Equilibrium Potentials and Additive Functionals

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1. Introduction. Throughout this introduction we shall assume that X is a “nice” Markov process—for definiteness a Hunt process with a reference measure. We refer the reader to [4] for all terminology and notation not explicitly explained here. It has been known for a long time that if A is a continuous additive functional (CAF) of X , then A admits a fine support, $\text{Supp } (A)$, which is finely perfect and nearly Borel (actually Borel if a reference measure exists.) Recall that a nearly Borel set B is finely perfect provided it is finely closed and each point of B is regular for B , that is $B = B^r$. See [13]. Thus it is natural to ask: Given a finely perfect Borel set F , does there exist a CAF whose fine support is F ? Gettoor [10] showed that if such an A exists it is unique up to a “time change.”

If F reduces to single point x_0 , then Blumenthal and Gettoor [5] showed the existence of a CAF, A , with $\text{Supp } (A) = \{x_0\}$. In this case A is a *local time* at x_0 and is unique up to a multiplicative constant. The method of proof consisted of showing that if $\phi_F(x) = E^x(e^{-T_F})$, then ϕ_F is a regular one potential and using the known representation theorem for such a potential in terms of a CAF. Shortly thereafter, Motoo [15] and Blumenthal and Gettoor, see [9], independently showed that if, in addition, $\bar{F} - F$ is polar, then the desired A exists. See also Section V-4 of [4]. Once again the method of proof (essentially) consisted in showing that ϕ_F is a regular one potential and then using the representation theorem. However, it is very easy to see that ϕ_F need not be a regular one potential in general. For example, if X is translation to the right at unit speed on the real line, then $F = [0, 1)$ is finely perfect and it is easy to see that ϕ_F is the one potential of

$$A_t = \int_0^t I_F(X_s) ds + I_{(T_1, \infty)}(t)$$

where $T_1 = T_{(1)}$. Clearly A is not continuous. But $I_F A$ defined by $(I_F A)_t = \int_0^t I_F(X_s) dA_s$ is continuous and has F for its fine support in this example.

Thus one is led to consider the following problems. Suppose that F is finely perfect and that $\phi_F = u_A^1$ with A a *natural* additive functional (NAF). Is it

the case that $I_F A$ is continuous, and, if so, is the fine support of $I_F A$ equal to F ? We shall show in Section 2 that the answer to the first question is yes. (Actually we prove a slightly more general result which is essentially contained in [8].) This result has also been obtained by Azema, [1] and [2], using quite different methods. However, the answer to the second question is no. Example (4.4) of the present paper shows that $I_F A$ may be zero and A purely discontinuous. Thus there is no possibility of finding a CAF with given fine support F by looking at ϕ_F . While the present manuscript was in preparation we received a copy of [3] in which Azema, building on earlier work by Dellacherie, constructs a CAF with given fine support F . Thus the answer to the question raised in the first paragraph is in the affirmative.

Still it is of interest to analyze the natural additive functional A associated with a finely perfect set F via the relationship $\phi_F = u_A^1$. This is the object of the present paper. In Section 2 we prove that $I_F A$ is the continuous part of A . In Section 3 we give an explicit representation for the purely discontinuous part of A and obtain additional information about the continuous part. Section 4 contains some examples. In this paper we often, but not always, omit the qualifying phrase "almost surely" in those places where it is obviously required.

2. The decomposition theorem. Throughout this section we assume that $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ is a special standard process with state space E , that is, X satisfies IV-4.1 of [4]. In addition we assume the existence of a reference measure ξ for X (V-1.1 of [4]). *We shall not mention these assumptions explicitly in the hypotheses of our results.* Let D be a finely closed set. Then under our basic assumptions $D = F \cup H$ where F is a finely perfect Borel set and H is semipolar, F and H are disjoint, and this decomposition is unique. See pages 180-183 of [13] and V-1.14 of [4]. If D is nearly Borel, then so is H . If B is any subset of E , then we define

$$Z_B(\omega) = \{t: X_t(\omega) \in B\}.$$

Since $B \subset E$, $t \in Z_B(\omega)$ implies that $t < \zeta$. If D is nearly Borel and finely closed, then almost surely $Z_D(\omega)$ is closed in the "right" topology on $\mathbf{R}^+ = [0, \infty)$. See XV-38 of [13]. If F is finely perfect (hence Borel), then almost surely the closure of Z_F in $\mathbf{R}^+$ is either empty or perfect and Z_F is itself either empty or uncountable. See XV-68 of [13]. Moreover, because Z_F is closed for the right topology, it is easy to see that $\bar{Z}_F - Z_F$ is countable. Consequently if $T_F < \infty$, then $[T_F, T_F + \epsilon] \cap Z_F$ is uncountable for every $\epsilon > 0$, where, as usual, $T_B = \inf \{t > 0: X_t \in B\}$ denotes the hitting time of a nearly Borel set B . Of course, if H is semipolar Z_H is countable (almost surely).

We now fix a finely closed nearly Borel set D and let $D = F \cup H$ be its decomposition into a finely perfect (Borel) set F and a semipolar (nearly Borel) set H . In addition we fix a bounded 1-excessive function u , and we assume that either $u = 1$ or u is the 1-potential, u_C^1 , of a continuous additive functional C whose fine support, $\text{Supp } (C)$, contains D . The two most important cases

for us are $u = 1$ and $u = U^1 1$. Next let $v = P_D^1 u$. For example, if $u = 1$ then $v(x) = \phi_D(x) = E^x(e^{-T_D})$ is the 1-equilibrium potential of D . Clearly v is 1-excessive and $v \leq u$. Finally we assume that there exists a natural additive functional A such that $v = u_A^1$. This is always the case if $u = u_C^1$ with C a CAF (or only an NAF). It is an additional assumption only in the case $u = 1$. It is satisfied in this case if $D \subset \{U^1 1 > \beta\}$ for some $\beta > 0$; for example, if $U^1 1$ is lower semicontinuous and D has compact closure in E . *The above assumptions and notation will be in force throughout the remainder of this section.* Of course, one may replace the parameter 1 by $\alpha > 0$ throughout, and even by $\alpha = 0$ under an appropriate transience condition—it may be necessary to exclude the case $u = 1$ when $\alpha = 0$.

Define

$$(2.1) \quad R = \inf \{t: A_t > 0\}.$$

The reader should check that R is an exact terminal time and that $R = \infty$ if $R \geq \zeta$.

$$(2.2) \text{ Proposition. } P_D^1 u \leq P_R^1 u.$$

Proof. Fix x in E . Then

$$(2.3) \quad v(x) = u_A^1(x) = E^x \int_{[R, \infty)} e^{-t} dA_t = E^x \{e^{-R} A_R + e^{-R} v(X_R)\}.$$

Now X is special standard and A is natural, and so $X_R = X_{R-}$ if $A(R) > 0$. Therefore R is accessible on $\{A(R) > 0\}$. See XIV-T 38 of [13] or [12]. Consequently since $A(R-) = 0$

$$A(R) = v(X_R)_- - v(X_R) \leq u(X_R)_- - v(X_R)$$

if $A(R) > 0$. See IV-4.29 of [4]. But u is regular and so $u(X_R)_- = u(X_R)$ if $A(R) > 0$. As a result we obtain

$$(2.4) \quad A(R) + v(X_R) \leq u(X_R)$$

if $A(R) > 0$; but (2.4) is obvious if $A(R) = 0$, and so it holds in general. Finally, (2.3), (2.4), and $v = P_D^1 u$ yield $P_D^1 u \leq P_R^1 u$, establishing (2.2).

We next want to show $R = T_D$. However, this is not true without some additional assumption on D . See Example 4.1. We shall show that it is true under an additional restriction on D . This requires a definition and some preliminary remarks. But the most important special case is when D itself is finely perfect (*i.e.*, $D = F$), and since the proof is very simple in this case we give it separately before treating the general case.

(2.5) *Proposition.* *If D is finely perfect, then almost surely $R = T_D$ and $A(R) = 0$.*

Proof. Since D is finely perfect $X(T_D)$ is regular for D if $T_D < \infty$, and so

$(P_D^1)^2 = P_D^1$. Therefore

$$P_D^1 v = P_D^1 P_D^1 u = P_D^1 u = v,$$

and consequently

$$0 = v(x) - P_D^1 v(x) = E^x \int_{[0, T_D]} e^{-t} dA_t.$$

But this implies that $A(T_D) = 0$ and hence $T_D \leq R$.

Next suppose $u = 1$. Then by (2.2), $E^x(e^{-T_D}) \leq E^x(e^{-R})$ and combining this with $T_D \leq R$ yields $T_D = R$. If $u = u_C^1$ with $D \subset \text{Supp}(C)$, then from (2.2) and $T_D \leq R$ we obtain $E^x \int_{T_D}^R e^{-t} dC_t = 0$. But this implies that $T_D = R$ because $X(T_D) \notin D \subset \text{Supp}(C)$ if $T_D < \infty$.

Therefore $T_D = R$ and since we saw in the first paragraph of the proof that $A(T_D) = 0$, the proof of (2.5) is complete.

As mentioned before we are now going to show that $R = T_D$ under a weaker assumption on the set D . The reader who is interested only in the case that D is finely perfect may skip to (2.10).

To begin we need a definition. If B is any subset of E define

$$(2.6) \quad L_B = \inf \{t > 0: X_{t-} \notin B\}.$$

To be more precise we should write $L_B = \inf \{t > 0: X_{t-} \text{ exists and is in } B\}$. (Of course, almost surely $t \rightarrow X_t$ has left limits on $(0, \zeta)$ and (ζ, ∞) , but X_{t-} need not exist.) It is well known (page 59 of [4]) that if B is nearly Borel, then almost surely

$$T_B = \inf \{t > 0: X_t \notin B \text{ or } X_{t-} \text{ exists and is in } B\}$$

whenever the right side is less than ζ . In particular $T_B \leq L_B$ on $\{T_B \wedge L_B < \zeta\}$. (If X is a Hunt process, then $T_B \leq L_B$.)

(2.7) Definition A nearly Borel set B is *admissible* provided $T_B = L_B$ almost surely on $\{T_B < \zeta\}$.

If B is finely perfect, then B is admissible because if $T_B < \infty$, $[T_B, T_B + \epsilon] \cap Z_B$ is uncountable for every $\epsilon > 0$ and $t \rightarrow X_t$ has at most a countable number of discontinuities. It is known that if X satisfies Hunt's hypothesis (B), then every nearly Borel set is admissible. See [16], for example. In fact, Azema [2] has shown that hypothesis (B) is equivalent to the statement that all compact sets are admissible, at least for Hunt processes.

The next result is our promised strengthening of (2.5). However, it can no longer be asserted that $A(R) = 0$.

(2.8) Proposition. *If D is admissible, then almost surely $R = T_D$.*

Proof. Suppose firstly that $x \notin D^r$. Then $P_D^1 v(x) = v(x)$, and so

$$0 = v(x) - P_D^1 v(x) = E^x \int_{[0, T_D]} e^{-t} dA_t.$$

Therefore $A(T_D) = 0$ and $T_D \leq R$ almost surely P^x for $x \in D^r$. Next suppose that x is not in D . Then there exists a decreasing sequence $\{G_n\}$ of open supersets of D such that $(T_{G_n} \wedge \zeta) \uparrow (T_D \wedge \zeta)$ almost surely P^x .

Let G be an open set containing D , and let $\Lambda = \{0 < T_D = T_G < \zeta\}$. On Λ , $X(T_D-) = X(T_G-) \in E - G$. But $L_D = T_D$ on Λ because D is admissible, and combining this with the fact that $X(L_D-) = X(T_D-) \notin D$ we obtain $L_D \circ \theta_{T_D} = 0$ on Λ . Consequently $T_D \circ \theta_{T_D} \leq L_D \circ \theta_{T_D} = 0$ on Λ , and so $X(T_D)$ is regular for D on Λ . We now apply this observation to each G_n to obtain

$$P_{G_n}^1 v(x) = P_{G_n}^1 P_D^1 u(x) = P_D^1 u(x) = v(x).$$

But this implies that $A(T_{G_n}) = 0$ almost surely P^x , and letting $n \rightarrow \infty$ we see that $A[(T_D \wedge \zeta)-] = 0$ almost surely P^x . Therefore $T_D \leq R$ almost surely P^x .

So far we have shown that $T_D \leq R$ almost surely P^x provided that x is not in the semipolar set $D - D^r$. The argument in the second paragraph of the proof of (2.5) now yields $T_D = R$ almost surely P^x provided x is not in $D - D^r$.

Finally, fix an x in $D - D^r$. Since $D - D^r$ is semipolar and hence of potential zero there exists a sequence $\{t_n\}$ of strictly positive numbers decreasing to zero such that $P^x[X_{t_n} \in D - D^r] = 0$ for each n . Therefore

$$P^x[R \circ \theta_{t_n} \neq T_D \circ \theta_{t_n}] = E^x\{P^{X(t_n)}[R \neq T_D]\} = 0,$$

and since R and T_D are exact terminal times it follows on letting $n \rightarrow \infty$ that $P^x[R \neq T_D] = 0$. This completes the proof of (2.8).

In the sequel we shall state it explicitly whenever we assume that D is admissible. We adopt the convention that $A_t = 0$ if $t < 0$. Next we shall study the relationships among the sets, Z_F , Z_D , and

$$(2.9) \quad \begin{cases} J(\omega) = \{t: A_{t+\epsilon}(\omega) - A_{t-\epsilon}(\omega) > 0 \text{ for all } \epsilon > 0\} \\ I(\omega) = \{t: A_{t+\epsilon}(\omega) - A_t(\omega) > 0 \text{ for all } \epsilon > 0\}. \end{cases}$$

It is evident that (almost surely) J is closed, I is closed for the right topology, $\bar{I} = J$, $J - I$ is countable, and that the support of the measure dA_t on $\mathbf{R}^+$ is J .

(2.10) Proposition. *Suppose that D is admissible, or only that $R = T_D$. Then almost surely $I \subset Z_D \subset J$.*

Proof. We begin by showing that $Z_D \subset J$ and $\bar{Z}_D = J$. To this end observe that

$$\{\omega: Z_D(\omega) \not\subset J(\omega)\} \subset \bigcup_{s < r} \{A_r(\omega) = A_s(\omega); s + T_D(\theta_s \omega) < r\}$$

where the union is over all pairs (s, r) of rationals satisfying $s < r$. But if $s < r$

$$P^x\{A_r = A_s; s + T_D \circ \theta_s < r\} = E^x\{P^{X(s)}[A_{r-s} = 0; T_D < r - s]\} = 0$$

because $T_D = R$. Consequently $Z_D \subset J$. Since J is closed, in order to show that $\bar{Z}_D = J$ it suffices to show $J \subset \bar{Z}_D$. But

$$\{\omega: J(\omega) \subsetneq \bar{Z}_D(\omega)\} \subset \bigcup_{s < r} \{A_r(\omega) > A_s(\omega); s + T_D(\theta_s \omega) > r\}$$

where again the union is over all pairs (s, r) of rationals with $s < r$, and

$$P^x\{A_r > A_s; s + T_D \circ \theta_s > r\} = E^x\{P^{X(s)}[A_{r-s} > 0; T_D > r - s]\} = 0$$

because $T_D = R$. Thus we have $Z_D \subset J$ and $\bar{Z}_D = J$.

Next we show that $I \subset Z_D$. Consider an ω such that $Z_D(\omega)$ is closed for the right topology and $\bar{Z}_D(\omega) = J(\omega)$. If $t \in I(\omega)$ and $t \notin Z_D(\omega)$, then $t \in \bar{Z}_D(\omega) - Z_D(\omega)$ because $I(\omega) \subset J(\omega) = \bar{Z}_D(\omega)$. Since $Z_D(\omega)$ is closed for the right topology, there exists an $h > 0$ such that $(t, t + h) \cap Z_D(\omega)$ is empty. But $t \in I(\omega)$ and so there exists a sequence $\{t_n\}$ decreasing to t with $t_n > t$ and $t_n \in J(\omega) = \bar{Z}_D(\omega)$ for each n . Thus $(t, t + h) \cap \bar{Z}_D(\omega)$ is not empty and this is a contradiction. Therefore $I \subset Z_D$.

(2.11) Remark. Since $J - I$ is countable so is $J - Z_D$. Also $Z_D - Z_F = Z_H$ is countable, and hence so is $J - Z_F$. Example (4.1) shows that (2.10) and the preceding statement are not valid without the assumption that $R = T_D$.

We turn our attention now to the decomposition of A into its continuous and pure jump parts. It is *not* assumed that D is admissible in the next proposition.

(2.12) Proposition. Let $w(x) = E^x \int_0^\infty e^{-t} I_{D^c}(X_t) dA_t$. Then w is a regular one potential and $B_t = \int_0^t I_{D^c}(X_s) dA_s$ is a continuous additive functional whose one potential is w .

Proof. For notational convenience we write $K = D^c$ in this proof only. Then $w = U_A^1 I_K$. Let $\{T_n\}$ be an increasing sequence of stopping times with limit T . Clearly w is 1-excessive. Thus to show that it is a regular one potential we must show that $P_{T_n}^1 w \rightarrow P_T^1 w$ as $n \rightarrow \infty$. Recall that since X is special standard we have $\lim_n g(X_{T_n}) \geq g(X_T)$ on $\{T < \infty\}$ for any 1-excessive function g . See IV-4.21 of [4].

First observe that $\lim_n u(X_{T_n}) = u(X_T)$ on $\{T < \zeta\}$ because u is regular. Therefore $\lim_n v(X_{T_n}) \leq \lim_n u(X_{T_n}) = u(X_T)$ on $\{T < \zeta\}$. But $v(x) = P_D^1 u(x) = u(x)$ if $x \in K = D^c$ and so $\lim_n v(X_{T_n}) \leq v(X_T)$ on $\{X_T \in K\}$. Combining this with the above remark we see that $\lim_n v(X_{T_n}) = v(X_T)$ if $X_T \in K$. Let $h = I_{E-K}$. Then $v = w + U_A^1 h$. If $X_T \in K$, then

$$\begin{aligned} w(X_{T_n}) &= v(X_{T_n}) - U_A^1 h(X_{T_n}) \\ &\rightarrow v(X_T) - \lim_n U_A^1 h(X_{T_n}) \\ &\leq v(X_T) - U_A^1 h(X_T) = w(X_T), \end{aligned}$$

and therefore $\lim_n w(X_{T_n}) = w(X_T)$ when $X_T \in K$. Now

$$(2.13) \quad P_{T_n}^1 w(x) = E^x\{e^{-T_n} w(X_{T_n}); X_T \in K\} + E^x\{e^{-T_n} w(X_{T_n}); X_T \notin K\},$$

and the first term on the right side of (2.13) approaches $E^x\{e^{-T} w(X_T); X_T \in K\}$

as $n \rightarrow \infty$. Therefore to prove that w is a regular 1-potential it suffices to show that if J_n denotes the second term on the right side of (2.13), then

$$(2.14) \quad \lim_n J_n = E^x \{e^{-T} w(X_T); X_T \notin K\}.$$

Let $J_n = J'_n + J''_n$ where J'_n and J''_n denote the integrals over $\{T < \zeta\}$ and $\{T \geq \zeta\}$ respectively. Since $w \leq v$, it is a natural one potential, and hence $\lim_n J''_n = 0$. See IV-4.20 of [4]. If $\Gamma \in \mathfrak{F}_{T_k}$ and $n \geq k$, then

$$(2.15) \quad E^x \{e^{-T_n} w(X_{T_n}); \Gamma\} = E^x \left\{ \int_{(T_n, \infty)} e^{-t} I_K(X_t) dA_t ; \Gamma \right\}.$$

Let $L = \lim_n e^{-T_n} w(X_{T_n})$. Then letting $n \rightarrow \infty$ in (2.15) we obtain

$$(2.16) \quad E^x \{L; \Gamma\} = E^x \left\{ \lim_n \int_{(T_n, \infty)} e^{-t} I_K(X_t) dA_t ; \Gamma \right\}$$

for all $\Gamma \in \mathfrak{F}_{T_k}$ and hence for all $\Gamma \in \bigvee \mathfrak{F}_{T_k}$. But X is special standard and so $\mathfrak{F}_T = \bigvee \mathfrak{F}_{T_k}$. Now use (2.16) with $\Gamma = \{X_T \notin K; T < \zeta\} \in \mathfrak{F}_T$ to obtain

$$(2.17) \quad \lim_n J_n = \lim_n J'_n = E^x \left\{ \lim_n \int_{(T_n, \infty)} e^{-t} I_K(X_t) dA_t ; X_T \notin K, T < \zeta \right\}.$$

If $T_n = T$ for all sufficiently large n , then

$$(2.18) \quad \lim_n \int_{(T_n, \infty)} e^{-t} I_K(X_t) dA_t = \int_{(T, \infty)} e^{-t} I_K(X_t) dA_t.$$

If, on the other hand, $T_n < T$ for all n and $X_T \notin K$, then the left side of (2.18) becomes

$$\int_{(T, \infty)} e^{-t} I_K(X_t) dA_t = \int_{(T, \infty)} e^{-t} I_K(X_t) dA_t.$$

Combining these observations with (2.17) we obtain

$$\begin{aligned} \lim_n J_n &= E^x \left\{ \int_{(T, \infty)} e^{-t} I_K(X_t) dA_t ; X_T \notin K, T < \zeta \right\} \\ &= E^x \{e^{-T} w(X_T); X_T \notin K\}, \end{aligned}$$

and in view of (2.14) this establishes the fact that w is a regular one potential.

Define $B_t = \int_0^t I_K(X_s) dA_s$. It is evident that B is a natural additive functional whose bounded one potential is w . It now follows from the familiar representation and uniqueness theorems for the potentials of additive functionals that B must, in fact, be continuous. This completes the proof of (2.12).

We come now to the main result of this section. Recall that if B is an additive functional and f a nonnegative measurable function on E , then fB denotes the functional $t \rightarrow \int_0^t f(X_s) dA_s$ and that fB is itself an additive functional if, for example, B is finite on $[0, \zeta)$ and f is bounded. As usual equality between additive functionals means equivalence.

(2.19) Theorem. *In the above notation $I_F A$ is a CAF and $I_F A = I_{D^*} A$. If, in addition, D is admissible, then $A = I_F A + I_{F^c} A$ is the decomposition of A into its continuous and purely discontinuous parts.*

Proof. By (2.12), $B = I_{D^*} A$ is a CAF. But $F \subset D^* \subset D$ and F is the largest finely perfect set contained in D . Therefore the fine support of B is contained in F . As a result $I_F A = I_{D^*} A$ and this also implies $I_{F^c} A = I_{(D^*)^c} A$.

Now suppose that D is admissible. Let $A = A^c + A^i$ be the decomposition of A into its continuous and purely discontinuous parts. From its definition, (2.9), $J(\omega)$ is the support of the measure $dA_t(\omega)$ on $[0, \infty)$. If for each ω , $\Gamma(\omega)$ is a subset of $[0, \infty)$, let $\varphi_\Gamma(t) = \varphi_{\Gamma(\omega)}(t)$ denote the indicator function of $\Gamma(\omega)$. Then as measures on $[0, \infty)$ we have

$$(2.20) \quad \begin{aligned} I_{F^c}(X_t) dA_t &= I_{F^c}(X_t) \varphi_J(t) dA_t = \varphi_{J-Z_F}(t) dA_t \\ &= \varphi_{J-Z_F}(t) dA_t^c + \varphi_{J-Z_F}(t) dA_t^i = \varphi_{J-Z_F}(t) dA_t^i, \end{aligned}$$

because $J - Z_F$ is countable by (2.10) and (2.11), and hence is not charged by dA_t^c . This implies $I_{F^c} A$ is purely discontinuous. Consequently $A^c = I_F A$ and $A^i = I_{F^c} A$. This completes the proof of Theorem 2.19.

(2.21) Remark. Since $A^c = I_F A = I_{D^*} A$ it follows that for admissible D

$$t \rightarrow \int_0^t \varphi_{J-Z_F}(X_s) dA_s = \int_0^t \varphi_{J-Z_{D^*}}(s) dA_s$$

is the purely discontinuous part of A , and that dA_t does not charge $Z_{D^*} - Z_F$. Moreover, we know from (2.10) that $J = \bar{Z}_D$. Example 4.3 shows that the second assertion in (2.19) is *not* valid for general D .

It is well known that the purely discontinuous part, A^i , of A is given by

$$(2.22) \quad A_t^i = \sum_{\substack{s \leq t \\ s < \tau}} [v(X_s)_- - v(X_s)] I_{\{X_s = X_{s-}\}},$$

and in the light of the above remark, if D is admissible the sum in (2.22) need only be taken over those s in $J - Z_{D^*} = \bar{Z}_D - Z_{D^*}$. In particular it follows that $t \rightarrow v(X_t)$ is continuous at all t not in $\bar{Z}_D - Z_{D^*}$ at which $t \rightarrow X_t$ is continuous.

In the course of the proof of (2.19) it was observed that the fine support of $A^c = I_F A$ is contained in F . However, even when D is itself finely perfect, *i.e.*, $D = F$, $I_F A$ may be zero. See Example 4.4.

3. Equilibrium potentials. In this section we specialize the situation in Section 2 to the case $u = 1$ and investigate in more detail the structure of A . Throughout this section we assume that D is a finely closed nearly Borel set which is admissible and is such that $\phi_D = P_D^1 1$ is the 1-potential of a natural additive functional A . This amounts to assuming that $u = 1$ in the notation of Section 2, and then $v = \phi_D$. We shall use the notation of Section 2 without special mention. In the present situation we are able to give a more precise description of the

purely discontinuous part of A . We emphasize that in this section we are assuming that D is *admissible* and that this will not be mentioned explicitly in the statements of our results.

(3.1) **Proposition.** *Let A^i denote the discontinuous part of A . Then*

$$(3.2) \quad A_t^i = \sum_{\substack{0 \leq s \leq t \\ s < \zeta, s \in \bar{Z}_D}} [1 - \phi_D(X_s)] I_{\{X_s = X_{s-}\}}.$$

Proof. First observe that if $s \in Z_D$, then $\phi_D(X_s) = 1$, and so the sum in (3.2) really extends only over $s \in \bar{Z}_D - Z_D$. In view of (2.21) and (2.22) we must show that $\phi_D(X_s)_- = 1$ if $0 < s < \zeta$, $X_s = X_{s-}$, and $s \in \bar{Z}_D$. For this it suffices to show that the projection of

$$M = \{(t, \omega) : t \in \bar{Z}_D, X_t = X_{t-}, \phi_D(X_t)_- < 1, 0 < t < \zeta\}$$

on Ω is almost surely null. Meyer, see [7], has shown that $\{(t, \omega) : t \in \bar{Z}_D(\omega)\}$ is well measurable (relative to the family $(\mathcal{F}_t^\mu)$ for each initial measure μ). The phrase in parentheses will be omitted in the sequel. As a result it is easy to see that M itself is well measurable. Suppose that for some μ the projection of M on Ω has positive P^μ measure. Then by the section theorem [7] there exists a stopping time T such that $[T] \subset M$ and $P^\mu(T < \infty) > 0$. Here $[T] = \{(T(\omega), \omega) : T(\omega) < \infty\} \subset \mathbf{R}^+ \times \Omega$ is the graph of T . But $[T] \subset M$ implies that $X_T = X_{T-}$ on $\{T < \zeta\} = \{T < \infty\}$, and so it follows from a result of Meyer, XIV-T38 of [13], see also [12], that T is accessible. Therefore T is actually previsible because X is special standard. Let $\{T_n\}$ be an increasing sequence of stopping times that increases to T strictly from below almost surely P^μ . Note that $T > 0$. Let

$$\Gamma_n = \{\omega : (T_n, T) \cap Z_D \text{ is empty, } X_T \in D, T < \infty\} \in \mathcal{F}_T.$$

On Γ_n , $X_T = X_{T-}$, $T < \zeta$, and $T = T_n + T_D \circ \theta_{T_n}$. Fix n and let $Q = T_n + T_D \circ \theta_{T_n}$. Then

$$\Gamma_n \subset \{X_Q = X_{Q-}; Q < \zeta\} \cap \{T = Q\},$$

and so

$$\begin{aligned} P^\mu\{\phi_D(X_T)_- < 1; \Gamma_n\} &\leq P^\mu\{\phi_D(X_Q)_- < 1, X_Q = X_{Q-}, Q < \zeta\} \\ &= E^\mu\{P^{X(T_n)}[\phi_D(X_{T_D})_- < 1, X_{T_D} = X_{T_D-}, T_D < \zeta]; T_n < \zeta\} = 0, \end{aligned}$$

because T_D is accessible on $\{X_{T_D} = X_{T_D-}, T_D < \zeta\}$ and it is well known that $\phi_D(X_{T_D})_- = 1$ on any set on which T_D is accessible and finite. See, for example, (5.9) of [6]. Since $\{T < \infty\}$ is contained in $\{\phi_D(X_T)_- < 1\}$ we have up to a set of P^μ measure zero $\{T < \infty\} = \{T < \infty\} \cap (\bigcap_n \Gamma_n^c)$. But

$$\Gamma_n^c \cap \{T < \infty\} \subset \{(T_n, T) \cap Z_D \text{ is not empty, } T < \infty\}$$

because $T \in \bar{Z}_D$ if $T < \infty$ and the fact that Z_D is closed for the right topology yield

$$\{X_T \notin D; T < \infty\} = \{T \notin Z_D, T \in \bar{Z}_D, T < \infty\} \\ \subset \{(T_n, T) \cap Z_D \text{ is not empty}\}.$$

Consequently if

$$\Lambda = \{(T_n, T) \cap Z_D \text{ is not empty for all } n, T < \infty\},$$

then $\Lambda \in \mathfrak{F}_T$ and $P^\mu(T < \infty) = P^\mu(\Lambda)$. Let $R_n = T_n + T_D \circ \theta_{T_n}$ and $R = \lim R_n$. On Λ , $T_n \leq R_n < T$ and $R = T$. If $\Gamma \in \mathfrak{F}_{T_k}$ and $n \geq k$

$$E^\mu\{e^{-R_n}; \Gamma\} = E^\mu\{e^{T_n} \phi_D(X_{T_n}); \Gamma\},$$

and letting $n \rightarrow \infty$ this becomes

$$(3.3) \quad E^\mu\{e^{-R}; \Gamma\} = E^\mu\{e^{-T} \phi_D(X_T); \Gamma\}$$

for all $\Gamma \in \mathfrak{F}_{T_k}$ and hence for all $\Gamma \in \mathfrak{F}_T$. Letting $\Gamma = \Lambda$ in (3.3) we see that $\phi_D(X_T)_- = 1$ almost surely P^μ on Λ . But $\Lambda \subset \{T < \infty\}$ and so $\phi_D(X_T)_- < 1$ on Λ . As a result $P^\mu(\Lambda) = 0$ contradicting $P^\mu(T < \infty) > 0$. Thus M has a null projection on Ω and (3.1) is established.

We turn next to an analysis of the continuous part A^c of A for an equilibrium potential ϕ_D . For simplicity we assume that D itself is finely perfect, that is, $D = F$, although most of what follows does not depend on this assumption. Obviously $\bar{Z}_F$ is contained in the interval $[T_F, \infty)$. Let $I_\mu = (a_\mu, b_\mu)$, $\mu \geq 1$, be the contiguous intervals of $\bar{Z}_F$ in $[T_F, \infty)$. Thus the I_μ together with $(-\infty, T_F)$ is the set of contiguous intervals of $\bar{Z}_F$ as a subset of the real line. Let Z_F and $\bar{Z}_F$ denote $\{(t, \omega); t \in Z_F(\omega)\}$ and $\{(t, \omega); t \in \bar{Z}_F(\omega)\}$ respectively. Clearly Z_F is well measurable and it follows from the previously mentioned result of Meyer that $\bar{Z}_F$ is also well measurable. Another much deeper result of Dellacherie [7] now tells us that

$$(3.4) \quad \bar{Z}_F - Z_F = \bigcup_{n \geq 1} [T_n]$$

where each T_n is a stopping time and the graphs, $[T_n]$, of the T_n are disjoint. For each T_n in (3.4) define $J_n = (T_n, T_n + T_F \circ \theta_{T_n})$. If $T_n(\omega) < \infty$, then $T_n(\omega) \in \bar{Z}_F(\omega) - Z_F(\omega)$ and so $J_n(\omega)$ is a contiguous interval of $\bar{Z}_F(\omega)$ in $[T_F(\omega), \infty)$. Let $J'_k(\omega)$, $k \geq 1$, denote the contiguous intervals of $\bar{Z}_F(\omega)$ in $[T_F(\omega), \infty)$ which are disjoint from the $J_n(\omega)$. Clearly the totality of the intervals $\{J_n(\omega), J'_k(\omega); n, k \geq 1\}$ coincides with $\{I_\mu(\omega)\}$. Moreover if $a'_k(\omega)$ is the left end point of $J'_k(\omega)$ we must have $a'_k(\omega) \in Z_F(\omega)$. Now

$$\phi_F(x) = E^x(e^{-T_F}) = E^x \int_{T_F}^{\infty} e^{-t} dt \\ = E^x \int_0^{\infty} e^{-t} I_F(X_t) dt + E^x \sum_n \int_{J_n} e^{-t} dt + E^x \sum_n \int_{J'_n} e^{-t} dt,$$

and

$$\begin{aligned}
E^x \sum_n \int_{J_n} e^{-t} dt &= \sum_n E^x \int_{T_n}^{T_{n+1} + T_F \circ \theta T_n} e^{-t} dt \\
&= E^x \sum_n e^{-T_n} [1 - \phi_F(X_{T_n})] = E^x \sum_{s \in \tilde{Z}_F} e^{-s} [1 - \phi_F(X_s)]
\end{aligned}$$

because $\phi_F(X_s) = 1$ if $s \in Z_F$. Recalling the form of A^i given in (3.1) and defining

$$(3.5) \quad A'_i = \sum_{\substack{s \leq i \\ s < \zeta, s \in \tilde{Z}_F}} [1 - \phi_F(X_s)] I_{\{X_s \neq X_{s-}\}}$$

we see that

$$\begin{aligned}
(3.6) \quad \phi_F(x) &= E^x \int_0^\infty e^{-t} I_F(X_t) dt + u_{A'}^1(x) + u_A^1(x) \\
&\quad + E^x \sum_n \int_{J_n} e^{-t} dt + \sum_n E^x \{e^{-\zeta}; \zeta = T_n\}.
\end{aligned}$$

Here we are tacitly assuming that A' is an additive functional. This will be established shortly. Suppose we could find a *continuous* additive functional B with the same one potential as A' . Then (3.6) would lead one to suppose that $A_i^c = \int_0^i I_F(X_s) ds + B_i + A_i^*$ where A^* is a CAF whose one potential is given by the sum of the two last terms in (3.6). However, this requires some justification since we don't know a priori that the two sums in (3.6) are the one potential of an additive functional.

The preceding discussion should be regarded as motivation for the following result. We let (N, H) be a fixed Lévy system for X , that is, $N(x, dy)$ is kernel and $H = (H_t)$ is a CAF with a bounded one potential such that for all non-negative Borel $\Phi(x, y)$ defined on $E \times E$ with $\Phi(x, x) = 0$ we have

$$(3.7) \quad E^x \sum_{s \leq t} \Phi(X_{s-}, X_s) = E^x \int_0^t N\Phi(X_s) dH_s$$

for all t and x where $N\Phi(x) = \int N(x, dy)\Phi(x, y)$. See [14] or [17].

(3.8) Proposition. *Let F be a finely perfect set such that $\phi_F(x) = E^x(e^{-T_F})$ is the one potential of a natural additive functional A . Then the continuous part A^c of A has the form*

$$A_i^c = \int_0^i I_F(X_s) ds + \int_0^i I_F(X_s) \int_{E-F} N(X_s, dy) [1 - \phi_F(y)] dH_s + A_i^*$$

where A^* is a CAF whose one potential is given by the sum of the last two terms in (3.6).

Proof. Assume for the moment that A' defined in (3.5) is given by

$$(3.9) \quad A'_i = \sum_{s \leq i} I_F(X_{s-}) I_{E-F}(X_s) [1 - \phi_F(X_s)];$$

more precisely, that A' and the process defined by the sum on the right side

of (3.9) are indistinguishable (with respect to P^μ for all μ). Then A' is (equivalent to) an additive functional, and it follows from (3.7) that for all t and x

$$(3.10) \quad E^x \int_0^t e^{-s} dA'_s = E^x \int_0^t e^{-s} I_F(X_s) \int_{E-F} N(X_s, dy) [1 - \phi_F(y)] dH_s.$$

Recall from (V-2.6) of [4] that if B is a CAF then $A^\circ = B + B^*$ with B^* a CAF if and only if

$$(3.11) \quad E^x \int_0^t e^{-s} dB_s \leq E^x \int_0^t e^{-s} dA_s^\circ$$

for all t and x . Now

$$E^x \int_0^t e^{-s} dA_s = \phi_F(x) - e^{-t} P_t \phi_F(x) = E^x \int_{T_F}^{t+T_F \circ \theta_t} e^{-s} ds.$$

But the interval $J_n = (T_n, T_n + T_F \circ \theta_{T_n})$ is contained in $[T_F, t + T_F \circ \theta_t]$ if and only if $T_n \leq t$ because if $T_n > t$ the fact that $T_n \in \bar{Z}_F - Z_F$ implies that $T_n > t + T_F \circ \theta_t$, and so

$$E^x \int_0^t e^{-s} dA_s \geq E^x \int_0^t e^{-s} I_F(X_s) ds + E^x \sum_{T_n \leq t} \int_{J_n} e^{-s} ds.$$

The argument leading to (3.6) now yields

$$(3.12) \quad E^x \int_0^t e^{-s} dA_s \geq E^x \int_0^t e^{-s} I_F(X_s) ds + E^x \int_0^t e^{-s} dA_s^i + E^x \int_0^t e^{-s} dA_s'.$$

Combining (3.10), (3.11), (3.12), and the fact that $A^\circ = A - A^i$ we obtain the decomposition asserted in (3.8). The evaluation of the potential of A^* then follows from (3.6).

Thus it remains to prove (3.9). For this in view of the definition of A' in (3.5) it suffices to show that for almost all ω

$$(3.13) \quad (\bar{Z}_F - Z_F) \cap \{s: X_{s-} \neq X_s; s < \zeta\} = \{s: X_{s-} \in F, X_s \in E - F\}.$$

A result of Meyer (see [18]) states that if f is 1-excessive, then almost surely $f(X_t)_- = f(X_{t-})$ for all $t \in (0, \zeta)$ at which $X_{t-} \neq X_t$. Now suppose t belongs to the left side of (3.13). Then $0 < t < \zeta$ and there exists a sequence $\{t_n\}$ which increases to t strictly from below and such that $t_n \in Z_F$ for each n . Therefore $1 = \phi_F(X_{t_n}) \rightarrow \phi_F(X_t)_- = \phi_F(X_{t-})$ by Meyer's result. Consequently $X_{t-} \in F$ and so t belongs to the right side of (3.13). Conversely suppose t belongs to the right side of (3.13). Then $t \notin Z_F$, $X_t \neq X_{t-}$, and $t < \zeta$. If t is not in $\bar{Z}_F$, then there exists $h > 0$ such that $(t-h, t+h) \cap Z_F$ is empty. Consequently the set of ω 's such that the right side of (3.13) is not contained in the left side is contained in

$$\bigcup_{0 < r < q} \{r + T_F \circ \theta_r > q; r + L_F \circ \theta_r < q < \zeta\}$$

where the union is over all rationals r, q with $0 < r < q$. But this last set is almost surely null because $T_F = L_F$ almost surely on $\{L_F < \zeta\}$. This establishes (3.9) and completes the proof of (3.8).

Of course, the most interesting component in the decomposition (3.8) is A^* about which we have a very little information in general. For example, if X is Brownian motion in one dimension and $F = \{0\}$, then $A = A^c = A^*$ and A is the local time of X at 0.

The assumption that $D = F$ was used only in establishing (3.9). Thus in the general case (assumptions as in the first paragraph of this section) $A_t^c = \int_0^t I_F(X_s) ds + A_t^*$ where A^* is a CAF.

This leads to the following question. Suppose $D = F \cup H$ with H not empty and D satisfying the conditions in the first paragraph of this section. Then we may apply the above discussion to both D and F , that is $\phi_D = u_A^1$ and $\phi_F = u_B^1$ where A and B are natural additive functionals and we know that their continuous parts are given by $A^c = I_F A$ and $B^c = I_F B$. What is the relationship between A^c and B^c ? Example 4.2 shows that they need not be the same. However, in that example the fine supports of A^c and B^c are the same. Whether or not this is true in general is an open question. It would be of even more interest to determine explicitly the fine supports of B^c and A^c .

We turn next to conditions that guarantee that A is itself continuous. As is well known, A is continuous if and only if ϕ_D is regular. In this case if x is in F , then $P^x(T_D = 0) = 1$, and in light of (2.8) this implies that $\text{Supp}(A) = F$. But then by the properties of $\text{Supp}(A)$, $R = T_F$, and hence using (2.8) again, $T_D = T_F$. Consequently $\phi_D = \phi_F$ and we may as well assume that $D = F$.

Before stating our result it is necessary to recall some definitions. A nearly Borel set B is called a d -set if whenever $\{T_n\}$ is an increasing sequence of stopping times with limit T

$$P^x[X(T_n) \in B \text{ for all } n, X(T) \notin B, T < \zeta] = 0$$

for all x . See [11] or [12]. A nearly Borel set B is *projective* if whenever $\{T_n\}$ is an increasing sequence of stopping times with limit T , then $\lim (T_n + T_B \circ \theta_{T_n}) = T + T_B \circ \theta_T$ almost surely on $\{\lim (T_n + T_B \circ \theta_{T_n}) < \zeta\}$. See (V-4.2) and (V-4.3) of [4]. Finally we shall say that a nearly Borel set B is *quasi-left-closed* if almost surely

$$(\bar{Z}_B - Z_B) \cap \{t: 0 < t < \zeta\} \subset \{t: X_{t-} \in B\}.$$

We are now able to state our next result.

(3.14) Proposition. *Let F be a finely perfect set. Then the following are equivalent:*

- (a) ϕ_F is regular.
- (b) F is a d -set.
- (c) F is projective.
- (d) F is quasi-left-closed.

Proof. Proposition 2.8 of [11] states that a finely closed nearly Borel set B

with $B - B'$ polar is projective if and only if it is a d -set. This obviously yields the equivalence of (b) and (c). Next we show that (c) implies (a). For this argument the assumption that F is finely perfect is not necessary and so we formulate what is actually proved as a lemma.

(3.15) *Lemma.* *Let B be a projective subset of E . Then ϕ_B is regular.*

Proof. Let $\psi(x) = E^x(e^{-t})$. Let $\psi_B = P_B^1 U^1 1$. Then $\psi_B = \phi_B - \psi$. It is well known that ψ is regular. (Recall that we are supposing that X is special standard.) Therefore to establish (3.15) it will suffice to show that ψ_B is regular. Let $\{T_n\}$ be an increasing sequence of stopping times with limit T . Then we must show that $\lim_n \psi_B(X_{T_n}) = \psi_B(X_T)$ almost surely on $\{T < \zeta\}$. Let L be the limit in question which exists on $\{T < \infty\}$. Then we know that $L \geq \psi_B(X_T)$ on $\{T < \infty\}$ by IV-4.21 of [4]. Let $R_n = T_n + T_B \circ \theta_{T_n}$ and $R = \lim R_n$. Then $T \leq R \leq T + T_B \circ \theta_T$ and the assumption that B is projective amounts to $R = T + T_B \circ \theta_T$ if $R < \zeta$. Let $\Lambda \in \mathcal{F}_{T_k}$ and $n \geq k$. Then

$$E^x\{e^{-T_n}\psi_B(X_{T_n}); \Lambda\} = E^x\left\{\int_{R_n}^{\infty} e^{-t} I_E(X_t) dt; \Lambda\right\},$$

and letting $n \rightarrow \infty$ we obtain

$$(3.16) \quad E^x\{e^{-T}L; \Lambda\} = E^x\left\{\int_R^{\infty} e^{-t} I_E(X_t) dt; \Lambda\right\}$$

for all $\Lambda \in \mathcal{V} \vee \mathcal{F}_{T_k} = \mathcal{F}_T$. The integral in t on the right side of (3.16) really extends over those values of t such that $R < t < \zeta$. If $R < \zeta$, then $R = T + T_B \circ \theta_T$ while if $R \geq \zeta$ so is $T + T_B \circ \theta_T$. Consequently we obtain from (3.16)

$$\begin{aligned} E^x\{e^{-T}L; T < \zeta\} &= E^x\left\{\int_{T+T_B \circ \theta_T}^{\infty} e^{-t} I_E(X_t) dt; T < \zeta\right\} \\ &= E^x\{e^{-T}\psi_B(X_T); T < \zeta\}, \end{aligned}$$

and since $L \geq \psi_B(X_T)$ if $T < \zeta$, this implies that $\psi_B(X_T) = L = \lim_n \psi_B(X_{T_n})$ if $T < \zeta$. This establishes (3.15).

We now show that (a) implies (d). Suppose $t \in \bar{Z}_F - Z_F$ and that $0 < t < \zeta$. Then there exists a sequence $\{t_n\}$ increasing to t strictly from below with $t_n \in Z_F$ for all n . Hence $1 = \phi_F(X_{t_n}) \rightarrow \phi_F(X_t)_-$. We now consider two cases. If $X_t \neq X_{t-}$, then by Meyer's result [18], $\phi_F(X_t)_- = \phi_F(X_{t-})$ and so $X_{t-} \in F$. If $X_t = X_{t-}$, then by the regularity of ϕ_F , $\phi_F(X_t) = \phi_F(X_t)_- = 1$ contradicting $t \in \bar{Z}_F - Z_F$. Consequently F is quasi-left-closed.

Finally we show that (d) implies (b). Let $\{T_n\}$ be an increasing sequence of stopping times with limit T such that $X(T_n) \in F$ if $T < \zeta$. If $T_n = T$ for all large n , then $X_T \in F$. On $\{T_n < T \text{ for all } n, T < \zeta\}$, $X(T_n) \rightarrow X(T-) = X(T)$ by the quasi-left-continuity of X . If $X(T) \notin F$, then $T \in \bar{Z}_F - Z_F$ and (d) implies that $X(T-) \in F$. Consequently F is projective, and (3.14) is established.

4. Some examples.

(4.1) *Example.* Introduce the following process X . The state space $E = \{0\} \cup [1, \infty)$. Starting from a point $x \geq 1$ the process is translation to the right at speed 1, while 0 is a holding point with parameter 1 from which the process jumps to the point 1. This is a Hunt process with a reference measure. Let $D = \{1\}$. Then D is finely closed and thin. Consequently F is empty. Obviously D is not admissible. One computes $\phi_D(x) = 0$ if $x \geq 1$ and $\phi_D(0) = 1/2$. Let $A_t^0 = \int_0^t I_{\{0\}}(X_s) ds$. Again a trivial computation shows that $\phi_D = u_A^1$, so that $A = A^0$ is continuous. But almost surely $P^0, R = 0$ and $T_D > 0$. Thus Proposition 2.8 is not true if one drops the assumption that D be admissible. Also $I_F A = 0$, and so $I_F A$ is not the continuous part of A in this case.

(4.2) *Example.* The process is the same as in (4.1). Let $\{a_n\}$ be a strictly decreasing sequence with $a_1 = 2$ and $\lim a_n = 1$. Let $D = \{0, 1\} \cup \{a_n, n \geq 1\}$. Then D is admissible and $F = \{0\}$. Now $\phi_F(0) = 1$ and $\phi_F(x) = 0$ if $x \geq 1$. Thus if we let $A^F = 2A^0$ where A^0 is defined in (4.1), we see that A^F is the continuous additive functional whose 1-potential is ϕ_F . Straightforward evaluations show that $\phi_D(0) = \phi_D(1) = 1$, $\phi_D(x) = e^{-(a_n - x)}$ if $a_{n+1} \leq x < a_n$, and $\phi_D(x) = 0$ if $x \geq 2$. Let $\gamma_1 = 1$ and $\gamma_n = 1 - e^{-(a_{n-1} - a_n)}$ if $n \geq 2$. Define

$$A_t^i = \sum_{n=1}^{\infty} \gamma_n I_{(T_n, \infty)}(t)$$

where $T_n = T_{\{a_n\}}$. Clearly A^i is a natural additive functional. Let $A^D = A^0 + A^i$. Then one easily computes that the one potential of A^D is ϕ_D . Thus the continuous part of A^D is A^0 while (the continuous part of) $A^F = 2A^0$. See the discussion following the proof of Proposition 3.8.

(4.3) *Example.* Once again X is the process defined in (4.1). This time let $D = \{1\} \cup [2, 3)$ so that $F = [2, 3)$. Clearly D is not admissible. One easily sees that $\phi_D(0) = 1/2$, $\phi_D(x) = e^{x-2}$ if $1 \leq x < 2$, $\phi_D(x) = 1$ if $2 \leq x < 3$, and $\phi_D(x) = 0$ if $x \geq 3$. A simple computation shows that ϕ_D is the 1-potential of A defined by

$$A_t = (1 - e^{-1})A_t^0 + \int_0^t I_F(X_s) ds + I_{(T_3, \infty)}(t)$$

where A^0 is the CAF defined in (4.1). In this example the continuous part of A is not given by $I_F A$. Therefore Theorem 2.19 is not valid without the assumption that D is admissible. Also in this example I is not contained in Z_D . See (2.10) and (2.11). These points are also illustrated by Example 4.1, but F is empty in that example.

(4.4) *Example.* Let X be translation to the right at unit speed along the real line. Let D be the standard Cantor set contained in $[0, 1]$. Let $I_n = (a_n, b_n)$, $n \geq 1$ be the contiguous intervals of D that are contained in $(0, 1)$. Clearly $F = D - (\{1\} \cup \{a_n, n \geq 1\})$. By the properties of the Cantor set $\phi_D = \phi_F$

in this case. Let

$$\gamma_n = 1 - e^{-(b_n - a_n)} = e^{a_n}[e^{-a_n} - e^{-b_n}]$$

and define

$$A_t = \sum_n \gamma_n I_{[T_{a_n}, \infty)}(t) + I_{[T_1, \infty)}(t)$$

where $T_a = T_{\{a\}}$. Clearly A is a natural additive functional and we claim that $\phi_F(x) = u_A^1(x)$. This is clear if $x \geq 1$ since both $\phi_F(x)$ and $u_A^1(x)$ are zero in this case. Suppose $x \leq 0$. Then $\phi_F(x) = e^x$, while

$$\begin{aligned} u_A^1(x) &= E^x \int_0^\infty e^{-t} dA_t = \sum_n \gamma_n e^{x-a_n} + e^{x-1} \\ &= e^x \left[\sum_n (e^{-a_n} - e^{-b_n}) + e^{-1} \right]. \end{aligned}$$

But the Cantor set has Lebesgue measure zero and so

$$1 - e^{-1} = \int_0^1 e^{-t} dt = \sum_n \int_{I_n} e^{-t} dt = \sum_n (e^{-a_n} - e^{-b_n}),$$

and therefore $u_A^1(x) = e^x = \phi_F(x)$ if $x \leq 0$. If $x \in [a_n, b_n]$, then $\phi_F(x) = e^{x-b_n}$ while

$$\begin{aligned} u_A^1(x) &= E^x \int_0^\infty e^{-t} dA_t = \sum_{k, a_k > a_n} \gamma_k e^{x-a_k} + e^{x-1} \\ &= e^x \left[\sum_{\substack{k \\ a_k > a_n}} (e^{-a_k} - e^{-b_k}) + e^{-1} \right] \\ &= e^x \left[\int_{b_n}^1 e^{-t} dt + e^{-1} \right] = e^{x-b_n}. \end{aligned}$$

Thus $\phi_F = u_A^1$ almost everywhere Lebesgue and hence everywhere since Lebesgue measure is a reference measure. Therefore the continuous part of A , $I_F A$, is zero and $A = A^i$. Propositions (3.1) and (3.14) should be viewed in the light of this example. Note that in the present case $\bar{Z}_D - Z_F = D - F = \{1\} \cup \{a_n; n \geq 1\}$.

On the other hand if μ is the ordinary Cantor measure and B is the CAF whose one potential is $U^1\mu$, then it is evident that $\text{Supp}(B) = F$.

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