Motions in two Dimensional Indefinite Finsler Spaces

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1. Introduction. The n -dimensional Finsler manifolds which have a metric tensor of signature $n - 2$ have been investigated in [1] where it is shown that they induce locally timelike metrics in the sense of Busemann [4]. In this paper we investigate the groups of motions of the two dimensional spaces which then have a metric tensor with one positive and one negative eigenvalue. These spaces are, of course, a generalization of the two dimensional pseudo-Riemannian spaces.

The Minkowskian spaces are found to have an infinite isotropic group only in the Lorentz case. This implies that an indefinite Finsler surface with an infinite isotropic group at each point is pseudo-Riemannian. A Minkowskian space with a pairwise transitive group of motions is shown to be Lorentzian. The homogeneous surfaces are shown to be either locally Minkowskian or locally quasihyperbolic.

2. Indefinite Finsler Surfaces. Let M be a two dimensional connected and paracompact differentiable manifold of class C^∞ . Denote the local coordinates of a point x by x^1, x^2 . In the tangent space $T(x)$ at x we take a natural frame and denote the components of a vector y in $T(x)$ by y^1, y^2 . Let $L(x, y)$ be a continuous function defined on the tangent bundle $T(M)$ of M which has the following properties:

- (A). The function $L(x, y)$ is of class C^∞ if $y \neq 0$.
- (B). $L(x, ky) = k^2 L(x, y)$ for all $k > 0$.
- (C). The metric tensor $g_{ij}(x, y) = \frac{1}{2} \partial^2 L / \partial y^i \partial y^j$ has one positive and one negative eigenvalue for all (x, y) with $y \neq 0$.
- (D). $|L(x, -y)| = |L(x, y)|$.

The Christoffel symbols $\gamma_{hk}^i(x, y)$ are defined in the usual way, and for each point x_0 there is a simple convex neighborhood $U(x_0)$. In $U(x_0)$ the indefinite distance between two points p and q is defined by $\rho(p, q) = \int |L(x, \dot{x})|^{1/2} dt$

where the integral is along the unique extremal from p to q lying in $U(x_0)$, see [1].

3. Minkowskian Spaces. If M is $\mathbb{R}^2$ and $L(x, y) = L(y)$ is only a function of y , then the space is called (indefinite) Minkowskian. In this case the ordinary straight lines are the extremals and $\rho(p, q)$ is defined for all points p and q . Furthermore, we let z denote the origin and identify the tangent space at z with M ; then we regard L as a function of x . We then have the formula $\rho(p, q) = |L(q - p)|^{1/2}$. Let $C = \{x \mid \rho(x, z) = 0\}$ denote the light cone and $K = \{x \mid \rho(x, z) = 1\}$ denote the unit circle. The set C consists of the union of a finite number of lines through z , see [1, p. 1038].

By, if necessary, choosing new affine coordinates we may assume that C contains the coordinate axes and does not contain any line of positive slope. Then by Lemma 8 of [1] we have

$$L_2(1, 0) = \left. \frac{\partial L}{\partial x^2} \right|_{(1,0)} \neq 0.$$

Lemma 1. *Let M be a Minkowskian plane and let C contain the coordinate axes but no line of positive slope. The component of K in the first quadrant can be given by $x^2 = f(x^1)$ where*

$$\lim_{x^1 \rightarrow \infty} x^1 f(x^1) = \pm [L_2(1, 0)]^{-1}.$$

Proof. The fact that the component of K in the first quadrant can be given by $x^2 = f(x^1)$ follows from [1, p. 1036]. We find $L(x^1, h)$ by expanding in a power series about the point $(x^1, 0)$ and using $L(x^1, 0) = L_1(x^1, 0) = 0$.

$$L(x^1, h) = L_2(x^1, 0)h + \frac{h^2}{2} L_{22}(x_1, \bar{h}) \quad \text{for } 0 < \bar{h} < h.$$

Since L is positive homogeneous of degree two, we have L_2 positive homogeneous of degree one and L_{22} positive homogeneous of degree zero.

$$L_2(x^1, 0) = x^1 L_2(1, 0).$$

Using $K = \{(x^1, h) \mid L(x^1, h) = \pm 1\}$ we find that the component we wish satisfies

$$\pm 1 = x^1 L_2(1, 0)h + \frac{h^2}{2} L_{22}(x_1, \bar{h}).$$

From [1, p. 1037] and [4, p. 30], it follows that $f(x^1) \rightarrow a$ for $a \geq 0$ as $x^1 \rightarrow \infty$. Setting $h = f(x^1)$ in the last equation we find $a = 0$. The lemma follows easily.

Let M and M' be Finsler manifolds with metric functions $L(x, y)$ and $L'(x', y')$. At each point x of M the function $L(x, y)$ induces a Minkowskian metric on $T(x)$ and similarly for each point x' of M' . A differentiable map $f: M \rightarrow M'$ is a local isometry if each of the tangent maps $f_*: T(x) \rightarrow T'(f(x))$ is a vector

space isomorphism which preserves the (indefinite) metric. A local isometry is an isometry if it is one to one and onto.

The group of isometries of a Finsler manifold M onto itself will be denoted $I(M)$. The subgroup of isometries which leaves x fixed is the isotropic subgroup at x and is denoted $I_x(M)$. If M is Minkowskian the isotropic groups at different points of M are isomorphic.

Theorem 2. *Let M be a two-dimensional Minkowskian space. If the isotropic group (at any point) is infinite, then M is the two-dimensional Lorentz space.*

Proof. Each element of the isotropic group at z is a centro-affinity which permutes the lines in C . Without loss of generality we may assume that C contains the coordinate axes and no line of positive slope. A simple argument shows we may also assume an infinite number of elements of $I_z(M)$ maps each axis on itself. Then there must be an element of $I_z(M)$ which has the matrix representation

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

where $a > 0$, $b > 0$ and $a \neq b$. We will take $a > b$ (if $b > a$ the argument is similar).

For fixed r and θ let $(r \cos \theta, r \sin \theta) \in K$. Then for each integer m the matrix A^m represents an element of $I_z(M)$. Consequently, $(ra^m \cos \theta, rb^m \sin \theta) \in K$. By Lemma 1, if $(r \cos \theta, r \sin \theta)$ is in the first quadrant, then

$$\lim_{m \rightarrow \infty} r^2 a^m b^m \cos \theta \sin \theta = \pm [L_2(1, 0)]^{-1}.$$

Thus, $b = a^{-1}$ and $r^2 \cos \theta \sin \theta = \pm [L_2(1, 0)]^{-1}$. Using similar reasoning in the third quadrant, it follows that K consists of the two conjugate hyperbolas $x^1 x^2 = \pm [L_2(1, 0)]^{-1}$ and that M is isometric to the two dimensional Lorentz space.

Corollary 3. *If M is a two-dimensional Finsler manifold which has an infinite isotropic group at each point, then M is pseudo-Riemannian.*

Proof. Each element of $I_x(M)$ induces an isometry of $T(x)$ on itself leaving x fixed. Since each $I_x(M)$ is infinite, the isotropic group of each $T(x)$ must be infinite and, by Theorem 2, $T(x)$ is Lorentzian. Thus, M is pseudo-Reimannian.

4. Homogeneous Spaces. The manifold M is homogeneous when $I(M)$ is transitive. Minkowskian spaces are always homogeneous since they have the affine translations as motions.

Let Γ denote the group of conformal mappings of the upper half plane on itself which leave the point at ∞ fixed, i.e.,

$$\Gamma : \begin{aligned} (x^1)' &= \alpha x^1 + \beta \\ (x^2)' &= \alpha x^2 \end{aligned}$$

$$\alpha > 0, -\infty < \beta < \infty.$$

We define an indefinite quasihyperbolic plane to be an indefinite Finsler metric defined on the upper half plane which is not Minkowskian and which has Γ as a subgroup of its group of isometries. Clearly, a quasihyperbolic plane is homogeneous. The definite quasihyperbolic metrics have been studied in [3] and [5].

A space is called locally quasihyperbolic (Minkowskian) if each point has a neighborhood isometric to an open set of a quasihyperbolic (Minkowskian) space. It is not hard to show that a two dimensional indefinite Finsler manifold with a transitive abelian group of isometries has a universal covering manifold which is Minkowskian, compare [2, p. 31] and [4, p. 33]. Thus, we obtain the following result.

Theorem 4. *A two-dimensional indefinite Finsler manifold with a transitive abelian group of isometries is locally Minkowskian.*

Of course, $I(M)$ may contain a transitive subgroup which is not abelian.

Theorem 5. *Let M be a homogeneous two-dimensional indefinite Finsler manifold. Then M is either locally Minkowskian or locally quasihyperbolic.*

Proof. Let $\tilde{M}$ be the universal covering manifold of M . Then $I(\tilde{M})$ is transitive on $\tilde{M}$ and the isotropic groups at two distinct points are isomorphic.

Assume $I_x(\tilde{M})$ is infinite at some point x of $\tilde{M}$. Then $\tilde{M}$ is pseudo-Riemannian by Corollary 3. Since $I(\tilde{M})$ is transitive, $\tilde{M}$ has constant curvature. If the curvature is zero, M is locally isometric to the Lorentz space. If the curvature is not zero, M is locally isometric to the quasihyperbolic metric given by

$$L(x, y) = k \frac{(y^1)^2 - (y^2)^2}{(x^2)^2}$$

for some k . Compare [6, p. 69] for another form of the line element.

If $I_x(\tilde{M})$ is finite at some point, the arguments in [3, pp. 369–370] can be used to establish that M is locally quasihyperbolic.

5. Timelike Minkowskian Spaces. An indefinite Minkowskian space M of signature $n - 2$ has an induced timelike metric which makes it a timelike Minkowskian space.

It is not hard to show, using Lemma 1 and result 5.4 of [4], that the following lemma is valid for the two dimensional case. A simple argument then establishes the result for higher dimensions. The finite compactness is in the sense of Busemann, see [4, p. 22].

Lemma 6. *A timelike Minkowski space which is induced by an indefinite Minkowskian metric is finitely compact.*

Using 7.5 and 7.10 of [4], one can obtain one more result on induced timelike Minkowskian spaces.

Theorem 7. *Let M be an n -dimensional timelike Minkowskian space which*

is induced by an indefinite Minkowskian space. Let M (as a timelike space) have a pairwise transitive group of motions. If $n = 2$ or 3 , then M is Lorentzian.

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