Conformal Metrics in $\mathbb{R}^2$ with Prescribed Curvature

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Let $\mathfrak{M}$ be a Riemannian manifold with metric g . The following problem has recently been considered by a number of authors (see bibliography in [3]).

Problem I. What are necessary and sufficient conditions on a (smooth) function K for K to be the Gaussian curvature of a conformal metric $g' = e^{-2u}g$ on $\mathfrak{M}$? This problem has been attacked by a number of authors using variational arguments ([1], [2], [3], [4], [7]). We would like to illustrate here how the problem can be attacked using the methods of [6]. We confine our remarks to problems on the plane, although the method may be extendable to the problem on general Riemannian manifolds.

The metric on the plane is $ds^2 = dx^2 + dy^2$ and we search for a metric of the form $ds^2 = e^{-2u}(dx^2 + dy^2)$ with prescribed curvature K on a given domain $\mathfrak{D}$ in the plane. The curvature K is given by (see [5], p. 278) $K = e^{2u} \Delta u$. This may be written as

$$(1) \quad \Delta u - Ke^{-2u} = 0.$$

Thus, the problem of finding a Riemannian metric on a region $\mathfrak{D}$ with prescribed curvature K is equivalent to finding a solution of (1) on $\mathfrak{D}$.

Theorem I. *If K is smooth and bounded and if $\mathfrak{D}$ is bounded, then equation (1) has a solution, hence Problem I is solvable in this case.*

We shall solve this problem by using the technique of upper and lower solutions developed in [6]. Accordingly, it suffices to find two functions ϕ and ψ on $\mathfrak{D}$ with the following properties:

- (i) $\psi < \phi$ on $\mathfrak{D}$.
- (ii) $\Delta\psi - Ke^{-2\psi} \geq 0$ on $\mathfrak{D}$.
- (iii) $\Delta\phi - Ke^{-2\phi} \leq 0$ on $\mathfrak{D}$.

ψ is called a lower solution and ϕ is called an upper solution. It follows from Theorem 2.1 of [6] that there exists a solution u of (1) with $\psi \leq u \leq \phi$ on $\mathfrak{D}$.

Let $K_0 \leq K \leq K_1$ on $\mathfrak{D}$. We may assume, without loss of generality, that $K_0 < 0$ and $K_1 > 0$. Then for any function u ,

$$(2) \quad \Delta u - K_1 e^{-2u} \leq \Delta u - K e^{-2u} \leq \Delta u - K_0 e^{-2u}.$$

Accordingly, to get lower and upper solutions ψ and ϕ it is sufficient to find solutions of

$$\Delta \psi - K_1 e^{-2\psi} = 0 \quad \text{and} \quad \Delta \phi - K_0 e^{-2\phi} = 0$$

where K_0 and K_1 are constants.

Now one manifold with constant positive curvature is the sphere in $\mathbf{E}^3$. By taking the stereographic projection of a sphere with South pole at the origin we obtain a metric for $\mathbf{R}^2$ with constant positive curvature (depending on the radius of the sphere (see [5], p. 314)). The appropriate metric is

$$ds^2 = g^{-2}(dx^2 + dy^2)$$

where $g = 1 + K_1(x^2 + y^2)/4$. Putting $\psi = \ln g$ we get a solution of $\Delta \psi - K_1 e^{-2\psi} = 0$.

To get an upper solution we might try $\phi = \ln (1 + K_0(x^2 + y^2)/4)$. This gives a Riemannian structure on the plane with constant negative curvature K_0 (the hyperbolic plane; see [5], p. 315). Unfortunately, ϕ has singularities at $r^2 = -4/K_0$, where it tends to $-\infty$. Moreover, $\phi < \psi$, contradicting (i). Instead we look for another manifold of constant negative curvature: the Poincaré half plane. We take

$$ds^2 = \frac{1}{-K_0(y - y_0)^2} (dx^2 + dy^2)$$

where y_0 is sufficiently negative. Accordingly $\phi = \ln ((-K_0)^{1/2}(y - y_0))$. For y_0 sufficiently negative $\phi > \psi$ on $\mathfrak{D}$ (since $\mathfrak{D}$ is bounded). Thus we have our functions ϕ and ψ and the proof of Theorem I is complete.

The problem is more delicate on unbounded domains in $\mathbf{R}^2$. The stereographic plane ([5], p. 314) gives an example of a Riemannian metric on the entire plane with positive Gaussian curvature, while the Poincaré half plane gives an example of an unbounded domain in the plane with constant negative curvature. However, it is impossible to find a conformal Riemannian metric on the entire plane with constant negative curvature. This is a consequence of

Theorem II. *Let K be a smooth function on the plane. A necessary condition that K be the Gaussian curvature of a Riemannian metric $ds^2 = e^{-2u}(dx^2 + dy^2)$ on the entire plane is that there do not exist constants $r_0 > 0$ and $c > 0$ such that $K \leq -cr^{-2}$ for $r \geq r_0$.*

Proof. Suppose the hypothesis false: that is, there exist $c, r_0 > 0$ such that $K \leq cr^{-2}$ for $r \geq r_0$.

Proceeding as in [8] we have

$$\int_0^r \int_0^{2\pi} \Delta u \rho \, d\rho \, d\theta = \int_0^r \int_0^{2\pi} K e^{-2u} \rho \, d\rho \, d\theta.$$

Using Green's theorem the left side becomes

$$(3) \quad rI'(r) = \frac{1}{2\pi} \int_0^r \int_0^{2\pi} K e^{-2u} \rho \, d\rho \, d\theta$$

where

$$I(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) \, d\theta.$$

Differentiating again we obtain

$$\frac{1}{r} (rI')' = \frac{1}{2\pi} \int_0^{2\pi} K e^{-2u} \, d\theta \leq -\frac{c}{r^2} \frac{1}{2\pi} \int_0^{2\pi} e^{-2u} \, d\theta$$

for $r \geq r_0$. Since e^{-2u} is convex,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-2u} \, d\theta \geq e^{-2I};$$

hence

$$\frac{1}{r} (rI')' \leq -\frac{c}{r^2} e^{-2I}.$$

Now putting $t = \log r$, $m(t) = I(e^t)$ we get $r(d/dr) = d/dt$, and

$$\ddot{m} \leq -ce^{-2m}.$$

From (3) $\dot{m} \leq 0$ for $r > r_0$ ($t > t_0 = \log r_0$); therefore

$$\frac{d}{dt} \left(\frac{\dot{m}^2}{2} \right) = 2\dot{m}\ddot{m} \geq c \frac{d}{dt} e^{-2m}$$

Integrating from t_0 to t

$$\begin{aligned} \dot{m}^2(t) - \dot{m}^2(t_0) &\geq c[e^{-2m(t)} - e^{-2m(t_0)}], \\ \dot{m}^2(t) &\geq \dot{m}^2(t_0) + c[e^{-2m(t)} - e^{-2m(t_0)}]. \end{aligned}$$

The right side is increasing and is equal to $\dot{m}^2(t_0)$ for $t = t_0$. Therefore

$$\dot{m}(t) \leq |\dot{m}(t)| \leq (A + ce^{-2m})^{1/2}$$

where $A = \dot{m}^2(t_0) - ce^{-2m(t_0)}$. Therefore

$$\frac{1}{(A + ce^{-2m})^{1/2}} \frac{dm}{dt} \leq -1.$$

Integrating both sides from t_0 to t we get

$$t \leq t_0 + \int_m^{m_0} \frac{du}{(A + ce^{-2u})^{1/2}} \leq t_0 + \int_{-\infty}^{m_0} \frac{du}{(A + ce^{-2u})^{1/2}} = T_0$$

which shows that m reaches $-\infty$ before $t = T_0$; in other words, there are no entire solutions.

The example $u = \log \log r$, $K = -1/r^2$ shows it is possible to construct a conformal metric in an exterior domain with negative curvature, but this solution has a singularity at $r = 1$. Even if we chose $K = -1/r^2$ for $r \geq r_0 > 1$ and modified K for $r < r_0$, Theorem II shows it is still impossible to construct an entire solution of (1) for which the curvature is $-r^2$ for $r \geq r_0 > 1$. The example $u = \log r \sin \alpha \theta$ for $\alpha = \frac{1}{2}$ gives an example of a metric with constant negative curvature ($K = -\alpha^2$) on the slit plane $0 < \theta < 2\pi$.

That the equation $\Delta u = e^u$ cannot have entire solutions on the plane was proved by Wittich [8]. Theorem II here, a slight extension of Wittich's result, does not contradict Theorem 3 of [3], since the metric obtained there need not be conformally related to the original metric. Comparison arguments of Theorem II have also been discussed by Osserman in [9].

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