Between Barreled and W -Barreled for $C(X)$

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1. For a completely regular Hausdorff space X , let $C(X)$ be the set of continuous real-valued functions on X , with the compact-open topology. A basis for this topology is given by $\{N(K, \epsilon, f): K \subset X \text{ is compact, } \epsilon > 0, f \in C(X)\}$, where $N(K, \epsilon, f) = \{g \in C(X): |f(t) - g(t)| \leq \epsilon \text{ for all } t \in K\}$.

In 1954 L. Nachbin and T. Shirota proved independently that $C(X)$ is barreled if and only if given any closed, non-compact subset F of X there exists $f \in C(X)$ which is unbounded on F [2, 3].

A space with this property will be referred to as an N - S space.

Wilansky has defined a linear topological space to be W -barreled if every closed, absorbing subset has non-empty interior [5, p. 224]. The purpose of this paper is to give a condition on X which is between $C(X)$ barreled and $C(X)$ W -barreled. In Section 2 we define the term L - W space, show that if X is an L - W space, $C(X)$ is barreled, show that if $C(X)$ is W -barreled, X is an L - W space, and give an example of a space X which is not an L - W space for which $C(X)$ is barreled. In Section 3 we study almost-discrete spaces in order to give an example of a space X which is an L - W space but for which $C(X)$ is not W -barreled. In Section 4 we introduce the concept of the center of a subset of $C(X)$ and show its value for barrels and lack of value for W -barrels.

2. The main result in this section is Theorem 2.6 which gives a necessary condition on X for $C(X)$ to be W -barreled.

Definition 2.1. A topological space X is said to be an L - W space if for every decreasing sequence of closed, non-compact subsets $\{F_n\}$ of X for which $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$, there exists f in $C(X)$ unbounded on each F_n .

Clearly, $C(X)$ is barreled whenever X is an L - W space.

Theorem 2.2. If $C(X)$ is category II, then X satisfies the stronger condition that for every decreasing sequence $\{F_n\}$ of closed, non-compact, non-empty subsets of X there exists f in $C(X)$ unbounded on each F_n .

Proof. For each n , let $A_n = \{f \in C(X): |f(t)| \leq n \text{ for all } t \in F_n\}$. Clearly

each A_n is closed. If no member of $C(X)$ is unbounded on each F_n , then $\bigcup_{n=1}^{\infty} \{A_n\} = C(X)$. Then A_n is somewhere dense for some n . But since X is completely regular, this implies that F_n is compact.

Every f in $C(X)$ may be considered as a map from X into $\mathbb{R}^+$. As such, f has a continuous extension, $\beta f: \beta X \rightarrow \mathbb{R}^+$. X is said to be realcompact if for every $p \in \beta X - X$ there exists $f \geq 0$ in $C(X)$ with $(\beta f)(p) = \infty$.

Theorem 2.3. *If X is a realcompact, locally compact space, then X is an L - W space. In fact, X satisfies the stronger condition of Theorem 2.2.*

Proof. Let $\{F_n\}$ be a decreasing sequence of closed, non-compact subsets of X . For each n , let H_n be the closure of F_n in βX . Since no F_n is compact, $H_n - X \neq \emptyset$ for each n , so that $\{H_n - X\}$ is a decreasing sequence of non-empty subsets of βX . Moreover each of these sets is closed [1, 6.9].

Choose $p \in \bigcap_{n=1}^{\infty} \{H_n - X\}$, and choose f in $C(X)$ with $(\beta f)(p) = \infty$. Clearly f is unbounded on F_n for each n .

Our next result reveals a wide class of topological spaces which are not L - W spaces.

Theorem 2.4. *Let X be a topological space which is first countable at a point p which has no compact neighborhood. Then X is not an L - W space.*

Proof. Let $\{F_n\}$ be a decreasing basic sequence of closed neighborhoods of p . No F_n is compact; and for f in $C(X)$, f is bounded on F_n where $F_n \subset f^{-1}[(f(p) - 1, f(p) + 1)]$.

Example 2.5. $\mathbb{Q}$, the space of rationals, is a realcompact, metric, N - S space but is not an L - W space.

Proof. $\mathbb{Q}$ is countable, hence Lindelöf, hence realcompact by [1, 8.2]. It is clearly an N - S space and is not an L - W space by Theorem 2.4.

This example shows that “ X is locally compact” may not be omitted from Theorem 2.3. As a corollary of Theorem 2.2 we learn that $C(\mathbb{Q})$ is category I.

Theorem 2.6. *If $C(X)$ is W -barreled, then X is an L - W space.*

Proof. Assume that X is not an L - W space. We may assume that X is an N - S space. Let $\{F_n\}$ be a decreasing sequence of closed, non-compact subsets of X with $F = \bigcap_{n=1}^{\infty} \{F_n\} \neq \emptyset$ and such that every f in $C(X)$ is bounded on some F_n . Clearly, F is compact.

Let $A_n = \{f \in C(X): |f(t)| \leq 1/n \text{ for all } t \in F_n\}$, $A = \bigcup_{n=1}^{\infty} A_n$, $B = \bar{A}$.

B is closed by definition and is easily seen to be absorbing.

Since $C(X)$ is W -barreled, B has interior, hence there exists $K \subset X$ compact, $\epsilon > 0$, and $f \in B$ such that $N(K, \epsilon, f) \subset B$.

Then $F \subset K$, for if $p \in F - K$, choose $g \in C(X)$ such that g vanishes on K and $g(p) = 2$. Then $f + g \in N(K, \epsilon, f)$, but $f + g \notin B$ since $N(\{p\}, 1, f + g) \cap A = \emptyset$.

Since each F_n is not compact, there exists a sequence $\{t_n\}$ with $t_n \in F_n - K$. Choose N such that $1/N < \epsilon/2$. Let g in $C(X)$ satisfy $g(t_n) = 2$ if $f(t_n) \geq 0$, $g(t_n) = -2$ if $f(t_n) < 0$ for $n = 1, 2, \dots, N$; $g(t) = \epsilon$ for all $t \in K$ if $\{p: f(p) > 0\} \cap F \neq \emptyset$, $g(t) = -\epsilon$ for all $t \in K$ if $\{p: f(p) > 0\} \cap F = \emptyset$.

Then $f + g$ does not belong to A since $|f(t_n) + g(t_n)| \geq 2$ for $n = 1, 2, \dots, N$ and $|f + g| \geq \epsilon$ for some point in $F \subset F_n$ if $n \geq N + 1$. Moreover $f + g \notin B$, for let $K' = F \cup \{t_1, \dots, t_N\}$. If $h \in N(K', \epsilon/3, f + g)$, then $h \notin A$. However, $f + g \in N(K, \epsilon, f)$, contradicting the assumptions.

Example 2.7. $C(Q)$ is barreled and bornological, but is not W -barreled.

3. Let $A = (a_{nk})$ be an infinite real matrix. The A -transform of the real sequence $x = \{x_k\}$ is the sequence $\{\sum_k a_{nk} x_k\}$ if these sums exist. Let c denote the Banach space of convergent sequences with $\|x\| = \sup_k |x_k|$. Let $c_A = \{x: Ax \in c\}$. The matrix A is said to sum the members of c_A . Define the functional $\lim_A$ on c_A by $\lim_A x = \lim Ax$. The matrix is called conservative if $c \subset c_A$ and regular if in addition $\lim = \lim_A$ on c . It is well known that A is regular if and only if $\lim_n a_{nk} = 0$ for all k , $\lim_n \sum_k a_{nk} = 1$, and $\sup_n \sum_k |a_{nk}| < \infty$. The matrix A is called positive if $a_{nk} \geq 0$ for all n, k .

Let $\mathcal{F}$ denote the set of finite subsets of $N - \{1\}$ where N is the set of positive integers. Let $\mathcal{R}$ be a ring of subsets of N which includes $\mathcal{F}$ and N . Let μ be a real finitely additive function on $\mathcal{R}$, $\mu \geq 0$, $\mu(N) = 1$, and $\mu = 0$ on $\mathcal{F}$. Let $\beta_\mu = \{U \cup \{1\}: U \in \mathcal{R}, \mu(U) = 1\}$. Then $\beta_\mu \cup \mathcal{F}$ is a base for a topology on N . The resulting space, N_μ , is Hausdorff, completely regular, and has at most one limit point. Actually, Snyder has observed that any countable space with these properties may be obtained as an N_μ space. [4]

For $U \subset N$, χ_U is the sequence $\chi_U(n) = 1$ if $n \in U$, $\chi_U(n) = 0$ if $n \notin U$. Let A be a positive regular matrix. Let $\mathcal{R} = \{U \subset N: \lim_A \chi_U = 1 \text{ or } 0\}$, $\mu^A(U) = \lim_A \chi_U$. Denote the resulting space by N_A . Let $\mu_n^A(U) = \sum_{k=1}^\infty a_{nk} \chi_U(k)$. When no confusion is possible μ^A and μ_n^A will be written μ and μ_n .

Lemma 3.1. *Let A be a positive regular matrix. There exists a positive regular matrix B whose rows are members of E^∞ and for which the identity map from N_A to N_B is a homeomorphism.*

Proof. Choose r_n such that $\sum_{k=r_n}^\infty a_{nk} < 1/2^n$, $n = 1, 2, 3, \dots$. Define B by $b_{nk} = a_{nk}$, $k < r_n$, $b_{nk} = 0$, $k \geq r_n$. Clearly B is regular, and $b_{nk} \in E^\infty$ for all n . For $U \subset N$, $|\mu_n^A(U) - \mu_n^B(U)| < 1/2^n$ so that $\mu^A(U) = \mu^B(U)$.

Definition 3.2. Let $A = (a_{nk})$ be a positive regular matrix. Define $a_A^n = \sup_k a_{nk}$ and $b_A^k = \sup_n a_{nk}$. When no confusion is possible we will use a^n and b^k in place of a_A^n and b_A^k .

Definition 3.3. A topological space is said to be pseudofinite if every compact subset is finite.

Lemma 3.4. *Let A be a positive regular matrix. Then N_A is pseudofinite if and only if $a^n \rightarrow 0$.*

Proof. In [4] Snyder proved that N_A is pseudofinite if and only if $b^k \rightarrow 0$, so assume $b^k \rightarrow 0$. Let $\epsilon > 0$. Choose K such that $b^k < \epsilon$ for all $k \geq K$, so that $a_{nk} < \epsilon$ for all n whenever $k \geq K$. Choose N such that for all $n \geq N$, $a_{nk} < \epsilon$ for $k = 1, 2, \dots, K$ by regularity of A . Whenever $n \geq N$, $a^n < \epsilon$.

A symmetrical argument proves the converse.

Definition 3.5. Let N_μ be a countable space in which 1 is the only non-isolated point. We say that N_μ has the dispersion property if N_μ is hemicompact and if for some choice $\{C_n\}$ as a cobase for the compact subsets of N_μ there exists a sequence $\{D_n\}$ of compact subsets of N with $C_n \cap D_n = \emptyset$ for all n and such that every neighborhood of 1 meets D_n for all but finitely many n .

Theorem 3.6. *Let A be a positive regular matrix for which N_A is pseudofinite. Then N_A has the dispersion property.*

Proof. Let $C_n = \{1, 2, \dots, n\}$. By Lemma 3.4 $a^n \rightarrow 0$. For each i choose N_i such that $N_i > i$ and $\sup_k a_{nk} < 1/2^i$ for all $n \geq N_i$. By Lemma 3.1 we may choose r_n such that $a_{nk} = 0$ whenever $k \geq r_n$. Let $D_n = \{n+1, n+2, \dots, r_{N_n}\}$ and observe that if $U \subset N$ fails to meet D_n , then $\mu_{N_n}(U) \leq n/2^n \leq 1/2$. Hence if U is a neighborhood of 1, U must meet D_n for all but finitely many n .

Theorem 3.7. *If N_μ has the dispersion property, then $C(N_\mu)$ is not W -barreled.*

Proof. Let $\{C_n\}$ and $\{D_n\}$ be as in Definition 3.5. We may assume that $1 \notin C_n$ for all n . Choose f_n in $C(N_\mu)$ satisfying $f_n(t) = 2$ if $t \in D_n$, $f_n(t) = 1/n$ if $t \in C_n$, $t \neq 1$, $f_n(t) = 0$ otherwise.

Let $A_n = \{g \in C(N_\mu) : |f_n(t) - g(t)| < 1/n \text{ for all } t \in D_n\}$. Note that A_n is open. Thus $B = C(N_\mu) - \bigcup_{n=1}^{\infty} A_n$ is closed.

B is absorbing. [Let $g \in C(N_\mu)$. Choose W , a neighborhood of 1, such that g is bounded on W . Choose k : $W \cap D_n \neq \emptyset$ for all $n \geq k$ and choose t_n in $W \cap D_n$ for $n \geq k$. Let M in $\mathbb{R}$ satisfy $g(t) \leq M$ for all $t \in \bigcup_{n=1}^k D_n$ and for all $t = t_n$ for $n > k$. Let $\lambda = 1/3M$. If $|\delta| \leq \lambda$, then $|\delta g(t)| \leq 1/3$ for $t \leq t_k$ or $t = t_n$ for $n > k$. Then $\delta g \notin A_n$ for each n ; hence $\delta g \in B$.]

B is nowhere dense. [Let $g \in C(N_\mu)$, $K \subset N$ compact, $\epsilon > 0$. Choose n : $K \subset C_n$. Define h in $C(N_\mu)$ by $h(t) = 2$ if $t \in D_n$, $h(t) = g(t)$ if $t \notin D_n$. Then $h \notin A_n$ so that $h \in N(K, \epsilon, g) - B$.]

Theorem 3.8. *Any pseudofinite, countable space with one non-isolated point is an L - W space. In fact, it satisfies the stronger condition of Theorem 2.2.*

Proof. Let $\{F_n\}$ be a decreasing sequence of closed, non-compact subsets. Choose t_n in F_n such that $t_m \neq t_n$ if $m \neq n$. The set $\{1, t_1, t_2, \dots\}$ is then infinite and cannot be compact. This means there must exist a neighborhood U of 1 such that $t_n \notin U$ for infinitely many n . Define f in $C(N_\mu)$ by $f(t_n) = n$ if $t_n \notin U$, $f(t) = 0$ otherwise. Then f is unbounded on each F_n .

Example 3.9. Let A be a positive regular matrix for which N_A is pseudofinite. Then N_A is an L - W space, but $C(N_A)$ is not W -barreled.

The following example shows that not every space with the properties of Theorem 3.8 has the dispersion property.

Example 3.10. Let $p \in \beta N - N$, and let $N \cup \{p\}$ have the relative topology of βN . Then $N \cup \{p\}$ does not have the dispersion property.

Proof. Let $\{C_n\}$ and $\{D_n\}$ be as in Definition 3.5. Let $\{D_{n_k}\}$ be a subsequence whose terms are pairwise disjoint. Let $M = \bigcup_{i=1}^{\infty} \{D_{n_{i+1}}\}$. Either $M \cup \{p\}$ or $(N - M) \cup \{p\}$ is a neighborhood of p , and both fail to meet D_n for infinitely many n .

We next see that N_A may have the dispersion property when N_A is not pseudofinite.

Definition 3.11. We say that N_A has basically one infinite compact subset if there are infinite subsets $\{p_k\}$ and $\{m_i\}$ of N_A satisfying

- i. $\{p_k\} \cup \{m_i\} = N$;
- ii. $\{p_k\} \cap \{m_i\} = \emptyset$;
- iii. $\liminf \{b^{p_k}\} > 0$;
- iv. $b^{m_i} \rightarrow 0$.

Condition iii says that $\{p_k\}$ is compact, and condition iv says that no infinite subset of $\{m_i\}$ is compact by [4, Lemma, ii; Theorem, i, iv].

Lemma 3.12. If $D \subset N_A$ is pseudofinite, then $\sup_k \{a_{nk} : k \in D\} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Define a matrix B as follows:

$$\begin{aligned} b_{nk} &= 0 \quad \text{if } k \notin D \\ &= a_{nk}/\mu(D) \quad \text{if } k \in D. \end{aligned}$$

It is easily seen that B is regular and that N_B is pseudofinite. Thus $a_B^n = \sup \{a_{nk} : k \in D\} \rightarrow 0$ as $n \rightarrow \infty$ by Lemma 3.4.

Theorem 3.13. Let A be a positive regular matrix such that N_A has basically one infinite compact subset. Let $\{p_k\}$ and $\{m_i\}$ be as in Definition 3.11 and satisfy in addition $\limsup_n \{\sum_{k=1}^{\infty} a_{npk}\} < 1$. Then N_A has the dispersion property.

Proof. Let $C_n = \{p_k\} \cup \{m_1, m_2, \dots, m_n\}$, let $\limsup_n \{\sum_{k=1}^{\infty} a_{npk}\} < a < 1$, and let $\delta = 1 - a$. For each integer i choose an integer N_i such that

- i. $N_i > i$,
- ii. $\sup_i a_{nm_i} < \delta/2^i$ for all $n \geq N_i$ (by Lemma 3.12),
- iii. $\sum_{k=1}^{\infty} a_{npk} < a$ for all $n \geq N_i$.

By Lemma 3.1 we may choose r_n such that $a_{nk} = 0$ for all $k \geq r_n$.

Let $D_n = \{m_{n+1}, m_{n+2}, \dots, m_{r_n}\}$, and observe that if $U \subset N_A$ fails to meet D_n , then $\mu_{N_n}(U) \leq a + n\delta/2^n \leq a + \delta/2 < 1$. Hence if U is a neighborhood of 1, then U meets D_n for all but finitely many n .

4. For $t \in X$ we denote the map $f \rightarrow f(t)$ by P_t .

Definition 4.1. Let $B \subset C(X)$. The center of B , $\psi(B)$, is the set $\{t \in X: P_t[B] \neq \mathbb{R}\}$.

Example 4.2. Let $\mathbb{Q}$ be the set of rational numbers. If B is a W -barrel in $C(\mathbb{Q})$, then $\psi(B)$ is compact.

Proof. Suppose $\psi(B)$ is not compact. Since B is absorbing we may assume that $\psi(B)$ is bounded so that $\psi(B)$ must have an irrational limit point w in $\mathbb{R}$. Choose an increasing (or decreasing) sequence $\{t_n\}$ in $\psi(B)$ converging to w . Choose r_n in $\mathbb{R}$ with $r_n \notin nP_{t_n}[B]$. For each n let a_n be an irrational number with $t_n < a_n < t_{n+1}$ and define f in $C(\mathbb{Q})$ as follows:

$$\begin{aligned} f(x) &= r_1 & \text{for } x < a_1, \\ f(x) &= r_n & \text{for } a_{n-1} < x < a_n, \\ f(x) &= 0 & \text{for } x > w. \end{aligned}$$

Then $f \notin nB$ for all n contradicting the assumption that B is absorbing.

Theorem 4.3. Let X be a topological space. Then X is an N - S space if and only if every barrel in $C(X)$ has compact center.

Proof. If X is an N - S space, $C(X)$ is barreled and so necessity follows from the obvious facts that $\psi(N(K, \epsilon)) = K$ and that if $B \subset B'$, $\psi(B') \subset \psi(B)$.

If X is not an N - S space, let F be a subset of X which is closed, non-compact, and on which every member of $C(X)$ is bounded. $\{f \in C(X): |f(t)| \leq 1 \text{ for all } t \in F\}$ is a barrel the center of which, F , is not compact.

Unfortunately, the concept of center is of no value when B is assumed to be a W -barrel only.

Theorem 4.4. If $C(X)$ is not W -barreled, then $C(X)$ has a W -barrel with empty (hence compact) center which is nowhere dense.

Proof. Let $B \subset C(X)$ be a nowhere dense W -barrel. $B' = B \cup \{f \in C(X): f \text{ is constant}\}$ is nowhere dense, and $\psi(B') = \emptyset$.

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