Note on an Asymptotic Expansion

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1. Introduction. In a recent issue of this journal, Dorning, Nicolaenko and Thurber [1] studied the asymptotic behavior (as $t \rightarrow \infty$) of Ramanujan's function

$$(1) \quad N(t) = \int_0^\infty \frac{e^{-tx} dx}{x\{\pi^2 + (\log x)^2\}},$$

which they had encountered in the theory of neutron transport. Some ten years ago I had occasion to do the same for the quite similar function

$$(2) \quad M(t) = \int_0^\infty \frac{e^{-tx} dx}{1 + (\log x)^2},$$

at the request of the late Professor B. Friedman [2]. Apparently, these two integrals are special cases of

$$(3) \quad F(t, a, \sigma) = \int_0^\infty \frac{x^{\sigma-1} e^{-tx} dx}{a^2 + (\log x)^2},$$

namely,

$$(4) \quad N(t) = F(t, \pi, 0), \quad M(t) = F(t, 1, 1).$$

Accordingly, the main purpose of the present note is to derive, in a very simple way, an asymptotic expansion of $F(t, a, \sigma)$ as $t \rightarrow \infty$ for $a > 0$ and $\sigma \geq 0$.

2. Asymptotic expansion of F . I shall prove the following

Theorem. *If $a > 0$ and $\sigma \geq 0$, then*

$$(5) \quad F(t, a, \sigma) \sim t^{-\sigma} \sum_{n=0}^{\infty} \varphi_n(a, \sigma) (\log t)^{-n-1} \quad (t \rightarrow \infty),$$

in which the coefficients φ_n can be obtained from the generating function

$$(6) \quad \frac{\sin(ax)}{a} \Gamma(\sigma + x) = \sum_{n=0}^{\infty} \varphi_n(a, \sigma) \frac{x^n}{n!}.$$

Proof. Let us introduce the function $g(s)$ defined by

$$(7) \quad g(s) = -\frac{1}{a} e^{ias} \int_0^\infty \frac{x^{s+\sigma-1} e^{-tx} dx}{ia + \log x}, \quad (s \geq 0, \sigma > 0).$$

Then, as is easy to verify by taking imaginary parts of both sides, one gets

$$(8) \quad F(t, a, \sigma) = \operatorname{Im} \{g(0)\}.$$

Further, since

$$\frac{d}{ds} (e^{ias} x^{s+\sigma-1}) = e^{ias} (ia + \log x) x^{s+\sigma-1},$$

we have

$$\frac{dg}{ds} = -\frac{1}{a} e^{ias} \int_0^\infty x^{s+\sigma-1} e^{-tx} dx = -\frac{1}{a} e^{ias} \frac{\Gamma(s+\sigma)}{t^{s+\sigma}},$$

and, by integration,

$$(9) \quad g(0) = \frac{1}{a} \int_0^s e^{iaz} \frac{\Gamma(x+\sigma)}{t^{x+\sigma}} dx + g(s).$$

Now, from (7),

$$(10) \quad |g(s)| \leq \frac{1}{a} \int_0^\infty \frac{x^{s+\sigma-1} e^{-tx} dx}{\{a^2 + (\log x)^2\}^{1/2}} < \frac{1}{a^2} \frac{\Gamma(s+\sigma)}{t^{s+\sigma}}.$$

Combination of (8), (9) and (10) then leads to

$$(11) \quad F(t, a, \sigma) = t^{-\sigma} \int_0^s \frac{\sin(ax)}{a} \Gamma(\sigma+x) e^{-x \log t} dx + \frac{\theta}{a^2} \frac{\Gamma(s+\sigma)}{t^{s+\sigma}},$$

in which $|\theta| \leq 1$, while $s \geq 0$ and $\sigma \geq 0$ (the case $\sigma = 0$ must be complemented by arguments of continuity as σ tends to zero from above; note that the integral (7) is divergent for $s = \sigma = 0$). The integral in (11) is a Laplace integral in the variable $\log t$, of a function that is regular at $x = 0$. By standard arguments its asymptotic expansion is found to be that of equation (5). The second term on the right of (11) is of smaller order. This completes the proof of our theorem.

3. Final remark. In the case of the function $N(t)$ of (1), the generating function (6) reduces to

$$\frac{\sin(\pi x)}{\pi} \Gamma(x) = \frac{1}{\Gamma(1-x)}.$$

It is customary [3] to define coefficients c_k by means of

$$(12) \quad \frac{1}{\Gamma(x)} = \sum_{k=1}^{\infty} c_k x^k.$$

From $\Gamma(1-x) = -x\Gamma(-x)$ it then follows that

$$\frac{1}{\Gamma(1-x)} = \sum_{n=0}^{\infty} (-1)^n c_{n+1} x^n.$$

Accordingly

$$(13) \quad N(t) \sim \sum_{k=1}^{\infty} \frac{a_k}{(\log t)^k} \quad (t \rightarrow \infty),$$

in which

$$(14) \quad a_k = (-1)^{k-1} (k-1)! c_k.$$

From the existing numerical data [3], the first few coefficients are found to be (γ is Euler's constant)

$$a_1 = 1$$

$$a_2 = -\gamma = -0.5772156649$$

$$a_3 = \gamma^2 - \frac{\pi^2}{6} = -1.311756143$$

$$a_4 = 0.252015810$$

$$a_5 = 3.996926673,$$

in agreement with the authors cited [1], except for last-digit errors in their values of a_4 and a_5 (see their footnote on page 434).

REFERENCES

1. J. J. DORNING, B. NICOLAENKO & J. K. THURBER, *J. Math. Mech.* **19** (1969), 429-438.
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