+\alpha}(\Omega)$ is connected. It is also unbounded since $\|\cdot\|_{2m+\alpha} \geq \|\cdot\|_{\alpha}$. By working in $C^{2m-1+\alpha}(\Omega)$ instead of $C^{\alpha}(\Omega)$, results similar to the above can be obtained if g depends on the derivatives of u up to order $2m - 1$ provided that it is an odd function of these additional arguments. Likewise the problem treated by Nirenberg can be handled by the same techniques.

Next a sharper version of Theorem 9 will be obtained. Our goal is to get a better estimate for the number of zeros of Φ on $\partial\mathcal{Q}$ in terms of χ . For that purpose the notion of *genus* [7–9] will be employed. For what follows if A and B are topological spaces, let $C(A, B)$ denote the space of continuous maps from A to B . Let $\mathcal{E}$ be a real Banach space and $\Sigma(\mathcal{E})$ the family of closed symmetric subsets of $\mathcal{E} - \{0\}$. For $A \in \Sigma(\mathcal{E})$, $A \neq \emptyset$, we say A has *genus* q , denoted by $\gamma(A) = q$, if there exists $\varphi \in C(A, \mathbb{R}^q - \{0\})$ where φ is odd and q is the smallest integer having this property. In addition $\gamma(\emptyset) \equiv 0$ and $\gamma(A) = \infty$ if there is no finite q as above.

The generalization of Theorem 9 we will obtain is:

Theorem 11. *Let Φ and $\mathcal{Q}$ be as in Theorem 9. If $A = \{u \in \partial\mathcal{Q} \mid \Phi(u) = 0\}$, then $\gamma(A) \geq \chi$.*

Thus in particular if $\chi > 1$, (1) possesses infinitely many solutions on $\partial\mathcal{Q}$ for each such $\mathcal{Q}$. It may occur that $\gamma(A) > \chi$, as for example when $g \equiv 0$ and the dimension of the cokernel of $\mathcal{L}$ is positive. Theorem 11 is a consequence of a general result for mappings having the properties of Φ . For its proof some preliminaries are required.

Lemma 12. *Let $A, B \in \Sigma(\mathcal{E})$.*

- 1°. *If there exists an odd $\varphi \in C(A, B)$, then $\gamma(A) \leq \gamma(B)$.*
- 2°. *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
- 3°. *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.*
- 4°. *If $\gamma(B) < \infty$, then $\gamma(A - B) \geq \gamma(A) - \gamma(B)$.*

- 5°. If A is compact, then $\gamma(A) < \infty$.
 6°. If A is compact, there exists a uniform neighborhood $N_\delta(A) = \{u \in \mathcal{E} \mid \|u - A\| \leq \delta\}$ of A such that $\gamma(N_\delta(A)) = \gamma(A)$.
 7°. If A is homeomorphic by an odd homeomorphism to the unit sphere in $\mathbf{R}^p$, then $\gamma(A) = p$.

The proofs of 1°–7° are fairly elementary and can be found in [8] or [9]. With the aid of Lemma 12 the following extension of Corollary 8 for finite dimensional $\mathcal{E}$ obtains:

Theorem 13. Suppose $\mathcal{O}$ is a symmetric bounded open neighborhood of 0 in $\mathbf{R}^q$ and $\varphi \in C(\partial\mathcal{O}, F)$ where φ is odd and F is a proper subspace of $\mathbf{R}^q$ which is isomorphic to $\mathbf{R}^p$. If $Z = \{x \in \partial\mathcal{O} \mid \varphi(x) = 0\}$, then $\gamma(Z) \geq q - p$.

Proof. By 6° of Lemma 12, there exists a neighborhood of Z , $N_\delta(Z)$ such that $\gamma(N_\delta(Z)) = \gamma(Z)$. Since φ is continuous on $\partial\mathcal{O}$ and Z is compact it can further be assumed that there exists an $\epsilon_0 > 0$ such that $N_\delta(Z) \supset Z_\epsilon = \{x \in \partial\mathcal{O} \mid |\varphi(x)| \leq \epsilon\}$ for all $\epsilon \leq \epsilon_0$. By 2° of Lemma 12, $\gamma(Z_\epsilon) = \gamma(Z)$. Let $Y_\eta = \{x \in \partial\mathcal{O} \mid |\varphi(x)| \geq \eta\}$ where $\eta > 0$, and let $r(x) = x/|x|$ for $x \in \mathbf{R}^q - \{0\}$. Then $r \circ \varphi$ is odd and in $C(Y_\eta, S_F)$ where S_F is the unit sphere in F . Since F is isomorphic to $\mathbf{R}^p$, by 1° and 7° of Lemma 12, $\gamma(Y_\eta) \leq \gamma(S_F) = p$. Hence by 4° of Lemma 12, $\gamma(\partial\mathcal{O} - Y_\eta) \geq \gamma(\partial\mathcal{O}) - \gamma(Y_\eta) \geq \gamma(\partial\mathcal{O}) - p$. We can extend φ , e.g., via the Tietze Extension Theorem to an element of $C(\bar{\mathcal{O}}, F)$. Then the definition of genus and Corollary 8 imply $\gamma(\partial\mathcal{O}) = q$. Thus $\gamma(\partial\mathcal{O} - Y_\eta) \geq q - p$. But $\partial\mathcal{O} - Y_\eta = Z_\eta$ so choosing $\eta = \epsilon \leq \epsilon_0$, we find $\gamma(Z_\epsilon) = \gamma(Z) \geq q - p$.

Next the infinite dimensional version of Theorem 13 will be obtained. Let $\mathcal{E} = \mathbf{R}^k \times \hat{\mathcal{E}}$ be a real Banach space with $\hat{\mathcal{E}}$ a closed linear subspace. For $u = (a, \hat{u}) \in \mathcal{E}$, let $\Psi(u) = u - K(u)$ where K is compact and odd on $\mathcal{E}$ and $K(u) = (A(u), T(u))$ with $A: \mathcal{E} \rightarrow \mathbf{R}^k$ and $T: \mathcal{E} \rightarrow \hat{\mathcal{E}}$.

Theorem 14. Let $\mathcal{O}$ be a symmetric bounded open neighborhood of 0 in $\mathcal{E}$. Suppose for all $u \in \partial\mathcal{O}$, $a = A(u) \in \mathbf{R}^j$ where $j < k$. If $Z = \{u \in \partial\mathcal{O} \mid \Psi(u) = 0\}$, then $\gamma(Z) \geq k - j$.

Proof. The form of Ψ implies Z is a compact symmetric subset of $\mathcal{E}$. Therefore by 5° of Lemma 12, $\gamma(Z) < \infty$ and by 6° Z has a uniform neighborhood $N_\delta(Z)$ such that $\gamma(N_\delta(Z)) = \gamma(Z)$. There exists an $\eta_0 > 0$ such that $N_\delta(Z) \supset Z_\eta = \{u \in \partial\mathcal{O} \mid \|\Psi(u)\| \leq \eta\}$ for all $\eta \leq \eta_0$, for otherwise there is a sequence $(u_a) \subset \partial\mathcal{O} - N_\delta(Z)$ such that $\|\Psi(u_a)\| \leq 1/q$. Since

$$(15) \quad u_a = K(u_a) + \Psi(u_a)$$

and K is compact, the right hand side of (15) has a convergent subsequence. Hence some subsequence $u_{a_i} \rightarrow v \in \partial\mathcal{O} - N_\delta(Z)$ and from (15) $\Psi(v) = 0$. But this is impossible since $v \notin Z$. Thus $N_\delta(Z) \supset Z_\eta$ and by 2° of Lemma 12, $\gamma(Z_\eta) = \gamma(Z)$.

T can be approximated by a finite dimensional odd map by modifying a somewhat standard argument ([10] or [11]). Since $\overline{T(\partial\mathcal{O})}$ is a compact subset of $\hat{\mathcal{E}}$, for all $\epsilon > 0$ there exist $f_i, f_{-i} = -f_i \in \hat{\mathcal{E}}, 1 \leq i \leq p(\epsilon)$ such that $w \in \overline{T(\partial\mathcal{O})}$ implies $\|w - f_i\| \leq \epsilon$ for some $i, 1 \leq |i| \leq p$. Define

$$\sigma_\epsilon(w) = \frac{\sum_{1 \leq |i| \leq p} b_i(w) f_i}{\sum_{1 \leq |i| \leq p} b_i(w)}$$

where $b_i(w) = \epsilon - \|w - f_i\|$ if $\|w - f_i\| \leq \epsilon$ and $b_i(w) = 0$ otherwise. Note that $\sigma_\epsilon(-w) = -\sigma_\epsilon(w)$. For $u \in \partial\mathcal{O}$, let $K_\epsilon(u) = (A(u), \sigma_\epsilon(T(u)))$. It is fairly easy to show ([10], [11]) that $\|K_\epsilon(u) - K(u)\| \leq \epsilon$ for all $u \in \partial\mathcal{O}$.

Let $\Psi_\epsilon(u) = u - K_\epsilon(u)$ and $\text{span}\{f_1, \dots, f_p\} = S_p$. Therefore $\Psi_\epsilon: \partial\mathcal{O} \rightarrow \mathbf{R}^j \times S_p$. In particular Ψ_ϵ is an odd continuous map from $\partial\mathcal{O} \cap (\mathbf{R}^k \times S_p) \rightarrow \mathbf{R}^j \times S_p$. By Theorem 13, $\gamma(\{u \in \partial\mathcal{O} \cap (\mathbf{R}^k \times S_p) \mid \Psi_\epsilon(u) = 0\}) \geq k - j$. Hence 2° of Lemma 12 implies

$$(16) \quad \gamma(\{u \in \partial\mathcal{O} \mid \|\Psi_\epsilon(u)\| \leq \epsilon\}) \geq k - j.$$

If $u \in \partial\mathcal{O}$ and $\Psi(u) = 0$, then $\|\Psi_\epsilon(u)\| \leq \|K(u) - K_\epsilon(u)\| \leq \epsilon$. Similarly if $u \in \partial\mathcal{O}$ and $\|\Psi_\epsilon(u)\| \leq \epsilon$, then $\|\Psi(u)\| \leq \|\Psi_\epsilon(u)\| + \|K_\epsilon(u) - K(u)\| \leq 2\epsilon$. Therefore

$$(17) \quad \gamma(\{u \in \partial\mathcal{O} \mid \|\Psi(u)\| \leq 2\epsilon\}) \geq \gamma(\{u \in \partial\mathcal{O} \mid \|\Psi_\epsilon(u)\| \leq \epsilon\}) \geq \gamma(Z).$$

Choosing $2\epsilon \leq \eta_0$, then $\gamma(Z) = \gamma(Z_{2\epsilon})$ and combining (16)–(17) then yields Theorem 14.

Remark. It is easy to extend Theorem 14 to the case where K is only defined and odd on $\partial\mathcal{O}$.

Theorem 11 now follows immediately from Theorem 14 with $s = \chi$.

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