Some Observations on Harmonic, Borel, Approximate, and L^p -Differentiability

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1. The notion of harmonic differentiability was introduced by E. M. Stein, see [2, p. 246], and proves to be an important tool in certain problems of differentiability. The definition of harmonic differentiability is the following. Let $f \in L^1(\mathbf{R}^n)$, $\text{supp } f \subset \{|x - x_0| \leq 1\}$, and let $u(x, y) = (P_y * f)(x)$, where $P_y(t) = c_n y / (|t|^2 + y^2)^{(n+1)/2}$ is the Poisson kernel in the upper half space $(t, y) \in \mathbf{R}_+^{n+1}$, i.e., $t \in \mathbf{R}^n$, $y > 0$. The constant

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}.$$

If u and $\partial u / \partial x_i$, $j = 1, \dots, n$, have non-tangential limits at x_0 , we say that f is harmonically derivable at x_0 .

The other notions of differentiability with which we are concerned are approximate, Borel, and L^p -differentiability. We say that a function f on $\mathbf{R}^n$ is approximately derivable at x_0 iff there is $s: \mathbf{R}^n \rightarrow \mathbf{R}$ linear such that for each $\epsilon > 0$ the set $E_\epsilon = \{t: |f(x_0 + t) - f(x_0) - s(t)| \geq \epsilon |t|\}$ has 0 as a point of dispersion, and f is L^p -derivable at x_0 iff $\int_{|t| \leq h} |f(x_0 + t) - f(x_0) - s(t)|^p dt = o(h^{n+2p})$. For $n = 1$, f has a Borel derivative L at x_0 iff $h^{-1} \int_0^h (f(x_0 + t) - f(x_0))/t dt \rightarrow L$ as $h \rightarrow 0$. It can be shown that L^p -differentiability at x_0 implies approximate and harmonic differentiability at x_0 . For the approximate case see [5] and for harmonic differentiability see [2, p. 247].

We will show in this paper that Borel differentiability implies harmonic differentiability, and from this we will deduce the perhaps surprising fact that harmonic differentiability does not imply approximate differentiability. Some further results on harmonic and L^p -differentiability are given, and the paper concludes with a generalization of the Fatou type theorem in [4].

2. Theorem 1. *If $f \in L^1(\mathbf{R}^1)$ has at x_0 a Borel derivative L , then f is harmonically derivable to L at x_0 .*

Remark. For a vertical limit or, in case of a circle, for a radial approach under the assumption of symmetric Borel differentiability see [1].

Proof. We may assume that $x_0 = f(x_0) = L = 0$ and that $f(x) = 0$, $|x| \geq \frac{1}{2}$. We have to show that

$$(1) \quad \int_{-1}^1 P_v(t) f(x+t) dt = o(1),$$

$$(2) \quad \int_{-1}^1 P'_v(t) f(x+t) dt = o(1)$$

as $|x| \leq \alpha y$, $y \rightarrow 0$, $\alpha > 0$. Here, and in the sequel, primes denote differentiation with respect to t . The letter c will be used for a constant not necessarily the same at each occurrence.

We will establish (2); (1) is done similarly and is simpler. We will prove $\int_0^1 P'_v(t) f(x+t) dt = o(1)$ non-tangentially.

For $\tau > 0$ we have $\int_0^{x+\tau} f(u)/u du = \{\int_0^\tau + \int_{-x}^0\} f(x+t)/(x+t) dt$, so that $\int_0^\tau f(x+t)/(x+t) dt = o(|x| + \tau)$. Let $\epsilon > 0$ be given, and choose $\delta > 0$, $\eta > 0$ such that

$$(3) \quad \left| \int_0^\tau \frac{f(x+t)}{x+t} dt \right| \leq \epsilon(|x| + \tau), \quad 0 < \tau \leq \delta, |x| \leq \eta.$$

We write

$$\int_0^1 P'_v(t) f(x+t) dt = \left\{ \int_0^\delta + \int_\delta^1 \right\} P'_v(t) f(x+t) dt = I_1 + I_2.$$

The part I_2 can readily be seen to be $o(1)$ as $y \rightarrow 0$ by letting $\phi(x, t) = \int_0^t f(x+\tau) d\tau$ and integrating by parts.

The estimation of I_1 is a bit more tedious. We write $I_1 = \int_0^\delta [P'_v(t)(x+t)] f(x+t)/(x+t) dt$, and if we let $Q(x, t) = P'_v(t)(x+t)$, $F(x, t) = \int_0^t f(x+\tau)/(x+\tau) d\tau$, and integrate by parts we obtain

$$|I_1| = \left| Q(x, \delta) F(x, \delta) - \int_0^\delta F(x, t) Q'(x, t) dt \right| \leq o(1) + \left| \int_0^\delta F(x, t) Q'(x, t) dt \right|.$$

For $y \leq \eta/\alpha$, the second term is by (3) majorized by $\epsilon \int_0^\delta (|x| + t) |Q'(x, t)| dt$. Hence it follows that

$$\begin{aligned} \left| \int_0^\delta F(x, t) Q'(x, t) dt \right| &\leq \epsilon \int_0^\delta (|x| + t)^2 |P''_v(t)| dt + \epsilon \int_0^\delta (|x| + t) |P'_v(t)| dt \\ &= J_1 + J_2. \end{aligned}$$

Since

$$(4) \quad y \int_0^\delta \frac{t^2}{(t^2 + y^2)^2} dt \leq \pi/2, \quad y \int_0^\delta \frac{|x|t}{(t^2 + y^2)^2} dt \leq \frac{\alpha}{2},$$

we see that $J_2 \leq c\epsilon$.

For J_1 we need to estimate three integrals obtained by expanding $(|x| + t)^2$. The integral

$$\epsilon \int_0^\delta t^2 |P_v''(t)| dt \leq c\epsilon \left\{ y^3 \int_0^\delta \frac{t^2 dt}{(t^2 + y^2)^3} + y \int_0^\delta \frac{t^4}{(t^2 + y^2)^3} dt \right\},$$

and the last integral in $\{ \}$ is $\leq y \int_0^\delta t^2/(t^2 + y^2)^2 dt \leq \pi/2$ by (4). The first integral is seen to be bounded by making the substitution $u = t/y$.

Next

$$\epsilon |x| \int_0^\delta t |P_v''(t)| dt \leq c\epsilon \left(y^3 |x| \int_0^\delta \frac{t}{(t^2 + y^2)^3} dt + y |x| \int_0^\delta \frac{t^3}{(t^2 + y^2)^3} dt \right).$$

Both integrals are $\leq c\alpha$.

Finally,

$$\epsilon x^2 \int_0^\delta |P_v''(t)| dt \leq c\epsilon \left(y^3 x^2 \int_0^\delta \frac{dt}{(t^2 + y^2)^3} + y x^2 \int_0^\delta \frac{t^2}{(t^2 + y^2)^3} dt \right).$$

Both terms can be estimated by the substitution $u = t/y$ and, since $|x| \leq \alpha y$, are $O(\alpha^2)$. This completes the proof of Theorem 1.

3. From the splitting theorem [2, p. 248] it follows that if f is harmonically derivable on E , then f is a.e. on E approximately derivable. However, unlike L^p -differentiability, this is no longer true at every point of E . We give the following example. Let $I_n = [1/2^{n+1}, 1/2^n)$ and let $k_n \rightarrow \infty$. Divide I_n into $2k_n$ equal intervals $J_1, \dots, J_{2k_n}$. Let $f_n(x) = 1/2^n$, $x \in J_{2i}$, $i = 1, \dots, k_n$, and define $f_n(x) = -\sigma_i$, $\sigma_i > 0$, $x \in J_{2i-1}$, where σ_i is chosen so that

$$\int_{J_{2i-1} \cup J_{2i}} f(t)/t dt = 0.$$

Let $f_n(x) = 0$, $x \notin I_n$, and let $f(x) = \sum f_n(x)$. We will show that $\int_0^h f(t)/t dt = o(h)$, so that f has a Borel derivative 0 at 0. If $1/2^{n+1} \leq h < 1/2^n$, then $|\int_0^h f(t)/t dt| \leq (2/2^n) \log(b_i/a_i)$, if $h \in J_{2i-1} \cup J_{2i}$ and $J_{2i} = [a_i, b_i]$. Since $1 < b_i/a_i \leq 1/2k_n + 1$, we see that the above integral is $o(h)$. By Theorem 1 f is harmonically derivable to 0 at 0, whereas $\{t: |f(t)| \geq \frac{1}{2} |t|\}$ has positive upper density at 0, and so f is not approximately derivable at 0.

4. In this section we will compare harmonic and L^1 -differentiability and specifically we will address ourselves to the question in what sense harmonic differentiability is more general than L^1 -differentiability.

Let $f: \mathbf{R}^n \rightarrow \mathbf{R}$, $x_0 \in \mathbf{R}$, and let $s: \mathbf{R}^n \rightarrow \mathbf{R}$ be linear. We denote by $I(f, x_0, s)$ the class of functions ϕ such that $|\phi(t)| \leq |f(x_0 + t) - f(x_0) - s(t)|$, $t \in \mathbf{R}^n$. An obvious property for ordinary, approximate, and L^p -differentiability is that, if s is the differential of f at x_0 , then each $\phi \in I(f, x_0, s)$ is differentiable (in the same sense) at 0 with differential 0. The example in the previous section shows

that this is no longer true for harmonic differentiability. In fact, it is precisely the property by which harmonic differentiability differs from L^1 -differentiability.

Theorem 2. *Let $f \in L^1(\mathbf{R}^n)$, $\text{supp } f \subset \{|x - x_0| \leq 1\}$. Let $s: \mathbf{R}^n \rightarrow \mathbf{R}$ be linear, and assume that for each $\phi \in I(f, x_0, s)$ there is $j = j(\phi)$ such that*

$$\int_{\mathbf{R}^n} P_v^j(t + h)\phi(-t) dt = o(1)$$

nontangentially, where the upper j denotes differentiation with respect to t_j . Then f has s as its L^1 -differential at x_0 .

Proof. Let $Q_1, \dots, Q_{2^n}$ be the "octants" of $\mathbf{R}^n$. We note that for each j , $\text{sgn } P_v^j(t) = -\text{sgn } t_j$, and $\text{sgn } t_j$ is constant on each Q_k . We let $E^+ = \{t: f(x_0 + t) - f(x_0) - s(t) \geq 0\}$, $E_k^+ = E^+ \cap Q_k$, and define ϕ_k by $\phi_k(t) = 0$, $-t \notin E_k^+$, and $\phi_k(t) = f(x_0 - t) - f(x_0) - s(-t)$, $-t \in E_k^+$. By hypothesis,

$$\int_{E_k^+} P_v^{j_k}(t + h)\phi_k(-t) dt = o(1)$$

for $|h| \leq \alpha y$, $y \rightarrow 0$, and some $1 \leq j_k \leq n$. Let $\epsilon_k = \text{sgn } P_v^{j_k}(t)$, $t \in E_k^+$, and let $h = (0, \dots, 0, h_{j_k}, 0, \dots, 0)$, where $h_{j_k} = -\epsilon_k \alpha y$ appears in the j_k -th position. If $\psi_k(t) = \epsilon_k \phi_k(t)$, we obtain

$$\begin{aligned} \int_{E_k^+} P_v^{j_k}(t + h)\psi_k(-t) dt &= cy \int_{E_k^+} \frac{|t_{j_k} + h_{j_k}|}{(|t + h|^2 + y^2)^{(n+3)/2}} |\psi_k(-t)| dt \\ &\geq c \frac{y^2}{y^{n+3}} \int_{E_k^+ \cap \{|t| \leq y\}} |f(x_0 + t) - f(x_0) - s(t)| dt. \end{aligned}$$

Thus

$$\sum_k \frac{1}{y^{n+1}} \int_{E_k^+ \cap \{|t| \leq y\}} |f(x_0 + t) - f(x_0) - s(t)| dt = \frac{1}{y^{n+1}} \int_{E^+ \cap \{|t| \leq y\}} \dots = o(1)$$

as $y \rightarrow 0$. Similarly, the same expression with E^+ replaced by $E^- = \{t: f(x_0 + t) - f(x_0) - s(t) \leq 0\}$ is $o(1)$, and this completes the proof.

Corollary. $\int_{|t| \leq y} |f(x_0 + t) - f(x_0) - s(t)| dt = o(y^{n+1})$ if and only if for some $1 \leq j \leq n$,

$$\int_{\mathbf{R}^n} |P_v^j(t + h)[f(x_0 + t) - f(x_0) - s(t)]| dt = o(1)$$

nontangentially.

Proof. The necessity is proved in [2, p. 248], and the sufficiency is an easy application of Theorem 2.

5. Certain types of differentiability of a function $f: \mathbf{R}^n \rightarrow \mathbf{R}$ at x_0 are equivalent with the corresponding differentiability of the identity. More precisely,

if $s: \mathbf{R}^n \rightarrow \mathbf{R}$ is linear and if $E_\epsilon = \{t: |f(x_0 + t) - f(x_0) - s(t)| \geq \epsilon |t|\}$, then the following two statements are equivalent for ordinary and approximate differentiability.

(S1) $s(t)$ is the differential of f at x_0 .

(S2) $t|_{E_\epsilon}$ has differential 0 at 0 for every $\epsilon > 0$, where $t|_{E_\epsilon} = t$, $t \notin E_\epsilon$, and 0 elsewhere.

For L^1 -differentiability one has the easy implication (S1) $\rightarrow$ (S2), whereas the converse is no longer true as the following example shows. Let $I_n = [1/2^n - 1/2^{3n}, 1/2^n]$, and let $E = \cup I_n$. Then $|E \cap [0, h]| = O(h^3)$, so that $t|_E$ has 0 as an L^1 -differential at 0. If we let $f(x) = 2^n$, $x \in I_n$, and $f(x) = 0$, $x \notin E$, then $c_1 h^2 \geq \int_0^h f(t) dt \geq c_2 h^2$, and f is not L^1 -derivable at 0. Since for $\epsilon > 0$, $E_\epsilon \approx E$, the condition (S2) holds.

We will use the technique of harmonic differentiability to prove

Theorem 3. *Let $f \in L^1(\mathbf{R}^n)$, $\text{supp } f \subset \{|x - x_0| \leq 1\}$, and assume that for some $p > 1$, $\int_{|t| \leq y} |f(x_0 + t) - f(x_0)|^p dt = O(y^{n+p})$. If $s: \mathbf{R}^n \rightarrow \mathbf{R}$ is linear with the property that for each $\epsilon > 0$, $t|_{E_\epsilon}$ has at $t = 0$ an L^1 -differential 0, then s is the L^1 -differential of f at x_0 .*

Proof. By the corollary we need to show that $\int_{|t| \leq 1} |P_y^1(t + h)[f(x_0 + t) - f(x_0) - s(t)]| dt = o(1)$ nontangentially. Let $\epsilon > 0$ be given. First

$$\begin{aligned} \int_{E_\epsilon} |P_y^1(t + h)[f(x_0 + t) - f(x_0) - s(t)]| dt &\leq c \int_{E_\epsilon} |t| |P_y^1(t + h)| dt \\ &\quad + \int_{E_\epsilon} |P_y^1(t + h)\epsilon(t)| dt, \end{aligned}$$

where $\epsilon(t) = f(x_0 + t) - f(x_0)$. By the corollary the first integral is $o(1)$ as $y \rightarrow 0$ non-tangentially. The second integral we write as $\int_{E_\epsilon \cap \{|t| < y\}} + \int_{E_\epsilon \cap \{y \leq |t| \leq 1\}} = I_1 + I_2$. By Hölder's inequality,

$$\begin{aligned} I_1 &\leq \frac{c}{y^{n+1}} \int_{E_\epsilon \cap \{|t| < y\}} |\epsilon(t)| dt \leq \frac{c}{y^{n+1}} \left\{ \int_{|t| < y} |\epsilon(t)|^p dt \right\}^{1/p} |E_\epsilon \cap \{|t| < y\}|^{1/q} \\ &= \frac{c}{y^{n+1}} O(y^{n/p+1}) o(y^{n/q}) = o(1). \end{aligned}$$

For the estimate of I_2 we use the technique of [2, p. 248] and write $I_2 = \sum \int_{A_k} |P_y^1(t + h)| |\epsilon(t)| dt$, where $A_k = \{t \in E_\epsilon : 2^k y \leq |t| < 2^{k+1} y\}$. Since for $|h| \leq \alpha y$, $|P_y^1(t + h)| \leq cy/|t|^{n+2}$, we see that by Hölder's inequality

$$\begin{aligned} I_2 &\leq cy \sum (y \cdot 2^k)^{-n-2} \left\{ \int_{A_k} |\epsilon(t)|^p dt \right\}^{1/p} |E_\epsilon \cap \{|t| \leq 2^{k+1} y\}|^{1/q} \\ &\leq cy \sum \frac{(2^{k+1} y)^{n/p+1}}{(2^k y)^{n+2}} |E_\epsilon \cap \{|t| \leq 2^{k+1} y\}|^{1/q}. \end{aligned}$$

Let $\eta > 0$ and choose $\delta > 0$ such that $|E_\epsilon \cap \{|t| \leq h\}| \leq \eta h^n$, $0 < h \leq \delta$. We split the above sum into two sums $\sum_1, \sum_2$, where $\sum_1$ is extended over all k with $2^{k+1}y \leq \delta$. Then $y \sum_1 \leq c\eta^{1/a} 2^{n+1} \sum 2^{-k}$ for any $y > 0$. Since $|E_\epsilon \cap \{|t| \leq 2^{k+1}y\}| \leq c(2^{k+1}y)^n$, we have $y \sum_2 \leq c2^{n+1} \sum 2^{-k}$, where the sum is extended over all $k \geq [\log_2 (\delta/y)]$. Hence $y \sum_2 \leq cy/\delta$, and $I_2 = o(1)$.

If $A_\epsilon = E_\epsilon^c$, the complement of E_ϵ , we have

$$\begin{aligned} \int_{A_\epsilon} |P_v^1(t+h)[f(x_0+t) - f(x_0) - s(t)]| dt &\leq \epsilon \int_{|t| \leq 1} |P_v^1(t+h)| |t| dt \\ &= \epsilon \left\{ \int_{|t| < y} + \int_{y \leq |t| \leq 1} \right\} \leq \epsilon \left\{ c + cy \sum \frac{(2^{k+1}y)^{n+1}}{(2^k y)^{n+2}} \right\} \leq c\epsilon, \end{aligned}$$

and the proof is complete.

6. In the previous sections we were concerned with unsymmetric differentiability. In case of symmetric differentiability the non-tangential limits of the Poisson integral need to be replaced by vertical approaches. For the ordinary symmetric derivative this is due to Fatou [6, p. 99] and for the symmetric Borel derivative the reader is referred to [1]. Various other notions of differentiability were introduced in [1] with a corresponding Fatou type theorem. However, as observed in [3], the proof given there is inadequate. In a succeeding paper [4] the suitable differentiability condition in this setting is presented with a Fatou type theorem. The differentiability in question is the following. Let $f \in L^1(\mathbf{R}^1)$ and assume there is L such that

(1) for every $\epsilon > 0$, $E_\epsilon = \{t: |f(x_0+t) - f(x_0-t) - L \cdot t| \geq \epsilon \cdot |t|\}$ has the property $|E_\epsilon \cap [0, y]| = o(y^2)$,

(2) there is $0 < M_0 < \infty$ such that, if $E_{M_0} = \{t: |f(x_0+t) - f(x_0-t)| \geq M_0\}$, then $\int_{x \cap [0, y]} |f(x_0+t) - f(x_0-t)| dt = o(y^2)$, where $T = E_{M_0}$.

Theorem 4. (1) and (2) imply that f has at x_0 a symmetric L^1 -derivative, i.e.,

$$\int_0^y |f(x_0+t) - f(x_0-t) - Lt| dt = o(y^2).$$

Proof. Let $\epsilon > 0$ be given. We write

$$\frac{1}{y^2} \int_0^y = \frac{1}{y^2} \left\{ \int_{E_\epsilon \cap [0, y]} + \int_{[0, y] - E_\epsilon} \right\} = \frac{1}{y^2} I' + \frac{1}{y^2} I''.$$

For any $y > 0$, $y^{-2} I'' \leq \epsilon/2$. Also

$$\frac{1}{y^2} I' \leq \frac{1}{y^2} \int_{E_\epsilon \cap [0, y]} |f(x_0+t) - f(x_0-t)| dt + o(1).$$

Let $E_1 = \{E_\epsilon \cap [0, y]\} - E_{M_0}$, $E_2 = E_\epsilon \cap [0, y] \cap E_{M_0}$. Then

$$\frac{1}{y^2} I' \leq o(1) + \frac{1}{y^2} \left\{ \int_{E_1} + \int_{E_2} \right\} |f(x_0+t) - f(x_0-t)| dt.$$

Since $y^{-2} \int_{E_1} \leq M_0 |E_1 \cap [0, y]|/y^2 = o(1)$ and $y^{-2} \int_{E_1} \leq y^{-2} \int_{T \cap [0, y]} = o(1)$ where $T = E_{M_0}$, the proof is complete.

The following theorem appears now as a generalization of the theorem in [4].

Theorem 5. Let $f \in L^1(\mathbf{R}^n)$, $\text{supp } f \subset \{|x - x_0| \leq 1\}$. Assume that $s(t) = \sum_1^n a_i t_i$ is the symmetric L^1 -differential of f at x_0 , i.e., $\int_{|t| \leq y} |f(x_0 + t) - f(x_0 - t) - s(t)| dt = o(y^{n+1})$. Let $u(x, y) = (P_y * f)(x)$. Then $(\partial u / \partial x_i)(x_0, y) \rightarrow a_i$, $j = 1, \dots, n$, as $y \rightarrow 0$.

Proof. The proof can be accomplished by the same technique as in [2, p. 247].

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