Periodic Orbits of the Second Kind

RICHARD B. BARRAR

Communicated by the Editors

Introduction. In Volume 1, Chapter 3 of his “*Les Méthodes Nouvelles de la Mécanique Céleste*”, Poincaré discusses periodic orbits. In particular, in Sections 42–48 he discusses periodic orbits of the second and third kind. At one point in his discussion Poincaré utilizes a result about doubly periodic functions that later P. Stäckel [4] showed to be false. Wintner [7, p. 47] also found fault with this work. Thus it is no surprise that mathematicians and astronomers have not utilized these sections although they pertain to a very important problem.

In recent years, starting with a paper of R. Arenstorf [1], mathematicians have returned to these problems, and have devised new methods to prove special cases of Poincaré’s results.

In the present paper we prove the existence of periodic orbits of the second kind for the three body problem, when the unperturbed primaries do not collide. Our analysis easily extends to periodic orbits of the third kind.

Our proof is based on a recent paper of A. Weinstein [6].

We begin with some notation and a statement of the problem.

Three body problem. The three body problems involve the notion of three points p_1 , p_2 , p_3 of masses m_1 , m_2 , m_3 respectively which attract each other according to the Newtonian laws of gravitation. For the purposes of the present paper we will assume that the three bodies move in the fixed x, y plane.

By utilizing the conservation of linear momentum we may assume that the center of mass is at rest at the center of coordinates. Hence to describe the motion it is sufficient to describe the relative position of two of the bodies. Let (x_1, x_2) be the position vector of p_2 with reference to p_1 , and let (x_3, x_4) be the position vector of p_3 with respect to the center of mass of p_1 and p_2 . Then as developed, for example, in Sternberg [5, Chapter IV, Section 8], with a normalization introduced by Jeffreys and Moser [2], the equations of motion are Hamiltonian in form

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial x_i}, \quad \frac{dX_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad i = 1, \dots, 4,$$

in phase space $\mathbf{R}^8 = \{(\mathbf{x}_i, \mathbf{X}_i), i = 1, \dots, 4\}$, with

$$(1) \quad H = \left[\frac{M_3}{2M_1} (X_1^2 + X_2^2) - \frac{m_1 m_2}{M_3 r_1} \right] \\ + \left[\frac{M_3}{2M_2} (X_3^2 + X_4^2) - \frac{m_3(m_1 + m_2)}{M_3 r_2} \right] + \frac{U}{M_3}, \\ U = -\frac{m_3(m_1 + m_2)}{r_2} + \frac{m_2 m_3}{\zeta_{23}} + \frac{m_1 m_3}{\zeta_{13}}$$

with

$$M_1 = \frac{m_1 m_2}{m_1 + m_2}, \quad M_2 = \frac{m_3(m_1 + m_2)}{m_1 + m_2 + m_3}, \quad M_3 = M_1(m_1 + m_2)^{2/3},$$

$r_1 = (x_1^2 + x_2^2)^{1/2}$, $r_2 = (x_3^2 + x_4^2)^{1/2}$ and where ζ_{12} , ζ_{23} , ζ_{13} represent the distance between the point masses.

The conservation of angular momentum equation is:

$$(2) \quad (X_1 x_2 - X_2 x_1) + (X_3 x_4 - X_4 x_3) = C.$$

We assume the masses of m_1 and m_3 are very small compared to the mass of m_2 and we let

$$\epsilon = \frac{m_1 + m_3}{m_1 + m_2 + m_3} \ll 1.$$

Then $U = O(\epsilon)$ is considered as a perturbation of the two body motions represented by the first two terms in the Hamiltonian.

Periodic orbits. Periodic orbits in the three body problem are best described in terms of polar coordinates since they are not truly periodic in terms of cartesian coordinates (see Poincaré [3, Section 39]). Let $\varphi_i = \arctan(x_{2i-1}/x_{2i})$, $r_i = ((x_{2i-1})^2 + (x_{2i})^2)^{1/2}$, $p_{r_i} = (X_{2i-1}x_{2i-1} + X_{2i}x_{2i})/r_i$, $p_{\varphi_i} = X_{2i-1}x_{2i} - X_{2i}x_{2i-1}$, $i = 1, 2$. A periodic orbit of the first or second kind of period T is one which satisfies:

$$\begin{aligned} r_i(T) &= r_i(0), & i &= 1, 2, \\ p_{r_i}(T) &= p_{r_i}(0), & i &= 1, 2, \\ p_{\varphi_i}(T) &= p_{\varphi_i}(0), & i &= 1, 2, \\ \varphi_i(T) &= \varphi_i(0) + \alpha T, & i &= 1, 2, \alpha \text{ arbitrary.} \end{aligned}$$

Thus to be periodic in cartesian coordinates a periodic orbit of the first or second kind might further require a rotation of the coordinate system. Periodic orbits of the second kind refer to those branching off from at least one elliptic orbit at $\epsilon = 0$, while periodic orbits of the first kind refer to those branching off from two circular orbits. In a similar fashion periodic orbits of the third kind refer to orbits branching off from orbits that are not planar.

Weinstein's Theorem and Corollary. The theoretical basis for our investigation is the following theorem of Weinstein [6]:

Theorem 1. *Let M be a symplectic manifold (not necessarily compact) with symplectic structure Ω , $f: M \rightarrow M$ a symplectic automorphism and Σ a compact regular non-degenerate fixed manifold for f . Then there is a neighborhood U of Σ in M and a C^1 neighborhood W of f in the space of symplectic automorphisms such that each $g \in W$ has at least $c(\Sigma)$ fixed points in U . In fact, there is a mapping P from W to the space of closed 1-forms on U and a mapping I from W to the space of embeddings of Σ in U with the following properties:*

- (i) *The set of fixed points of g in U is equal to the zero set of the pullback of $P(g)$ to the image $I(g)$.*
- (ii) *$I(f)$ is the identity, and $P(f)$ has Σ as a non-degenerate zero manifold.*
- (iii) *I and P are continuous in the C^1 topologies and preserve smooth dependence on parameters.*

The following definitions are used in the statement of the theorem:

- (a) Let M be any manifold $U \subseteq M$ an open subset, $\Sigma \subseteq U$ a submanifold, and $f: U \rightarrow M$ a differentiable mapping. Σ is called fixed if the restriction of f to Σ is the identity.

If f is a fixed manifold for f , $T_{\Sigma}f: T_{\Sigma}M \rightarrow T_{\Sigma}M$ is a bundle automorphism covering the identity and leaving $T\Sigma \subseteq T_{\Sigma}M$ fixed. The fixed submanifold Σ is called non-degenerate if the fixed set of $T_{\Sigma}f$ is equal to $T\Sigma$. The fixed submanifold is called regular if there exists a symplectic bundle isomorphism $R: T_{\Sigma}M \rightarrow T_{\Sigma}M$ such that $R - 1$ and $R - T_{\Sigma}f$ are invertible.

- (b) For any 1-form ω on a manifold Σ , denote by $\#(\omega)$ the number of points of Σ at which ω is zero. Denote by $c(\Sigma)$ the minimum value of $\#(\omega)$, where ω ranges over all closed 1-forms on Σ .

For our purposes it will be convenient to modify the Weinstein Theorem to the following form:

Corollary 1. *Assume in the above theorem:*

- (a) *M is contractible, hence there is a 1-form ω on M such that $-d\omega = \Omega$;*
- (b) *g is such that $\omega - g^*\omega = dS$, where S is a function on M .*

Then $P(g) = dS_1$ where S_1 is a function on U . Hence, in particular, the Lusternik-Schnirelmann theory of category may be applied to find the zero set of the pullback of $P(g)$ to the image of $I(g)$.

Proof. In Weinstein's notation, $\omega_{\times} = \pi_1^*\omega - \pi_2^*\omega$ is a 1-form on $M \times M$, with $d\omega_{\times} = -\Omega_{\times}$; and $\tilde{\Omega}^*\omega_M$ is a 1-form on $T(M)$ with $d\tilde{\Omega}^*\omega_M = -\tilde{\Omega}^*\Omega_M$, where ω_M and $\Omega_M = -d\omega_M$ are the canonical 1-form and canonical symplectic structure on T^*M .

Weinstein proves that there is a symplectic diffeomorphism Λ_R from a neighborhood A of Z (the zero section) in TM onto a neighborhood B of Δ (the diagonal) in $M \times M$.

Thus, since

$$\Omega_{\times} = (\Lambda_R^{-1})^* \tilde{\Omega}^*(\Omega_M)$$

in B , there exists a function S_2 defined in B such that

$$(3) \quad \omega_{\times} = (\Lambda_R^{-1})^* \tilde{\Omega}^*(\omega_M) + dS_2 .$$

Further, proceeding as Weinstein does in his Proposition 2.2.1, one finds for the graph of g , $(1_M \times g)(M)$:

$$(4) \quad \begin{aligned} (1_M \times g)^* \omega_{\times} &= (1_M \times g)^* (\pi_1^* \omega - \pi_2^* \omega) \\ &= \omega - g^* \omega = dS . \end{aligned}$$

Weinstein defines

$$P(g) = \tilde{\Omega} \circ \Lambda_R^{-1} \circ (1_M \times g) \circ (\pi \circ \Lambda_R^{-1} \circ (1_M \times g))^{-1}: U \rightarrow T^*M .$$

Thus by the characterizing property of the 1-form ω_M :

$$\begin{aligned} P(g) &= P(g)^*(\omega_M) \\ &= [\pi \circ \Lambda_R^{-1} \circ (1_M \times g)]^{-1*} \circ (1_M \times g)^* \circ (\Lambda_R^{-1})^* \circ \tilde{\Omega}^*(\omega_M) \\ &= [\pi \circ \Lambda_R^{-1} \circ (1_M \times g)]^{-1*} \circ (1_M \times g)^* [\omega_{\times} - dS_2] \\ &= [\pi \circ \Lambda_R^{-1} \circ (1_M \times g)]^{-1*} \circ [dS - (1_M \times g)^* dS_2] . \end{aligned}$$

Finally since the neighborhoods W and U are so chosen that $(1 \times g)(U) \subset B$ for $g \in W$, the proof is complete.

Basic assumption. Before stating our main theorem, let us introduce some notation.

If the unperturbed orbits of equation (1) are for $\epsilon = 0$ of the form:

$$(5) \quad \frac{1}{2\nu_i} (X_{2i-1}^2 + X_{2i}^2) - \frac{\nu_i \mu_i}{r_i} = \frac{-\nu_i^3 \mu_i^2}{2L_i^2} ,$$

$$G_i = X_{2i-1} x_{2i} - X_{2i} x_{2i-1} , \quad i = 1, 2 ,$$

then in terms of the semi-major axis a_i , and eccentricity e_i , of the resulting elliptic orbit, $L_i = \nu_i \mu_i^{1/2} a_i^{1/2}$, $(G_i)^2 = L_i^2 (1 - e_i^2)$. Further, the period of the elliptic orbit is $2\pi/n_i$, with $n_i = \nu_i^3 \mu_i^2 / L_i^3$.

As is illustrated in Figure 1, if for example $L_2 \gg L_1$, and C is properly chosen, then both e_1 and e_2 are bounded away from 1; hence $r_2 > r_1$, and the unperturbed orbits do not intersect irrespective of φ_i or g_i = angle of perihelion. This is the basic situation with which we will deal.

Main theorem.

Theorem 2. Consider for $\epsilon = 0$, the subset $\Sigma \subseteq \mathbf{R}^8$, with $\Sigma = \{P \mid P \in \mathbf{R}^8, L_i(P) = L_i^0, G_1(P) + G_2(P) = C\}$, with L_1^0, L_2^0, C such that

FIGURE 1

- (a) the unperturbed orbits do not intersect,
- (b) $e_1^2(P) + e_2^2(P) \neq 0$,
- (c) $T = 2\pi p_i/n_i$, $i = 1, 2$, where p_1, p_2 are relatively prime integers, and T is thus the total period of the commensurable unperturbed orbits on Σ .

Then for sufficiently small ϵ , periodic orbits of the second kind exist with period T in a neighborhood of Σ .

Proof. The fact that periodic orbits of the second kind are not truly periodic in cartesian coordinates introduces some difficulties that can be overcome in the following fashion:

We introduce two new canonical variables α and β and a new Hamiltonian H_N by the formula:

$$(6) \quad H_N = H - \alpha[(X_1x_2 - X_2x_1) + (X_3x_4 - X_4x_3)].$$

Then (for $r_1 \neq 0, r_2 \neq 0$), in terms of the canonical variables $(\alpha, \beta), (p_{r_i}, r_i), (p_{\varphi_1} + p_{\varphi_2}, \varphi_1), (p_{\varphi_2}, \varphi_2 - \varphi_1)$, and noting that H depends on φ_1, φ_2 only in the combination $(\varphi_2 - \varphi_1)$, we find for the equations of motion:

$$(7) \quad \begin{aligned} \frac{d\alpha}{dt} &= 0, & \frac{d\beta}{dt} &= p_{\varphi_1} + p_{\varphi_2} = C, \\ \frac{dr_i}{dt} &= \frac{\partial H_N}{\partial p_{r_i}} = \frac{\partial H}{\partial p_{r_i}}, & \frac{dp_{r_i}}{dt} &= -\frac{\partial H_N}{\partial r_i} = -\frac{\partial H}{\partial r_i}, \\ \frac{d(p_{\varphi_1} + p_{\varphi_2})}{dt} &= 0, & \frac{d\varphi_1}{dt} &= \frac{\partial H}{\partial p_{\varphi_1}} + \alpha, \\ \frac{dp_{\varphi_2}}{dt} &= -\frac{\partial H}{\partial(\varphi_2 - \varphi_1)}, & \frac{d(\varphi_2 - \varphi_1)}{dt} &= \frac{\partial H}{\partial p_{\varphi_2}}. \end{aligned}$$

Therefore α introduces a rotation of angle αT in the fixed coordinate system.

More precisely, we introduce additional coordinates $\beta \pmod{CT}$ and $-\pi/T \leq \alpha < \pi/T$ or $x_5 = (2(\alpha + \pi/T))^{1/2} \cos \beta/CT$, $X_5 = (2(\alpha + \pi/T))^{1/2} \sin \beta/CT$, and consider H_N to be defined in $M^{10} = \mathbb{R}^8 \times [x_5^2 + X_5^2 < 4\pi/T]$. Since H depends on φ_1, φ_2 only in the combination $\varphi_2 - \varphi_1$ our discussion above shows that if there is a periodic orbit with Hamiltonian N_N in M^{10} , (i.e., $x_i(T) = x_i(0)$, $X_i(T) = x_i(0)$, $i = 1, \dots, 5$), the corresponding motion with Hamiltonian H in $\mathbb{R}^8$ will be a periodic orbit of the second kind.

The solution of a canonical set of differential equations defined in all of M^{10} gives rise to a transformation, mapping $x_i(0), X_i(0)$ to $x_i(T), X_i(T)$, $i = 1, \dots, 5$, which is a symplectic automorphism satisfying the assumptions of Corollary 1. (See S. Sternberg [5, Part II, page 128].) Hence Theorem 2 will follow if we prove:

Corollary 2. Consider for $\epsilon = 0$ the subset $\tilde{\Sigma}$ of M^{10} with $\tilde{\Sigma} = \{P \mid P \in M^{10}, L_i(P) = L_i^0, G_1(P) + G_2(P) = C, \alpha(P) = 0\}$. Then

- (a) H can be redefined so as to be C^∞ in all of $\mathbb{R}^8$, without affecting the solution of the modified equations (7) in some neighborhood of $\tilde{\Sigma}$ for $0 \leq t \leq T$, $0 \leq \epsilon \leq \delta$,
- (b) $\tilde{\Sigma}$ is a compact, regular, non-degenerate manifold for the symplectic automorphism $x_i(0), X_i(0) \rightarrow x_i(T), X_i(T)$, $i = 1, \dots, 5$, resulting from the solution of the modified equations (7) when $\epsilon = 0$.

Proof. Since orbits on Σ do not intersect and do not assume the value $r = 0$, to prove part (a) of Corollary 2 it is only necessary to replace the $1/\zeta$ and $1/r$ terms in the original Hamiltonian (1) with functions which agree with them for $\zeta > \delta_1$ and $r > \delta_1$ but are constant for $\zeta \leq \delta_1/2$, and $r \leq \delta_1/2$.

For part (b) of Corollary 2 we must consider three types of points $P \in \tilde{\Sigma}$. Type 1: $e_2(P) = 0$, $0 < e_1(P) < 1$. Type 2: $0 < e_2(P) < 1$, $0 < e_1(P) < 1$. Type 3: $0 < e_2(P) < 1$, $e_1(P) = 0$. We will confine our discussion to Type 1. The other two types can be treated similarly.

That $\tilde{\Sigma}$ is a compact manifold follows from the following lemma:

Lemma 1. Let $P \in \mathbb{R}^8$ be such that $e_2(P) = 0$, $0 < e_1(P) < 1$. Then the following canonical variables form the coordinates of a chart of a neighborhood of P : (L_1, ℓ_1) , $(L_2, \ell_2 + g_2 - g_1)$, $(G_1 + G_2, g_1)$, $(\xi = (2(L_2 - G_2))^{1/2} \cos(g_1 - g_2), \eta = (2(L_2 - G_2))^{1/2} \sin(g_1 - g_2))$ with ℓ_i = mean anomaly.

Proof. [If $0 < e_1(P) < 1$, $0 < e_2(P) < 1$, the Delaunay variables form the coordinates of a chart of a neighborhood of P (see Sternberg [Chapter IV, Section 3]). However if the eccentricity vanishes there is not an adequate discussion in the literature to show that Poincaré variables form the coordinates of a chart of a neighborhood of P .]

We begin by noting that since

$$e_2 \cos(g_1 - g_2 + a) = \left(1 - \frac{(L_2 - G_2)}{2L_2}\right)^{1/2} \left(\frac{2(L_2 - G)}{L_2}\right)^{1/2} \cos(g_1 - g_2 + a)$$

we can replace ξ, η in our discussion by $e_2 \cos (g_2 - g_1)$ and $e_2 \sin (g_2 - g_1)$.

Next, it follows from an integration procedure in Wintner [8, p. 178] that the following formulas are valid for equation (5):

$$G_i X_{2i} = \nu_i^2 \mu_i \left[\frac{x_{2i-1}}{r_i} + e_i \cos g_i \right] G_i X_{2i-1} = -\nu_i^2 \mu_i \left[\frac{x_{2i}}{r_i} + e_i \sin g_i \right].$$

These equations, together with equation (5), show that knowing $x_i, X_i, j = 2i - 1, 2i$, determines $L_i, e_i \cos g_i, e_i \sin g_i$; while conversely knowing $e_i \cos g_i, e_i \sin g_i$ and x_{2i-1}, x_{2i} determines X_{2i-1}, X_{2i} .

Hence, in particular, knowing L_1, ℓ_1, G_2, g_1 determines via Kepler's equation E_1 , the eccentric anomaly, f_1 , the true anomaly, and finally x_1, x_2, X_1, X_2 and conversely.

Knowing g_1 together with the coordinates of the lemma is equivalent to knowing $L_2, e_2 \cos g_2, e_2 \sin g_2$ and $\ell_2 + g_2$. From our previous discussion it follows that to complete the proof of the lemma, we must show that these variables determine r_2 and $f_2 + g_2$ and conversely.

For this purpose we first note that Kepler's equation $E - e \sin E = \ell$ can be written in the form:

$$\ell + g = (E + g) - e \cos g \sin (E + g) + e \sin g \cos (E + g).$$

Thus by the implicit function theorem one finds for sufficiently small $e \sin g, e \cos g$:

$$E + g = K(\ell + g, e \sin g, e \cos g).$$

Hence $\ell + g$ determines $E + g$ and conversely $E + g$ determines $\ell + g$ with parameters $e \sin g, e \cos g$.

On the other hand, $E + g, e \sin g, e \cos g$, determine $f + g, e \sin g, e \cos g$ and conversely, via the formulas (see Sternberg [5, Chapter 2, Section 2]):

$$\frac{r}{a} = 1 - e \cos E,$$

$$r \cos f = a(\cos E - e),$$

$$r \sin f = a(1 - e^2)^{1/2} \sin E,$$

$$e.g., \quad \frac{r}{a} = 1 - e \cos g \cos (E + g) - e \sin g \sin (E + g),$$

$$\begin{aligned} \cos (f + g) &= \cos f \cos g - \sin f \sin g = \frac{a}{r} [(\cos g \cos E \\ &\quad - \sin g \sin E) - e \cos g - ((1 - e^2)^{1/2} - 1) \sin E \sin g] \\ &= \frac{a}{r} \left[\cos (E + g) - e \cos g + \frac{e \sin g}{1 + (1 - e^2)^{1/2}} (e \cos g \sin (E + g) \right. \\ &\quad \left. - e \sin g \cos (E + g)) \right]. \end{aligned}$$

This completes the proof of the lemma.

Finally, to show that $\tilde{\Sigma}$ is a regular non-degenerate manifold let us study the solution of the equations of motion when $\epsilon = 0$. In the coordinates of the lemma, they are:

$$\begin{aligned} L_i(t) &= L_i(0), & \alpha(t) &= \alpha(0), & \beta(t) &= \beta(0) + [G_1(0) + G_2(0)]t, \\ \ell_1(t) &= \frac{\gamma_1 t}{(L_1(0))^3} + \ell_1(0), & (\ell_2 + g_2 - g_1)(t) &= \frac{\gamma_2 t}{(L_2(0))^3} + (\ell_2 + g_2 - g_1)(0), \\ (G_1 + G_2)(t) &= (G_1 + G_2)(0), & g_1(t) &= g_1(0) + \alpha(0)t, \\ \xi(t) &= \xi(0), & \eta(t) &= \eta(0) \end{aligned}$$

where γ_1 and γ_2 are constants, which in the Hamiltonian (1) reduce to $\gamma_1 = 1$, $\gamma_2 = (m_3/m_1)^3$.

Thus if T is the period, we find $\tilde{\Sigma}$ is a fixed manifold and:

$$\begin{aligned} \frac{\partial}{\partial A(0)} &= \frac{\partial}{\partial A(T)} \quad \text{for } A = \ell_1, \beta, \xi, \eta, g_1, \ell_2 + g_2 - g_1, \\ (8) \quad \frac{\partial}{\partial L_1(0)} &= \frac{\partial}{\partial L_1(T)} - \frac{3\gamma_1 T}{(L_1(0))^4} \frac{\partial}{\partial \ell_1(T)}; \\ \frac{\partial}{\partial (G_1 + G_2)(0)} &= \frac{\partial}{\partial (G_1 + G_2)(T)} + T \frac{\partial}{\partial \beta(T)} \\ \frac{\partial}{\partial L_2(0)} &= \frac{\partial}{\partial L_2(T)} - \frac{3\gamma_2 T}{(L_2(0))^4} \frac{\partial}{\partial (\ell_2 + g_2 - g_1)(T)}; \\ \frac{\partial}{\partial \alpha(0)} &= \frac{\partial}{\partial \alpha(T)} + T \frac{\partial}{\partial g_1(T)}. \end{aligned}$$

From equation (8) it follows that $T\tilde{\Sigma}$ is non-degenerate, and since the matrix D_{ii} of the transformation (8) can be put in triangular form with $D_{ii} = 1$, it follows by taking $R = -1$ that $\tilde{\Sigma}$ is regular. This completes the proof of Corollary 2 and Theorem 2.

REFERENCES

1. R. F. ARENSTORF, *Periodic solutions of the restricted three body problem representing analytic continuations of Keplerian elliptic motion*, Amer. J. Math. **85** (1963), 27–35.
2. W. H. JEFFREYS & J. MOSER, *Quasi-periodic solutions for the three-body problem*, Astronom. J. **71** (1966), 568–578.
3. H. POINCARÉ, *Les méthodes nouvelles de la Mécanique céleste*, Vol. 1, Gauthier-Villars, Paris 1892.
4. P. STÄCKEL, *Periodic Lösungen zweiter art beim Dreikörperproblem*, Jahresberichte der Deutsche mathematiker-vereinigung, Leipzig **28** (1919), 180–181.
5. S. STERNBERG, *Celestial mechanics*, Part I, Part II, W. A. Benjamin, Inc., New York, 1969.
6. A. WEINSTEIN, *Periodic orbits of Hamiltonian systems via critical point theory*, (to appear) also available as a manuscript, Berkeley, 1971.

7. A. WINTNER, *Über eine Revision der Sortentheorie des restringierten Dreikörperproblems*, Sitzungsberichte der Sächsischen Akademie der Wissenschaften zu Leipzig **82** (1930), 3-56.
8. A. WINTNER, *The analytic foundations of celestial mechanics*, Princeton University Press, Princeton, 1941.

This research was supported in part by NSF Grant GP-28111.

University of Oregon

Date communicated: JANUARY 31, 1972