Operator Algebras of Finite Strict Multiplicity

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Communicated by the Editors

Introduction. A subalgebra $\mathbf{A}$ of the algebra $\mathbf{L}(\mathbf{X})$ of all (bounded linear) operators from the complex Banach space $\mathbf{X}$ into itself is said to be “*of finite strict multiplicity*” ($\mathbf{A}$ is an a.f.s.m.) if there exists a finite set $\Gamma = \{x_1, \dots, x_n\} \subset \mathbf{X}$ such that

$$(1) \quad \mathbf{X} = \text{linear span } \{Ax_i : A \in \mathbf{A}; \quad j = 1, 2, \dots, n\} = \mathbf{A}(\Gamma).$$

Here, and in what follows, *algebra* means *strongly closed* subalgebra of $\mathbf{L}(\mathbf{X})$ containing the identity I of $\mathbf{L}(\mathbf{X})$; *operator* and *subspace* mean *everywhere defined bounded linear transformation* (from $\mathbf{X}$ into $\mathbf{X}$) and *closed linear manifold*, respectively.

This article attempts to be an introductory survey of the algebras of finite strict multiplicity; thus, the sections are not necessarily connected.

Results include the study of the properties of an algebra $\mathbf{A}$ whose commutant is an a.f.s.m., the spectrum of the operators in $\mathbf{A}$ and their adjoints, *etc*; in particular, it is shown that the uniform and the strong operator topologies coincide on $\mathbf{A}$. The case when $\mathbf{A}$ is generated by the rational functions in a single operator is especially considered, and the *third section* is completely devoted to bilateral and unilateral weighted shifts which generate a separated s.c.a. It is also shown that every closed densely defined linear map commuting with a set of generators of an a.f.s.m. is necessarily bounded.

The last two sections deal with general commutative operator algebras, open problems and examples that illustrate that some of the results of the article (and related papers) are “sharp.”

In many aspects, this article is a sequel of [9].

The author wishes to thank M. Imina and A. Lambert for many helpful conversations and the referee for correcting a mistake in the original version.

§1. Preliminary results. The definition of “multiplicity of an operator” given in [21] can be extended to arbitrary algebras as follows:

(2) $\mu(\mathbf{A}) = \text{multiplicity of } \mathbf{A} = \inf \{\text{card } (\Gamma); \mathbf{X} = \text{closure } \mathbf{A}(\Gamma)\}$ (where Γ runs over all subsets of $\mathbf{X}$ and $\mathbf{A}(\Gamma)$ is interpreted as in (1)).

The definition of *strict multiplicity* “copies” definition (2) (see [9]).

(2') $\bar{\mu}(\mathbf{A}) = \text{strict multiplicity of } \mathbf{A} = \inf \{\text{card } (\Gamma) : \mathbf{X} = \mathbf{A}(\Gamma)\}.$

If $\bar{\mu}(\mathbf{A}) = 1$, we shall say that $\mathbf{A}$ is a *strictly cyclic algebra* (s.c.a.; see [11; 15; 18]). This means, of course, that $\mathbf{X} = \mathbf{A}(\{x_0\})$ for some (strictly cyclic) vector $x_0 \in \mathbf{X}$; in the case when x_0 can be chosen in such a way that:

$$Ax_0 = 0, \quad A \in \mathbf{A} \text{ implies } A = 0,$$

we shall say that x_0 separates points of $\mathbf{A}$ (or, $\mathbf{A}$ is a separated s.c.a.).

We shall need some results of [9]; for the sake of convenience, those results will be listed here:

Theorem 1. *Let $\mathbf{A}$ be an a.f.s.m. on $\mathbf{X}$, let T be a densely defined linear map commuting with $\mathbf{A}$ and let $\mathbf{A}' = \{L \in \mathbf{L}(\mathbf{X}) : LA = AL, \text{ for all } A \in \mathbf{A}\}$ be the commutant of $\mathbf{A}$. Then:*

i) *The only dense linear manifold of $\mathbf{X}$ invariant under (every operator of) $\mathbf{A}$ is the whole space $\mathbf{X}$.*

ii) *$D(T)$ (= domain of T) coincides with $\mathbf{X}$, T is bounded and, moreover, $T \in \mathbf{A}'$.*

iii) *If $T \in \mathbf{A}'$, $\lambda \in \sigma(T)$ (= the spectrum of T) and $T \neq \lambda I$, then either $\mathbf{M} = \ker(T - \lambda I)$ or $\mathbf{N} = \text{clos.}[\text{ran}(T - \lambda I)]$ is a nontrivial hyperinvariant subspace of T ; i.e., either $\{0\} \neq \mathbf{M} \neq \mathbf{X}$ or $\{0\} \neq \mathbf{N} \neq \mathbf{X}$ and $\mathbf{M}, \mathbf{N}$ are invariant under every operator that commutes with T .*

Note. A linear map $T: D(T) \rightarrow \mathbf{X}$, where $D(T) = \text{domain of } T$ is a linear manifold of $\mathbf{X}$, is said to commute with a subset $\Sigma \subset \mathbf{L}(\mathbf{X})$ whenever, for all $A \in \Sigma$ and for all $x \in D(T)$,

$$AD(T) \subset D(T) \quad \text{and} \quad ATx = TAx.$$

Two corollaries are immediate:

Corollary 1.

- i) *For every algebra $\mathbf{A} \subset \mathbf{L}(\mathbf{X})$, $\mu(\mathbf{A}) \leq \bar{\mu}(\mathbf{A})$.*
- ii) *If $\bar{\mu}(\mathbf{A}) < \infty$, then $\mu(\mathbf{A}) = \bar{\mu}(\mathbf{A})$.*
- iii) *If $\bar{\mu}(\mathbf{A}) \leq \aleph_0$, then $\bar{\mu}(\mathbf{A}) < \infty$.*

Proof. The first statement follows trivially from (2) and (2').

Assume that $\bar{\mu}(\mathbf{A}) = n < \infty$ and that $\mathbf{X} = \text{clos. } \mathbf{A}(\Gamma)$, where $\text{card } (\Gamma) = n - 1$; but then $\mathbf{A}(\Gamma)$ is dense in $\mathbf{X}$ and, since this linear manifold is clearly invariant under $\mathbf{A}$, Thm. 1, i) implies that $\mathbf{A}(\Gamma) = \mathbf{X}$. Therefore $\bar{\mu}(\mathbf{A}) \leq \text{card } (\Gamma) = n - 1 < n$, contradicting our assumption. This proves that $\mu(\mathbf{A}) = \bar{\mu}(\mathbf{A}) = n$.

Finally, if $\bar{\mu}(\mathbf{A}) \leq \aleph_0$, then there exists a sequence $\Gamma = \{x_n : n = 1, 2, \dots\}$ in $\mathbf{X}$ such that $\mathbf{X} = \mathbf{A}(\Gamma)$. Consider the Banach spaces $\mathbf{A}^{[n]} = \{A_1, \dots, A_n\} : A_i \in \mathbf{A}\}$ (under the norm $\|(A_1, \dots, A_n)\|_{[n]} = \max \{\|A_j\| : 1 \leq j \leq n\}$) and the bounded linear maps $S_n : \mathbf{A}^{[n]} \rightarrow \mathbf{X}$ defined by $S_n(A_1, \dots, A_n) = A_1x_1 + \dots + A_nx_n$, for $n = 1, 2, \dots$.

If $\mathbf{X}_n = \text{ran } S_n$, then our hypothesis implies that

$$\mathbf{X}_1 \subset \mathbf{X}_2 \subset \cdots \subset \mathbf{X}_n \subset \mathbf{X}_{n+1} \subset \cdots \subset \mathbf{X} = \bigcup_{n=1}^{\infty} \mathbf{X}_n.$$

According to [22, p. 201], either $\mathbf{X}_n = \mathbf{X}$ or $\mathbf{X}_n$ is a first category subset of $\mathbf{X}$. Now, if $\mathbf{X}_n \neq \mathbf{X}$ for all n , then $\mathbf{X} = \bigcup_{n=1}^{\infty} \mathbf{X}_n$ is a first category subset of the Banach space $\mathbf{X}$, a contradiction. Therefore, there exists a first index, n_0 , such that $\mathbf{X} = \mathbf{X}_{n_0}$; we conclude that $\bar{\mu}(\mathbf{A}) \leq n_0 < \infty$. Q.E.D.

Corollary 2. *If $\mathbf{A}$ is an a.f.s.m. and $T \in \mathbf{A}'$, then:*

i) $\sigma(T) = \sigma_p(T) \cup \sigma_p(T^*)$,

where T^* is the adjoint of T acting on the dual space $\mathbf{X}^*$, of $\mathbf{X}$, and σ_p denotes the point spectrum of an operator.

ii) Moreover, boundary $\sigma(T) \subset \sigma_p(T^*)$.

Remark. In case when $\mathbf{X}$ is a Hilbert space and the adjoint T^* of T is defined in the usual way ("via" inner product), then $\sigma_p(T^*)$ must be replaced by $\sigma_p(T^*)^* = \{\bar{z}: z \in \sigma_p(T^*)\}$.

Proof.

i) Let $\lambda \in \sigma(T)$ and let $\mathbf{M}$ and $\mathbf{N}$ be the subspaces defined in *Thm. 1, iii*). If $\mathbf{M} \neq \{0\}$, then $\lambda \in \sigma_p(T)$. If $\mathbf{N} \neq \mathbf{X}$, then

$$\ker (T^* - \lambda I) = \mathbf{N}^\perp = \{x^* \in \mathbf{X}^*: x^*(x) = 0, \text{ for all } x \in \mathbf{N}\}$$

is a non-zero subspace of $\mathbf{X}^*$; hence $\lambda \in \sigma_p(T^*)$.

Finally, if $\mathbf{M} = \{0\}$ and $\mathbf{N} = \mathbf{X}$, then $(T - \lambda I)$ is invertible, a contradiction.

ii) If $\lambda \in \sigma(T) \setminus \sigma_p(T^*)$, then $\mathbf{N} = \text{clos. } (T - \lambda I)\mathbf{X} = \mathbf{X}$; since the dense linear manifold $(T - \lambda I)\mathbf{X}$ is invariant under $\mathbf{A}$, it follows from *Thm. 1, i*) that $(T - \lambda I)\mathbf{X} = \mathbf{X}$ and $\ker (T - \lambda I) \neq \{0\}$. But this cannot happen if λ is a boundary point of $\sigma(T)$ (see [5; chap. VII]). We conclude that $\text{bdry. } \sigma(T) \subset \sigma_p(T^*)$. Q.E.D.

§2. The commutant of an A.F.S.M. Suppose $\mathbf{B}$ is an a.f.s.m. and $\Gamma = \{x_1, \dots, x_n\}$ satisfies (1). Then from the *proof* of *Lemma 1* in [9] it follows that there is a constant C such that for each x in $\mathbf{X}$ it is possible to choose $B_1, B_2, \dots, B_n \in \mathbf{B}$ so that

$$x = \sum_{i=1}^n B_i x_i \quad \text{and} \quad \sum_{i=1}^n \|B_i\| \leq C \|x\|.$$

Then we can apply the same argument as in [11, *Thm. 1, ii*)] (with minor modifications), to prove:

Theorem 2. *Let $\mathbf{A}$ be a subalgebra of $\mathbf{L}(\mathbf{X})$ and assume that $\mathbf{A}'$ is an a.f.s.m.; then the uniform and the strong operator topologies coincide on $\mathbf{A}$.*

In particular, if $\Sigma \subset L(X)$ is a set of strong generators of A , in the sense that

$$A = \text{strong clos. } (\Sigma_0),$$

where Σ_0 is the set of all finite linear combinations of all finite products (in any order) of the elements of $\Sigma \cup \{I\}$, then

$$A = \text{uniform clos. } (\Sigma_0).$$

Corollary 3. Let A be a commutative a.f.s.m. strongly generated by $\Sigma \subset L(X)$ and let Σ_0 be defined as in Thm. 2. Then $A = \text{unif. clos. } (\Sigma_0)$.

Proof. $A \subset A'$ and therefore A' is also an a.f.s.m. Now the result follows from Thm. 2. Q.E.D.

Theorem 3. Let A be a commutative algebra with a cyclic vector $x_0 \in X$ and assume that A' contains an a.f.s.m. B . Then $A = A' \supset B$ and A is a maximal abelian separated s.c.a.

Proof. Since x_0 is a cyclic vector for A , it separates points of A' . Since $A' \supset B \cup A$, A' is an a.f.s.m. with a separating cyclic vector x_0 . Therefore, A' is a separated s.c.a., and the strong and the uniform topologies coincide on A' ; moreover, as a Banach space, A' is isomorphic to X (see [11, §2]).

On the other hand, $A(\{x_0\})$ is dense in $X = A'(\{x_0\})$ and A is strongly closed in A' ; it follows that $A = A'$ and therefore A is a maximal abelian separated s.c.a. Q.E.D.

Corollary 4. Let A be a separated s.c.a. and let $T \in A$. Then the strong closure A_T of the polynomials in T coincides with the uniform closure of those polynomials; moreover, $A_T = A$ if and only if T has a cyclic vector.

If A is not singly generated (in particular, if A is not abelian), then no $T \in A$ has a cyclic vector.

Proof. Observe that A_T is abelian and $A'_T \supset A'$, an a.f.s.m. (use [11]). Therefore, by Thm. 2, the uniform and the strong topology coincide on A_T (this result is already contained in [11]).

If T has a cyclic vector x_0 , then Thm. 3 implies that $A_T = A'_T$ is a maximal abelian algebra contained in A . Hence, $A' \subset A_T \subset A$ and therefore A' is abelian. According to [11, Thm. 3, i)], A' is also a separated s.c.a.; from these two results and Cor. 3.2 of [15] (every abelian separated s.c.a. is a maximal abelian algebra), it follows that $A' = A_T = A$.

The remaining statements follow easily from the previous result. Q.E.D.

Let A_T^a be the strong closure of the rational functions in T , (with poles outside $\sigma(T)$); this is the "analytic algebra" generated by T ; [12]); then: $A_T \subset A_T^a \subset A''_T \subset A'_T$.

Theorem 4. Let $T \in L(X)$ and assume that A'_T is an a.f.s.m. Then:

i) If A''_T has a cyclic vector, then $A''_T = A'_T$ is a maximal abelian separated s.c.a.

ii) If $\mathbf{A}_T^a$ has a cyclic vector, then $\mathbf{A}_T^a = \mathbf{A}'_T = \mathbf{A}_T$, the maximal ideal space $\Phi(\mathbf{A}'_T)$ of $\mathbf{A}'_T$ (endowed with the Gelfand topology; [18]) is homeomorphic to $\sigma(T)$, $\sigma(T) = \sigma_p(T^*)$ and every eigenspace of T^* is one-dimensional.

iii) If $\mathbf{A}_T$ has a cyclic vector, then $\mathbf{A}_T = \mathbf{A}_T^a = \mathbf{A}'_T = \mathbf{A}_T$ and $\sigma(T)$ does not separate the complex plane.

Proof. i) and the first parts of ii) and iii) are trivial corollaries of *Thm. 2*. Moreover, in ii) (iii), resp.) $\mathbf{A}'_T$ coincides with the uniform closure of the rational functions (the polynomials, resp.) in T . The details of the rest of the proof can be easily checked by using the results of [18, *Chaps. I, II, III*].

ii) Let $\mathbf{A}_T^a = \mathbf{A}'_T$ be the uniform closure of the rational functions in T . Then $\mathbf{A}'_T$ is an abelian Banach algebra with identity and every non-zero multiplicative linear functional ϕ on $\mathbf{A}'_T$ is uniquely determined by its values on T and $(T - \lambda I)^{-1}$, $\lambda \notin \sigma(T)$. Moreover, $\phi((T - \lambda I)^{-1}) = [\phi(T - \lambda I)]^{-1} = [\phi(T) - \lambda]^{-1}$, and therefore ϕ is uniquely determined by $\alpha = \phi(T)$; clearly, $\alpha \in \sigma(T)$.

On the other hand, if $\alpha \in \sigma(T)$, then $(T - \alpha I)$ is not invertible in $\mathbf{A}'_T$. Hence $(T - \alpha I)$ is contained in a (proper) maximal ideal of $\mathbf{A}'_T$; in other words, there exists a multiplicative functional $\phi_\alpha \neq 0$ such that $(T - \alpha I) \in \ker \phi_\alpha$; moreover, this ϕ_α is the unique functional with those properties.

Therefore, $\Phi(\mathbf{A}'_T)$ can be identified with $\sigma(T)$ as a set; moreover, if $\Phi(\mathbf{A}'_T)$ is endowed with the Gelfand topology, then the map $\alpha \rightarrow \ker \phi_\alpha$ is a homeomorphism from $\sigma(T)$ to $\Phi(\mathbf{A}'_T)$.

For each $\alpha \in \sigma(T)$, $\ker \phi_\alpha = \text{clos. } \{(T - \alpha I)A : A \in \mathbf{A}'_T\} \in \Phi(\mathbf{A}'_T)$. Hence, $\dim (\mathbf{A}'_T / \ker \phi_\alpha) = 1$.

If x_0 is any separating strictly cyclic vector of $\mathbf{A}'_T$, then $S: \mathbf{A}'_T \rightarrow \mathbf{X}$, defined by $S(A) = Ax_0$ is an isomorphism of Banach spaces (see [11, *Thm. 3*]). It follows that $\text{clos. ran } (T - \alpha I) = S(\ker \phi_\alpha) = \ker \phi_\sigma(\{x_0\}) \neq \mathbf{X}$; moreover,

$$\dim [\mathbf{X} / \text{clos. ran } (T - \alpha I)] = 1.$$

Therefore, $\alpha \in \sigma_p(T^*)$ and the eigenspace

$$\{x^* \in \mathbf{X}^*: T^*x^* = \alpha x^*\}$$

is one-dimensional.

iii) If $\mathbf{A}_T$ has a cyclic vector, then $\mathbf{A}'_T = \mathbf{A}_T^a = \mathbf{A}_T$ is the uniform closure of the polynomials in T ; i.e., $\mathbf{A}'_T$ is the minimal Banach algebra (under the norm of $\mathbf{L}(\mathbf{X})$) containing T . Thus, according to [18, *Thm. (1.6.13)*], the spectrum $\Phi(\mathbf{A}'_T)$ of this algebra is equal to the polynomially convex hull of $\sigma(T)$, i.e.,

$$\Phi(\mathbf{A}'_T) \simeq \widehat{\sigma(T)} = \text{complement of the unbounded component of } [\mathbf{C} \setminus \sigma(T)].$$

On the other hand, ii) implies that $\Phi(\mathbf{A}'_T) \simeq \sigma(T)$. Therefore, $\sigma(T) = \widehat{\sigma(T)}$ does not separate the plane. Q.E.D.

Lemma 5. *Let $\mathbf{A}$ be an a.f.s.m. and let $\mathbf{M}, \mathbf{N}$ be two nontrivial invariant subspaces of $\mathbf{A}$; assume that $\mathbf{M} \cap \mathbf{N} = \{0\}$ and $\text{clos. } (\mathbf{M} \oplus \mathbf{N}) = \mathbf{X}$. Then:*

i) $\mathbf{M} \oplus \mathbf{N} = \mathbf{X}$ and the projection P , defined by $P(x_m + x_n) = x_m$ ($x_m \in \mathbf{M}$, $x_n \in \mathbf{N}$), belongs to $\mathbf{A}'$.

ii) If, moreover, $\mathbf{A}$ is an abelian s.s.c.a., then the maximal ideal space, $\Phi(\mathbf{A})$ of $\mathbf{A}$ is disconnected. In particular, if $\mathbf{A} = \mathbf{A}_T^a$, for some T , then $\sigma(T)$ is disconnected.

Proof. i) If $\mathbf{M} \oplus \mathbf{N}$ is dense in $\mathbf{X}$, then $\mathbf{M} \oplus \mathbf{N} = \mathbf{X}$ follows from *Thm. 1, i*); hence P is bounded and it is trivial to check that $P \in \mathbf{A}'$. (see [12, §5]).

ii) If $\mathbf{A}$ is an abelian separated s.c.a. and $\hat{P}(\phi)$ denotes the Gelfand-Fourier transform of P , then $\hat{P}(\Phi(\mathbf{A})) = \{0, 1\}$. Since $\hat{P}(\phi)$ is continuous in $\Phi(\mathbf{A})$, it follows that $\Phi(\mathbf{A})$ must be disconnected (see [18, Chap. III]).

Finally, observe that (by *Thm. 4, ii*)), if $\mathbf{A} = \mathbf{A}_T^a$, then $\Phi(\mathbf{A}) \simeq \sigma(T)$. Thus, the previous result shows that $\sigma(T)$ must be disconnected. Q.E.D.

§3. Bilateral weighted shifts on Banach spaces. Assume that $\mathbf{X}$ has a Schauder basis $\{e_n\}$ ($-\infty < n < +\infty$); i.e., for every $x \in \mathbf{X}$ there exists a unique two-sided sequence of complex numbers $\{c_n\}$ such that

$$\lim \left\| x - \sum_{-N}^{+M} c_n e_n \right\| = 0 \quad (N, M \rightarrow +\infty).$$

(then we shall write $x = \sum c_n e_n$).

Following [6] we shall say that $T \in \mathbf{L}(\mathbf{X})$ is a *bilateral weighted shift* in $\mathbf{X}$ if there is a Schauder basis $\{e_n\}$ ($\|e_n\| = 1$, for all n) and a (necessarily bounded) two-sided sequence of non-zero complex numbers $\{\alpha_n\}$ such that

$$(3) \quad T(\sum c_n e_n) = \sum c_n \alpha_n e_{n+1}.$$

Let

$$R_1 = \lim_{n \rightarrow \infty} \left(\sup_m \prod_{i=m+1}^{m+n} |\alpha_i| \right)^{1/n}, \quad R_2 = \lim_{n \rightarrow \infty} \left(\inf_m \prod_{i=m+1}^{m+n} |\alpha_i| \right)^{1/n},$$

$$I^+ = \lim_{n \rightarrow \infty} \inf \left(\prod_{j=0}^{n-1} |\alpha_j| \right)^{1/n}, \quad I^- = \lim_{n \rightarrow \infty} \sup \left(\prod_{j=1}^n |\alpha_{-j}| \right)^{1/n}.$$

It is well-known (see [7; 8; 9; 16]) that for every bilateral weighted shift T ,

$$(4) \quad \sigma(T) \supset \{\lambda: R_2 \leq |\lambda| \leq R_1\},$$

$$\{\lambda: I^- < |\lambda| < I^+\} \subset \sigma_p(T^*) \subset \{\lambda: I^- \leq |\lambda| \leq I^+\}$$

and $\mathbf{A}_T' = \mathbf{A}_T'^a$, i.e., $\mathbf{A}_T'$ is a maximal abelian algebra.

Clearly, $R_2 \leq I^- \leq R_1$ and $R_2 \leq I^+ \leq R_1$ in all cases and, since $\alpha_j \neq 0$, for all j , $0 \notin \sigma_p(T^*)$.

Using the results of the previous sections, it is possible to obtain a slightly improved version of a result due to Ralph Gellar ([7; *Thm. 16*]).

Corollary 6. *Let T be the bilateral weighted shift defined by (3) and assume that $\mathbf{A}_T'$ is an a.f.s.m. Then T is invertible, the limits R_1 , R_2 , I^+ , I^- satisfy the*

relations

$$(5) \quad \begin{aligned} 0 < I^- = R_2 \leq R_1 = I^+ (< \infty), \quad \text{and} \\ \sigma(T) = \sigma_p(T^*) = \{\lambda: 0 < R_2 \leq |\lambda| \leq R_1\}, \end{aligned}$$

$\mathbf{A}'_T$ is a separated s.c.a. and, moreover, for every $L \in \mathbf{A}'_T$, there exists a two-sided sequence $\{c_n\}$ of complex numbers such that

$$\lim \left\| L - \sum_{-N}^{+M} c_n T^n \right\| = 0, \quad (M, N \rightarrow +\infty).$$

Furthermore, $\mathbf{A}'_T$ cannot be (strongly) generated by a single operator.

Proof. Let $\mathbf{A}$ be the strong closure of the polynomials in T and T^{-1} ; then $\mathbf{A}_T \subset \mathbf{A} \subset \mathbf{A}_T^s \subset \mathbf{A}'_T$. Since $\mathbf{A}$ has cyclic vectors (namely, each of the e_n 's is cyclic for $\mathbf{A}$), it follows from *Thm. 4, ii*) that $\mathbf{A} = \mathbf{A}_T^s = \mathbf{A}'_T$, $\Phi(\mathbf{A}'_T) \simeq \sigma(T) = \sigma_p(T^*)$. From these results, the previous observations and (4), we obtain that $0 \notin \sigma(T)$ and the relations (5). Moreover, $\mathbf{A}'_T = \mathbf{A}$ is the uniform closure of the polynomials in T and T^{-1} . To check that every $L \in \mathbf{A}'_T$ is actually equal to the (uniformly convergent) sum of a Laurent series in T , it is enough to use the isomorphism between $\mathbf{A}'_T$ and $\mathbf{X}$ given by [11, *Thm. 3*], and the fact that $\{e_n\}$ is a Schauder basis in $\mathbf{X}$.

That $\mathbf{A}'_T$ is a maximal abelian separated s.c.a. is clear. Finally, observe that $\Phi(\mathbf{A}'_T) \simeq \sigma(T)$ is an annulus (use (5)) and separates the plane; hence, it follows from *Thm. 4, iii*) that $\mathbf{A}'_T$ cannot coincide with $\mathbf{A}_L$, for any $L \in \mathbf{A}'_T$. Q.E.D.

Corollary 7. *Let T be a unilateral weighted shift in a Banach space $\mathbf{X}$ with a Schauder basis $\{e_n\}$ ($n \geq 0$) (T is defined by $T(\sum_0^\infty c_n e_n) = \sum_0^\infty c_n \alpha_n e_{n+1}$, where $\{\alpha_n\}$ is a suitable bounded sequence of non-zero complex numbers) and assume that $\mathbf{A}'_T$ is an a.f.s.m. Then*

$$(5') \quad \begin{aligned} 0 \leq R_1 = I^+ (< \infty), \quad \text{and} \\ \sigma(T) = \sigma_p(T^*) = \{\lambda: |\lambda| \leq R_1\}. \end{aligned}$$

Moreover, $\mathbf{A}'_T$ is a separated s.c.a. and every $L \in \mathbf{A}'_T$ is actually equal to the (uniformly convergent) sum of a Taylor series in T .

This last result, applied to the ‘‘Donoghue weighted shift’’ (see *Section 6* below) improves a result due to Eric Nordgren ([17]).

According to *Lemma 5, ii*), $\mathbf{A}'_T$ (where T satisfies either the requirements of *Cor. 6*, or those of *Cor. 7*) cannot have a pair of invariant subspaces $\mathbf{M}$, $\mathbf{N}$, such that $\{0\} \neq \mathbf{M} \neq \mathbf{X}$, $\mathbf{M} \cap \mathbf{N} = \{0\}$ and $\text{clos.}(\mathbf{M} \oplus \mathbf{N}) = \mathbf{X}$. This improves (for these particular operators) a result in [6] about decomposing invariant subspaces. However, as it was shown in [10], every bilateral weighted shift (in ‘‘a large class’’ of Banach spaces) such that $\sigma_p(T^*) \neq \emptyset$, has two invariant subspaces satisfying the above conditions; clearly, the invariant subspaces of [10] are *not* invariant under T^{-1} , in the case when T is invertible.

As R. Gellar pointed out, the conclusion is even stronger for a unilateral weighted shift. In fact, if T satisfies the conditions of *Cor. 7*, then $\mathbf{A}'_T$ is actually an *integral domain* (see [6]) and (identifying $\mathbf{A}'_T$ with $\mathbf{X}$ and invariant subspaces with closed ideals as in [11]) it is possible to prove that, if $\mathbf{M}$ and $\mathbf{N}$ are two non-zero invariant subspaces of T , then $\mathbf{M} \cap \mathbf{N}$ is also a non-zero invariant subspace.

§4. Closed linear maps commuting with the generators of an A.F.S.M. Recall that (by definition) a linear map $T: D(T) \rightarrow \mathbf{X}$ is *closed* if its graph, $G(T) = \{(x, Tx): x \in D(T)\}$, is a closed subset of $\mathbf{X} \oplus \mathbf{X}$.

Let Σ be a set of strong generators for the a.f.s.m. $\mathbf{A}$ and let Σ_0 be defined as in *Thm. 2*; thus, if $\mathbf{A}(\Gamma) = \mathbf{X}$, $\Gamma = \{x_1, \dots, x_n\}$, then $\{Ax_i : A \in \Sigma_0; j = 1, 2, \dots, n\}$ is dense in $\mathbf{X}$.

Moreover, it follows from the *proof* of *Lemma 1* in [9] that the set Γ can be replaced in (1) by $\Gamma' = \{y_1, \dots, y_n\}$ provided $\|x_i - y_i\|$ is "sufficiently small", for $j = 1, 2, \dots, n$.

The following result can be considered as an extension of [14, *Lemma 3.2*].

Theorem 5. *Every densely defined closed linear map that commutes with a set of strong generators of an a.f.s.m. is everywhere defined and bounded, and commutes with $\mathbf{A}$.*

Proof. Let $\mathbf{A}$, Γ , Σ and Σ_0 be as above and let T be a densely defined closed linear map commuting with Σ . According to the previous comments, we can assume (without loss of generality) that $\Gamma \subset D(T)$.

Let $y_i = Tx_i$, $j = 1, \dots, n$. Then, for all $A \in \Sigma_0$, we have

$$(6) \quad TAx_i = ATx_i = Ay_i.$$

Let $x \in \mathbf{X}$. Then there exists $B_1, \dots, B_n \in \mathbf{A}$ such that

$$(7) \quad x = B_1x_1 + \dots + B_nx_n.$$

By definition of Σ_0 , we can choose n sequences $\{A_{ki}\}$ ($k = 1, 2, \dots; j = 1, 2, \dots, n$) of elements of Σ_0 such that

$$(8) \quad \begin{aligned} \lim_{k \rightarrow \infty} \|(A_{ki} - B_i)x_i\| &= 0 \quad \text{and} \\ \lim_{k \rightarrow \infty} \|(A_{ki} - B_i)y_i\| &= 0 \quad (\text{as } k \rightarrow \infty), \end{aligned}$$

for $j = 1, 2, \dots, n$.

From (6), (7), and (8), we obtain

$$(9) \quad \lim_{k \rightarrow \infty} T\left(\sum_{i=1}^n A_{ki}x_i\right) = \sum_{i=1}^n B_iTx_i = \sum_{i=1}^n B_iy_i = y.$$

Thus, from (8) and (9)

$$\lim_{k \rightarrow \infty} \sum_{i=1}^n A_{ki}x_i = x, \quad \lim_{k \rightarrow \infty} T\left(\sum_{i=1}^n A_{ki}x_i\right) = y$$

and therefore, (since T is closed), $x \in D(T)$ and $Tx = y$.

Since $x \in \mathbf{X}$ can be arbitrarily chosen, it follows that $D(T) = \mathbf{X}$; then, by the *closed graph theorem* ([5, p. 57]), $T \in \mathbf{L}(\mathbf{X})$. Clearly, T commutes with $\mathbf{A}$. Q.E.D.

§5. Unbounded linear maps commuting with an Abelian algebra.

Lemma 8. *If the abelian algebra $\mathbf{A}$ has a cyclic vector x_0 and $\mathbf{A}'$ is not abelian, then $\mathbf{A}$ commutes with a densely defined linear map U which is not even closable; i.e., there is no closed linear map T such that $D(T) \supset D(U)$ and $Tx = Ux$, for all $x \in D(U)$.*

Proof. Since x_0 is cyclic for $\mathbf{A}$, it separates points of $\mathbf{A}'$; thus, if $\mathbf{B}$ is a maximal abelian algebra containing $\mathbf{A}$, and $y \in \mathbf{B} \setminus \{\mathbf{A}\}$, then there exists a unique $B_y \in \mathbf{B}$ such that $B_y x_0 = y$. Moreover, if $L \in \mathbf{A}' \setminus \mathbf{B}$ and $x = Lx_0$, then $x \notin \mathbf{B}(\{x_0\})$.

As in [15], define the linear map U_x by

$$D(U_x) = \mathbf{B}(\{x_0\}) \quad \text{and} \quad U_x y = B_y x,$$

where $B_y \in \mathbf{B}$ is defined as above (since $\mathbf{A}'$ is not abelian, the existence of $L \in \mathbf{A}' \setminus \mathbf{B}$ is granted). Then (see [15, Lemma 3.1 and Cor. 3.2]), U_x is densely defined, commutes with $\mathbf{B}$ and is unbounded because $x \notin \mathbf{B}(\{x_0\})$. But, on the other hand, U_x commutes with $\mathbf{A} \subset \mathbf{B}$ and, for every $y \in \mathbf{A}(\{x_0\})$, $y = A_y x_0$ ($A_y \in \mathbf{A}$)

$$U_x y = U_x A_y x_0 = A_y U_x x_0 = A_y x = A_y Lx_0 = L A_y x_0 = Ly.$$

Therefore, the restriction of U_x to $\mathbf{A}(\{x_0\})$ coincides with the restriction L_0 of L to the same subset.

Let T be a closed extension of U_x ; then T is also a closed extension of L_0 . Since $D(L_0) = \mathbf{A}(\{x_0\})$ is dense, and T is closed, it follows that $T = L$ and T is a bounded linear map. Therefore, T cannot be an extension of U_x . This contradiction proves that U_x is not closable. Q.E.D.

Corollary 9. *If $\mathbf{A}$ is an abelian algebra with a cyclic vector, such that every densely defined linear map commuting with $\mathbf{A}$ is closable, then $\mathbf{A}'$ is a maximal abelian algebra.*

§6. Examples and open problems.

A) Let D be the "Donoghue weighted shift" in ℓ_+^2 ([3]); i.e., if $\{e_n\}$ ($n \geq 0$) is the canonical orthonormal basis of ℓ_+^2 , then D is the unilateral weighted shift defined by $De_n = 2^{-n}e_{n+1}$. Let D^* be the adjoint of D ; using the results of [4; 7; 12; 14; 17; 19] and previous sections it is possible to prove that:

- i) $\mathbf{A} = \mathbf{A}_{D \oplus (I + D^*)} = \mathbf{A}_D \oplus \mathbf{A}_{D^*}$;
- ii) $\mathbf{A} = \mathbf{A}'$ is a maximal abelian subalgebra of $\mathbf{L}(\ell_+^2 \oplus \ell_+^2)$;
- iii) $\mathbf{A}_D = \mathbf{A}_{D'}$ is an abelian separated s.c.a.;
- iv) $\mathbf{A}_D$ and $\mathbf{A}_{D^*}$ coincide with the uniform closure of the polynomials in the elements of a strong set of generators;
- v) For every $L \in \mathbf{A}_D$ (or $\mathbf{A}_{D^*}$), $\sigma(L)$ consists of a single element.

vi) $\mathbf{A}^* = \{T^*: T \in \mathbf{A}\}$ is unitarily equivalent to $\mathbf{A}$;

vii) $\mathbf{A}$ has a cyclic vector.

However, $\mathbf{A}$ is not an a.f.s.m. because $\mathbf{M} = \ell_+^2 \oplus D^*\ell_+^2$ is a *dense*, but *not closed* linear manifold invariant under $\mathbf{A}$ (now, use *Thm. 1, i*).

Thus, the *sufficient* condition of *Thm. 2* is *not necessary* in order to imply that an algebra $\mathbf{A}$ coincides with the uniform closure of the polynomials in the elements of any set of strong generators.

B) If $\mathbf{B} = \mathbf{A}_{D \oplus D}$ and $T = D \oplus (I + D)$, then $T \in \mathbf{B}'$. Clearly, $\mu(\mathbf{B}) = 2$ and therefore, by *Thm. 3*, $\mathbf{A}_T = \mathbf{A}_T' \supset \mathbf{B}$. However (see [12; 19]), $\mathbf{A}_T = \mathbf{A}_D \oplus \mathbf{A}_D \neq \mathbf{B}$; thus, the statement of *Thm. 3* cannot be improved to claim that $\mathbf{A} = \mathbf{A}' = \mathbf{B}$.

C) In [17], Eric Nordgren proved that every (not necessarily densely defined!) closed linear map commuting with D^* is bounded. This result cannot be improved; in fact, there exists an unbounded everywhere defined linear map in ℓ_+^2 that commutes with D^* (see [13, Ex. (iv)]).

D) In [14], Alan Lambert proved that every densely defined closed linear map commuting with D is bounded. This result is also sharp in this sense: if T is defined by $D(T) = D\ell_+^2$ and $Tx = D^{-1}x$, for all $x \in D(T)$, then T is closed and commutes with D (clearly, $D(T)$ is not dense in ℓ_+^2).

E) ‘‘Closedness’’ is also essential in *Thm. 5*. In fact: if $\mathbf{X} = \mathbf{C}([0, 1])$ (continuous functions in the interval $[0, 1]$, under the sup.-norm), and $\mathbf{M} = \{p(x) = \sum_{n=0}^N c_n x^n\}$, $\mathbf{N} = \{e^x p(x) : p(x) \in \mathbf{M}\}$, then $\mathbf{M} \cap \mathbf{N} = \{0\}$ and $\mathbf{M} \oplus \mathbf{N}$ (algebraic direct sum) is dense in $\mathbf{X}$. Define

$$T: \mathbf{M} \oplus \mathbf{N} \rightarrow \mathbf{X} \text{ by } T(p + e^x q) = p$$

(p, q are polynomials in x).

It is clear that T is unbounded, not closable (observe that $\mathbf{M}$ and $\mathbf{N}$ are dense in $\mathbf{X}$) and T commutes with the operator $A =$ multiplication by x , which (uniformly) generates the s.s.c.a. $\mathbf{A} = \{\text{Mult. by } f : f \in \mathbf{X}\} \subset \mathbf{L}(\mathbf{X})$.

F) It follows from *Cor. 7* and A)iii) that every $L \in \mathbf{A}_{D^*}$ can be written as $c_0 I + D^* T$, where $c_0 \in \mathbf{C}$ and $T \in \mathbf{A}_{D^*}$; therefore $\text{ran } (L - c_0 I) \subset \text{ran } D^*$. Using this fact, it is not difficult to check that the unbounded everywhere defined linear map of C), actually commutes with $\mathbf{A}_{D^*} = \mathbf{A}_{D^*}'$ (see [13]). Clearly, this map is not even closable.

On the other hand, if $\mathbf{A}$ is the algebra of all multiplications by $L^\infty([0, 1], dx)$ functions in $L^2([0, 1], dx)$, then ([1]) every densely defined linear map commuting with $\mathbf{A} = \mathbf{A}'$ is closable, even though $\mathbf{A}$ is not an a.f.s.m.

These two examples show that the problem analyzed in *Section 5* becomes ambiguous when $\mathbf{A}'$ is abelian.

G) Straightforward computations show that

$$\mathbf{A}'_D = \{\lambda I: \lambda \in \mathbf{C}\} \oplus \left\{ \sum_{k=1}^{\infty} c_k D_k \right\} = \{\lambda I\} \oplus \mathbf{A}'_D(0),$$

where $\sum_{k=1}^{\infty} |c_k|^2 < \infty$ and $D_k e_n = 2^{-kn} e_{n+k} = 2^{k(k+1)/2} D^k e_n$ ($k, n = 0, 1, 2, \dots$).

If $L = \sum_{k=1}^{\infty} c_k D_k \in \mathbf{A}'_D(0)$ and P_n denotes the orthogonal projection of ℓ_+^2 onto the one-dimensional subspace spanned by e_n , $n = 0, 1, 2, \dots$, then the Schatten p -distance ($0 < p < \infty$; see [20]) from L to 0 is bounded by

$$|L|_p \leq \sum_{n=0}^{\infty} |LP_n|_p \leq \sum_{n=0}^{\infty} 2^{-pn} \left\{ \sum_{k=1}^{\infty} |c_k|^2 \right\}^{p/2} \leq (1 - 2^{-p})^{-p} \|L\|^p.$$

Thus, the strong and the uniform topologies coincide with the topology induced by the p -distance on $\mathbf{A}'_D(0)$ (for any p , $0 < p < \infty$!). In particular (by Cor. 7),

$$\lim \left\| L - \sum_{k=1}^N c_k D_k \right\|_p = 0 \quad (N \rightarrow \infty).$$

This shows that D is an example of an operator with an “extremely small” commutant.

H) *Open problems:*

i) As N. Salinas pointed out to us, if $\mathbf{H}$ is a *non-separable* Hilbert space with orthonormal basis $\{e_\alpha\}$, then the algebra of all “diagonal” operators (with respect to that basis) is a maximal abelian algebra of normal operators with *no separating vectors* (moreover, no denumerable set of vectors can separate operators in this algebra).

Does a commutative algebra $\mathbf{A}$ on a *separable* Banach space always have a separating vector? Clearly, it will be enough to prove this in the case when $\mathbf{A}$ is *maximal* abelian and has no cyclic vectors. That a maximal abelian algebra on a separable Banach space need not have a cyclic vector, even in the case when $\mathbf{A}$ is singly generated, follows from [2, Ex. 3].

ii) *Conjecture:* If $\mathbf{A}$ is an abelian a.f.s.m., then $\mathbf{A}'' = \mathbf{A}$. (The result is false for non abelian algebras; consider the algebra of all upper triangular matrices in $\mathbf{C}^2$.)

iii) If $[0, \Omega]$ denotes the initial segment of the ordinal numbers, where Ω is the first non denumerable ordinal, then the Banach algebra $\mathbf{X} = \mathbf{C}([0, \Omega])$ (= continuous function in the compact Hausdorff space $[0, \Omega]$) is not separable. Therefore, $\mathbf{X}$ is not denumerably generated. Thus, if $\mathbf{A}$ denotes the algebra of all multiplications by the elements of $\mathbf{X}$, then $\mathbf{A}$ is a maximal abelian s.s.c.a. which cannot be strongly generated by a denumerable set of operators; hence $\bar{\mu}(\mathbf{A}) = 1$; $\inf. \{\text{card}(\Sigma): \Sigma \text{ generates (uniformly) } \mathbf{A}\} > \aleph_0$.

Is there any relation between $\bar{\mu}(\mathbf{A})$, $\bar{\mu}(\mathbf{A}')$ and the above infimum? For example: Is it always true that $\bar{\mu}(\mathbf{A}') \leq \inf. \text{card}(\Sigma)$ when $\mathbf{A}'$ is an a.f.s.m. and $\mathbf{A}$ is abelian?

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This research was supported by National Science Foundation Grant GU 3171.

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Date communicated: JANUARY 25, 1972