A uniform law of the iterated logarithm for functions C -uniformly distributed mod 1

OLAF P. STACKELBERG

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1. Introduction and results. A real-valued integrable function f defined for $t \geq 0$ is said to be C -uniformly distributed mod 1 if for every interval $[\alpha, \beta) \subset [0, 1]$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \chi(\{f(t)\}; \alpha, \beta) dt = \beta - \alpha,$$

where $\chi(\cdot; \alpha, \beta)$ is the characteristic function of the interval $[\alpha, \beta)$, and $\{y\}$ is the fractional part of y . Let $(C, \mathcal{C}, W)$ be the probability space of continuous functions on $t \geq 0$, with uniform topology, $\mathcal{C}$ the Borel sets, and let W be Wiener measure. Then $W\{f \in C: f(0) = 0\} = 1$. H. Hlawka [4] has proved that almost all continuous functions are C -uniformly distributed mod 1. In probabilistic terms Hlawka's theorem is a uniform law of large numbers. We extend this result and prove a uniform law of the iterated logarithm.

Theorem. Let $(C, \mathcal{C}, W)$ be the probability space of continuous functions with Wiener measure. Given $\epsilon > 0$, for almost all $f \in C$ there exists a positive number $T_0(\epsilon)$ such that

$$(1.1) \quad \frac{\left| \int_0^T (\chi(\{f(t)\}; \alpha, \beta) - (\beta - \alpha)) dt \right|}{(2T \log \log T)^{1/2}} < \sigma(\alpha, \beta) + \epsilon,$$

for all $0 \leq \alpha < \beta \leq 1$, $T > T_0$ where $\sigma(\alpha, \beta) = 3^{-1/2}(\beta - \alpha)(1 - (\beta - \alpha))$. Furthermore,

$$(1.2) \quad W \left\{ f \in C : \limsup_{T \rightarrow \infty} \frac{\int_0^T (\chi(\{f(t)\}; \alpha, \beta) - (\beta - \alpha)) dt}{(2T \log \log T)^{1/2}} \geq \sigma(\alpha, \beta) \right. \\ \left. \text{for all } 0 \leq \alpha < \beta \leq 1 \right\} = 1.$$

Remark. There is only one null (exceptional) set for all pairs (α, β) , whereas in the traditional iterated logarithm law we expect an exceptional set for each pair (α, β) . Since the pairs (α, β) are uncountable, the uniform law does not follow from the traditional law.

The discrepancy D_T of the process is defined as (see [2; §11])

$$D_T = \sup_{\alpha, \beta} \left\{ \frac{1}{T} \int_0^T \chi(\{f(t)\}; \alpha, \beta) dt - (\beta - \alpha) : 0 \leq \alpha < \beta \leq 1 \right\}.$$

Then we have the following corollary to the theorem above:

Corollary. For almost all $f \in C$

$$\limsup_{T \rightarrow \infty} \frac{T D_T}{(2T \log \log T)^{1/2}} = \frac{3^{1/2}}{12}.$$

J. W. S. Cassels [1] was the first to establish a uniform iterated logarithm law, namely for a family of sequences of independent Bernoulli variables. The idea of his proof is to approximate the family of sequences by not too large subfamilies, and to establish the iterated logarithm law for these subfamilies by standard methods. Our theorem relies on a recent generalization of Cassel's theorem by W. Philipp [8; Section 1.3] to certain weakly stationary, strongly mixing families of sequences of random variables. Let $\{x_n\}$ be a sequence of random variables, and denote by $\mathfrak{M}_{S, T}$ the σ -field generated by the variables x_n with $S \leq n \leq T$. The sequence x_n is strongly mixing if, for any event $A \in \mathfrak{M}_{0, N}$ and any event $B \in \mathfrak{M}_{N+M, \infty}$,

$$(1.3) \quad |PAB - PAPB| \leq PAPB\psi(M)$$

where $\psi(M) \rightarrow 0$ monotonically as $M \rightarrow \infty$.

Theorem P. ([8, Theorem 1.3.1 and Theorem 1.3.2]). Let $\mathfrak{A} = \{x_n(\gamma), n = 1, 2, \dots; \text{ with } \gamma \in \Gamma\}$ be a family of weakly stationary sequences of random variables $x_n(\gamma)$ uniformly bounded by 1. Set

$$(1.4) \quad S_N^2(\gamma) = S_N^2(x_n(\gamma)) = E \left(\sum_{n \leq N} (x_n(\gamma) - E x_n(\gamma)) \right)^2,$$

the variance of the partial sum, and assume that

$$(1.5) \quad \lim_{N \rightarrow \infty} \frac{1}{N} S_N^2(\gamma) = \sigma^2(\gamma) = \sigma^2(x_n(\gamma)) < 1 \quad \text{uniformly for } \gamma \in \Gamma.$$

Suppose that for each non-negative integer t there is a subclass $\Gamma_t \subset \Gamma$ which defines a subfamily $\mathfrak{A}_t \subset \mathfrak{A}$ whose elements we denote by $\{x_n(t); n = 1, 2, \dots\}$. Suppose further that for each t the family $\mathfrak{A}_t$ satisfies the following conditions:

- i) $1 \leq \text{card } \mathfrak{A}_t \leq \exp(c_1, t)$ for some constant $c_1 > 0$.
- ii) Each process $\{x_n(t); n = 1, 2, \dots\} \in \mathfrak{A}_t$ satisfies the mixing conditions (1.3) with

$$\psi_t(M) = \begin{cases} 1 & \text{for } 1 \leq M \leq t \\ c \exp(-\lambda(M-t)^\delta) & \text{for } t < M \end{cases}$$

where c , λ and δ are positive constants independent of t .

iii) For each random variable $x_n(\gamma)$ there are two sequences of random variables $x_n(0), x_n(1), \dots$; and $\bar{x}_n(0), \bar{x}_n(1), \dots$; such that for any $s \geq 0$,

$$(1.6) \quad y_n(s) = \sum_{t=0}^s x_n(t) \leq x_n(\gamma) \leq \sum_{t=0}^s \bar{x}_n(t) = \bar{y}_n(s)$$

and

$$(1.7) \quad 0 \leq E(\bar{y}_n(s) - y_n(s)) \leq \exp(-c_2 s)$$

with c_2 a positive constant. For each t both $\{x_n(t); n = 1, 2, \dots\}$ and $\{\bar{x}_n(t); n = 1, 2, \dots\}$ are members of $\mathfrak{A}_t$.

iv) Using notation (1.4)

$$S_N^2(t) = O(t^{-6}N) \text{ uniformly in } t \leq \log N.$$

v) Each process $\{y_n(s); n = 1, 2, \dots\}$, and each $\{\bar{y}_n(s); n = 1, 2, \dots\}$ satisfies the mixing condition (1.3) with $\psi_s(M)$ defined in ii).

vi) $\lim (1/N)S_N^2(y_n(s)) = \sigma^2(y_n(s))$ uniformly in s .

Then, for any $\epsilon > 0$, there almost always is an $N_0(\epsilon, c, \lambda, \delta)$ such that

$$(1.8) \quad \frac{\left| \sum_{n=1}^N (x_n(\gamma) - Ex_n(\gamma)) \right|}{(2N \log \log N)^{1/2}} < \sigma(\gamma) + \epsilon, \text{ for all } \gamma \in \Gamma, N > N_0.$$

Furthermore,

$$(1.9) \quad P \left\{ \limsup_{N \rightarrow \infty} \frac{\sum_{n=1}^N (x_n(\gamma) - Ex_n(\gamma))}{(2N \log \log N)^{1/2}} \geq \sigma(\gamma), \text{ for all } \gamma \in \Gamma \right\} = 1$$

The weakly stationary requirement is not needed in its full strength in Theorem P. It is apparent from the proof of Theorem P that the following condition on the variables $y_n(s)$ will suffice:

$$E \left(\sum_{n=1}^N y_n(s) - Ey_n(s) \right)^2 = N^2 \sigma^2(y_n(s)) + O(1)$$

uniformly in s as $N \rightarrow \infty$. (See Lemma 1.3.1 and 1.3.5 of [8]). Similarly for the $\bar{y}_n(s)$. In our application the $y_n(s)$ and $\bar{y}_n(s)$ are themselves processes in $\mathfrak{A}$, and condition vi) is the same as (1.5). It is also clear from the proof of Theorem P that $0 \leq E(\bar{y}_n(s) - y_n(s)) \leq 1$ for all s , and (1.7) for all s greater than some fixed s_0 independent of n can replace (1.7) in iii). Finally, in our application

$$\psi_t(M) = e^{-M} \text{ in ii).}$$

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2. Definition of processes and moments. For $0 \leq \alpha < \beta \leq 1$ and $f \in C$, define

$$(2.1) \quad x_N(f; \alpha, \beta) = \int_{N-1}^N \chi(\{f(t)\}; \alpha, \beta) dt \quad \text{for } N = 1, 2, \dots$$

where $\chi(\{f(t)\}; \alpha, \beta)$ is defined in Section 1. We write x_N for $x_N(f, \alpha, \beta)$, and $\chi(t)$ for $\chi(\{f(t)\}; \alpha, \beta)$ whenever the context is clear. Notice that for $0 \leq \alpha \leq \beta \leq \omega \leq 1$,

$$(2.2) \quad x_N(f; \alpha, \beta) + x_N(f; \beta, \omega) = x_N(f; \alpha, \omega).$$

The x_N are random variables on the probability space $(C, \mathcal{C}, W)$, and

$$\sum_{k=1}^N x_k = \int_0^N \chi(t) dt.$$

It can be shown that the x_N are continuous functions of f , but we will not need this fact in what follows.

Let Γ be the class of all pairs (α, β) with $0 \leq \alpha \leq \beta \leq 1$. Define the family

$$(2.3) \quad \mathfrak{A} = \{ \{x_n(f; \alpha, \beta); n = 1, 2, \dots\}, \{-x_n(f; \alpha, \beta); n = 1, 2, \dots\} \mid (\alpha, \beta) \in \Gamma \}.$$

For fixed α and β we compute the first two moments of x_N .

$$Ex_N = \int_{N-1}^N E\chi(t) dt$$

by Fubini's theorem. For $t > 0$,

$$\begin{aligned} E\chi(t) &= W[\alpha \leq \{f(t)\} < \beta] \\ &= \sum_{n=-\infty}^{\infty} W[n + \alpha \leq f(t) < n + \beta] \\ &= \sum_n \frac{1}{(2\pi t)^{1/2}} \int_{n+\alpha}^{n+\beta} \exp\left(-\frac{x^2}{2t}\right) dx \\ &= \frac{1}{(2\pi t)^{1/2}} \int_{\alpha}^{\beta} \sum_n \exp\left(-\frac{(x+n)^2}{2t}\right) dx \end{aligned}$$

The interchange of summation and integration follows from standard limit theorems, and we will feel free to perform the numerous interchanges hereafter without further comment.

We take the following remarkable identity from the theory of theta functions (See [6, page 277]): For $z > 0$ and x real

$$(2.4) \quad z^{1/2} \sum_{n=-\infty}^{\infty} \exp(-(n+x)^2 \pi z) = \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2 \pi}{z}\right) \cos(2\pi n x).$$

Setting $z = 1/2\pi t$, we get from (2.4)

$$(2.5) \quad \frac{1}{(2\pi t)^{1/2}} \sum_n \exp\left(-\frac{(x+n)^2}{2t}\right) = \sum_n \exp(-2\pi^2 n^2 t) \cos(2\pi n x) \\ = \sum_n \exp(-2\pi^2 n^2 t) \exp(2\pi n i x).$$

Then

$$E\chi(t) = \sum_n \exp(-2\pi^2 n^2 t) \int_{\alpha}^{\beta} \cos(2\pi n x) dx \\ = \beta - \alpha + 2 \sum_{n=1}^{\infty} \exp(-2\pi^2 n^2 t) \int_{\alpha}^{\beta} \cos(2\pi n x) dx.$$

Now

$$\left| \int_{\alpha}^{\beta} \cos(2\pi n x) dx \right| \leq \min\{(\beta - \alpha), 1 - (\beta - \alpha)\}$$

and we have

$$|Ex_N - (\beta - \alpha)| < 2 \min\{\beta - \alpha, 1 - (\beta - \alpha)\} \\ \cdot \sum_{n=1}^{\infty} \frac{\exp(-2\pi^2 n^2 (N-1)) - \exp(-2\pi^2 n^2 N)}{2\pi^2 n^2} \\ = \min\{\beta - \alpha, 1 - (\beta - \alpha)\} O(e^{-N}).$$

The sum

$$\sum_{n=1}^{\infty} \frac{\exp(-2\pi^2 n^2 (N-1)) - \exp(-2\pi^2 n^2 N)}{2\pi^2 n^2} < \sum_{n=1}^{\infty} \frac{1}{2\pi^2 n^2} = \frac{1}{12}, \\ \text{for } N = 1, 2, \dots$$

and we get

$$(2.6) \quad Ex_N < \frac{7}{6} (\beta - \alpha) \quad \text{for } N = 1, 2, \dots$$

Then

$$(2.7) \quad E \sum_{n=1}^N x_n = (\beta - \alpha)N + \min\{\beta - \alpha, 1 - (\beta - \alpha)\} O(1) \quad \text{as } N \rightarrow \infty$$

and

$$E \int_0^T \chi(t) dt = (\beta - \alpha)T + \min\{\beta - \alpha, 1 - (\beta - \alpha)\} O(1) \quad \text{as } T \rightarrow \infty.$$

The computation for the second moments are only somewhat more complicated. For fixed $0 \leq \alpha < \beta \leq 1$, we consider

$$\begin{aligned}
 (2.8) \quad E\left(\int_0^T \chi(t) dt - E \int_0^T \chi(t) dt\right)^2 &= E\left(\int_0^T (\chi(t) - (\beta - \alpha)) dt + O(1)\right)^2 \\
 &= E\left(\int_0^T (\chi(t) - (\beta - \alpha)) dt\right)^2 + O(1).
 \end{aligned}$$

Now

$$\begin{aligned}
 (2.9) \quad E\left(\int_0^T (\chi(t) - (\beta - \alpha)) dt\right)^2 \\
 = 2 \int_0^T \int_0^t E(\chi(s) - (\beta - \alpha))(\chi(t) - (\beta - \alpha)) ds dt
 \end{aligned}$$

and for positive s and t ,

$$\begin{aligned}
 (2.10) \quad E(\chi(s) - (\beta - \alpha))(\chi(t) - (\beta - \alpha)) \\
 = E\chi(s)\chi(t) - (\beta - \alpha)(E\chi(s) + E\chi(t)) + (\beta - \alpha)^2.
 \end{aligned}$$

Set $p(x, t) = 1/(2\pi t)^{1/2} \cdot \exp(-x^2/2t)$. For $0 < s < t$,

$$\begin{aligned}
 (2.11) \quad E\chi(s)\chi(t) &= W[\alpha \leq \{f(s)\} < \beta; \alpha \leq \{f(t)\} < \beta] \\
 &= \sum_n \sum_m W[n + \alpha \leq f(s) < n + \beta; m + \alpha \leq f(t) < m + \beta] \\
 &= \sum_n \sum_m \int_{n+\alpha}^{n+\beta} \int_{m+\alpha}^{m+\beta} p(x, s)p(y - x, t - s) dy dx \\
 &= \sum_n \sum_m \int_\alpha^\beta \int_\alpha^\beta p(x + n, s)p(y + m - x - n, t - s) dy dx
 \end{aligned}$$

where both m and n range over all integers. Up to here the development is standard. (See [5; chapter 4.]).

Set

$$(2.12) \quad q(n, x, t) = \exp(-2\pi^2 n^2 t) \cos(2\pi nx).$$

We change the summation variable m in (2.11) and get

$$\begin{aligned}
 (2.13) \quad E\chi(s)\chi(t) &= \sum_n \sum_m \int_\alpha^\beta \int_\alpha^\beta p(x + n, s)p(y - x + m, t - s) dy dx \\
 &= \sum_n \sum_m \int_\alpha^\beta \int_\alpha^\beta q(n, x, s)q(m, y - x, t - s) dy dx
 \end{aligned}$$

where the last equality follows (after interchanging summation and integration) from (2.12) and the identity (2.5) applied twice. We break (2.13) into parts as follows:

$$(2.14) \quad \sum_n \sum_m = \sum_n \sum_{\substack{m \\ |m| \neq |n|}}' + \sum_{m=n}' + \sum_{m=-n}' + \sum_{m=0}' + \sum_{n=0}' \\ + (\text{term with } m = n = 0)$$

where the prime on the sum indicates the summation variable in question is not zero. The function q is even in its first variable, and the second and third sum on the right of the equality sign in (2.14) are equal. We can write

$$(2.15) \quad E\chi(s)\chi(t) = 4 \sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} q(n, x, s) q(m, y - x, t - s) dy dx \\ + 4(\beta - \alpha) \sum_{n=1}^{\infty} \int_{\alpha}^{\beta} q(n, y, t) dy$$

$$(2.16) \quad + 2(\beta - \alpha) \sum_{n=1}^{\infty} \int_{\alpha}^{\beta} q(n, x, s) dx \\ + 2 \sum_{m=1}^{\infty} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} q(m, y - x, t - s) dy dx + (\beta - \alpha)^2.$$

Now

$$(2.17) \quad \frac{1}{2} \text{ times (2.15) } = (\beta - \alpha) E\chi(t) - (\beta - \alpha)^2 \quad \text{and}$$

$$(2.16) = (\beta - \alpha) E\chi(s) - (\beta - \alpha)^2.$$

Combining (2.10), (2.15), (2.16), and (2.17) we get

$$(2.18) \quad \int_0^T \int_0^t E(\chi(s) - (\beta - \alpha))(\chi(t) - (\beta - \alpha)) ds dt \\ = 4 \int_0^T \int_0^t \sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} q(n, x, s) q(m, y - x, t - s) dy dx ds dt \\ + 2(\beta - \alpha) \int_0^T \int_0^t \sum_{n=1}^{\infty} \int_{\alpha}^{\beta} q(n, y, t) dy ds dt \\ + 2 \int_0^T \int_0^t \sum_{m=1}^{\infty} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} q(m, y - x, t - s) dy dx ds dt \\ = I + II + III, \quad \text{respectively.}$$

We estimate I :

$$(2.19) \quad |I| \leq 4 \sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \int_0^T \int_0^t \exp(-2\pi^2 n^2 s) \exp(-2\pi^2 m^2 (t - s)) ds dt \\ \leq \frac{1}{\pi^4} \sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{1}{|n^2 - m^2|} \left(\frac{1}{m^2} + \frac{1}{n^2} \right)$$

$$= \frac{2}{\pi^4} \sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{1}{|n^2 - m^2| n^2}.$$

Now for fixed $n > 1$,

$$\sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{1}{|n^2 - m^2|} = \sum_{m=1}^{n-1} \frac{1}{n^2 - m^2} + \sum_{m=n+1}^{\infty} \frac{1}{m^2 - n^2}.$$

But

$$\sum_{m=1}^{n-1} \frac{1}{n^2 - m^2} \leq \frac{n}{2n-1} < 1$$

and

$$\sum_{m=n+1}^{\infty} \frac{1}{m^2 - n^2} = \sum_{k=1}^{\infty} \frac{1}{2nk + k^2} < \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

which implies (2.19) converges, and we have

$$(2.20) \quad I = O(1) \text{ as } T \rightarrow \infty, \text{ independent of } \alpha \text{ and } \beta.$$

Next we estimate II :

$$(2.21) \quad |II| \leq 2 \sum_{n=1}^{\infty} \int_0^T \int_0^t \exp(-2\pi^2 n^2 t) ds dt = O(1) \text{ as } T \rightarrow \infty.$$

Straightforward computation gives

$$(2.22) \quad \begin{aligned} III &= \frac{T}{2\pi^4} \sum_{m=1}^{\infty} \frac{1 - \cos(2\pi m(\beta - \alpha))}{m^4} + O(1) \\ &= \frac{T}{6} (\beta - \alpha)^2 (1 - (\beta - \alpha))^2 + O(1) \text{ as } T \rightarrow \infty. \end{aligned}$$

The last equality follows from the identity

$$\frac{1}{\pi^{2k}} \sum_{m=1}^{\infty} \frac{\cos 2\pi mx}{m^{2k}} = (-1)^{k+1} \frac{2^{2k}}{2(2k)!} B_{2k}(x), \quad (0 \leq x < 1)$$

where $B_{2k}(x)$ is the $(2k)$ th Bernoulli function. (See, for instance, [7; pages 18 and 65].) In (2.22) $k = 2$, and

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},$$

and we get (2.22). Set

$$(2.23) \quad \sigma(\alpha, \beta) = \frac{1}{3^{1/2}} (\beta - \alpha)(1 - (\beta - \alpha)).$$

Then recalling (2.9), (2.10), (2.18), (2.20), (2.21), (2.22) and (2.23) we have

$$(2.24) \quad E\left(\int_0^T (\chi(t) - (\beta - \alpha)) dt\right)^2 = T\sigma^2(\alpha, \beta) + O(1) \text{ as } T \rightarrow \infty,$$

and the asymptotic variance (1.4), by (2.24), is

$$(2.25) \quad S_N^2(x_n(f; \alpha, \beta)) = N\sigma^2(\alpha, \beta) + O(1).$$

Remark. It can be shown that

$$E\left(\int_0^T (\chi(t) - (\beta - \alpha)) dt\right)^{2p} \sim \frac{(2p)!}{p! 2^p} (\sigma^2(\alpha, \beta)T)^p \quad \text{as } T \rightarrow \infty$$

for $p \leq \log \log T$. The computations are tedious, and since we do not need asymptotic estimates of high moments in what follows we omit the details.

3. The mixing condition. In this section we establish the strong mixing condition (1.3) for the sequences x_n given by (2.1). We define σ -fields $\mathfrak{N}_{S,T}$ which include the σ -fields $\mathfrak{M}_{S,T}$ as subfields, and show (1.3) for “elementary events” defined below generating $\mathfrak{N}_{0,N}$ and $\mathfrak{N}_{N+M,\infty}$.

For fixed $0 \leq \alpha < \beta \leq 1$, let $\mathfrak{N}_{S,T}$ be the σ -fields generated by the sets of functions

$$A_i = \{f \in C: \alpha \leq \{f(t)\} < \beta\} \quad \text{for } S \leq t \leq T.$$

For any sequence $\{t_n\}$ with $S \leq t_i \leq T$ for all i , we have $\bigcup_n A_{t_n} \in \mathfrak{N}_{S,T}$ and taking $\{t_n\}$ dense in any subinterval I of $[S, T]$, it follows from the continuity of f that

$$\{f \in C: \alpha \leq \{f(t)\} < \beta \quad \text{for all } t \in I\} \in \mathfrak{N}_{S,T}.$$

From (2.1) we have

$$x_N(f; \alpha, \beta) = \lambda\{N-1 \leq t \leq N: \alpha \leq \{f(t)\} < \beta\}$$

where λ in Lebesgue measure. To show the variables x_N are $\mathfrak{N}_{N-1,N}$ -measurable, it suffices to show that the sets

$$C_\eta = \{f \in C: \lambda\{N-1 \leq t \leq N: \alpha \leq \{f(t)\} < \beta\} > \eta\} \in \mathfrak{N}_{N-1,N}$$

for all $\eta > 0$. But $f \in C_\eta$ if and only if there are finitely many closed, non-overlapping intervals with rational endpoints, $I_1, I_2, \dots, I_r$, with sum of lengths greater than η and

$$f \in \{f \in C: \alpha \leq \{f(t)\} < \beta \quad \text{for all } t \in I_k\} \quad \text{for } k = 1, 2, \dots, r.$$

Hence C_η in a countable union of finite intersections of sets in $\mathfrak{N}_{N-1,N}$, and is therefore a member of $\mathfrak{N}_{N-1,N}$. We have then $\mathfrak{M}_{0,N} \subset \mathfrak{N}_{0,N}$ and $\mathfrak{M}_{N+M,\infty} \subset \mathfrak{N}_{N+M,\infty}$, and it follows that if (1.3) holds for each $A \in \mathfrak{N}_{0,N}$ and $B \in \mathfrak{N}_{N+M,\infty}$, then (1.3) holds for each $A \in \mathfrak{M}_{0,N}$ and $B \in \mathfrak{M}_{N+M,\infty}$.

For fixed A , the class of sets

$$\{B: |WAB - WAWB| \leq \epsilon WAWB\},$$

and for fixed B , the class of sets

$$\{A: |WAB - WAWB| \leq \epsilon WAWB\}$$

are both monotone classes. Hence it suffices to establish (1.3) for all $A \in F_{0,N}$ and $B \in F_{N+M,\infty}$, where $F_{0,N}$ and $F_{N+M,\infty}$ are fields generating the σ -fields $\mathfrak{H}_{0,N}$ and $\mathfrak{H}_{N+M,\infty}$, respectively. (See [3, p. 27].)

Let $0 \leq t_1 < t_2 < \dots < t_n$. The "elementary events" are sets of functions of the form $B_{t_1} B_{t_2} \dots B_{t_n}$, where $B_{t_i} = A_{t_i}$ or $A_{t_i}^c$, the complement of A_{t_i} , with respect to C , for $i = 1, 2, \dots, n$. A finite intersection of elementary events is again an elementary event, and we take the field $F_{S,T}$ to consist of all finite unions of elementary events with $S \leq t_1 < \dots < t_n \leq T$. Now it is easy to see that if (1.3) holds for elementary events, then it also holds for finite unions of elementary events.

Suppose $0 \leq s_r < s_{r-1} < \dots < s_1 \leq N < N + M \leq t_1 < \dots < t_r$ with $M \geq 1$. Then the sets

$$A = B_{s_r} \dots B_{s_1} \quad \text{and} \quad B = B_{t_1} \dots B_{t_r}$$

are two elementary events in $\mathfrak{H}_{0,N}$ and $\mathfrak{H}_{N+M,\infty}$, respectively. Using the identity (2.5) successively (as (2.5) was used twice in (2.13)) and recalling the notation (2.12), we have

$$\begin{aligned} (3.1) \quad WAB &= W[\{f(s_r)\} \in D_r, \dots, \{f(s_1)\} \in D_1; \\ &\quad \cdot \{f(t_1)\} \in E_1, \dots, \{f(t_r)\} \in E_r] \\ &= \sum_{n_r, \dots, n_1} \int_{D_r} \dots \int_{D_1} q(n_r, x_r, s_r) \\ &\quad \cdot q(n_{r-1}, x_{r-1} - x_r, s_{r-1} - s_r) \dots q(n_1, x_1 - x_2, s_1 - s_2) \\ &\quad \cdot \sum_{m_1, \dots, m_r} \int_{E_1} \dots \int_{E_r} q(m_1, y_1 - x_1, t_1 - s_1) \\ &\quad \cdot q(m_2, y_2 - y_1, t_2 - t_1) \dots q(m_r, y_r - y_{r-1}, t_r - t_{r-1}) \\ &\quad \cdot dy_r dy_{r-1} \dots dy_1 dx_1 \dots dx_r, \end{aligned}$$

where D_i and E_i are either $[\alpha, \beta]$ or $[\alpha, \beta]^c$, the complement with respect to $[0, 1)$. $WAWB$ has the same form as WAB , except that the term $q(m_1, y_1 - x_1, t_1 - s_1)$ is replaced by $q(m_1, y_1, t_1)$ in (3.1). We prove

$$(3.2) \quad |WAB - WAWB| \leq (WAWB)e^{-M}$$

by induction on the indices of the s 's. We can write

$$(3.3) \quad WB = \int_{E_1} \left(1 + 2 \sum_{m=1}^{\infty} q(m, y, t_1) \right) g(y) dy$$

where

$$(3.4) \quad g(y) = \sum_{m_2, \dots, m_r} \int_{E_2} \dots \int_{E_r} q(m_2, y_2 - y, t_2 - t_1) \dots \\ \cdot q(m_r, y_r - y_{r-1}, t_r - t_{r-1}) dy_r \dots dy_2$$

Now g is non-negative for all y as can be seen by interchanging summation and integration, and applying (2.5) to the $r - 1$ sums. Then

$$(3.5) \quad WB \geq \int_{E_1} (1 - 2 \left| \sum_{m=1}^{\infty} q(m, y, t_1) \right|) g(y) dy > \frac{9}{10} \int_{E_1} g(y) dy$$

since $t_1 \geq 1$ and

$$\left| \sum_{m=1}^{\infty} q(m, y, t_1) \right| \leq \sum_{m=1}^{\infty} \exp(-2\pi^2 m^2) < \frac{\exp(-2\pi^2)}{1 - \exp(-2\pi^2)} < \frac{1}{20}.$$

Now, for fixed x ,

$$\begin{aligned} & \left| \int_{E_1} \sum_m (q(m, y - x, t_1 - s) - q(m, y, t_1)) g(y) dy \right| \\ & \leq \int_{E_1} \sum_m |q(m, y - x, t_1 - s_1) - q(m, y, t_1)| g(y) dy \\ & = \int_{E_1} \sum'_m \exp(-\pi^2 m^2 M) \\ & \quad \cdot \left| q\left(m, y - x, t_1 - s_1 - \frac{M}{2}\right) - q\left(m, y, t_1, -\frac{M}{2}\right) \right| g(y) dy \\ & \leq 4 \exp(-\pi^2 M) \sum_{m=1}^{\infty} \exp(-\pi^2 m^2) \int_{E_1} g(y) dy \\ & < WBe^{-M} \quad \text{for } M \geq 1, \end{aligned}$$

by (3.5). Premultiplying both sides of the inequality

$$(3.6) \quad \left| \int_{E_1} \sum_m (q(m, y - x, t_1 - s_1) - q(m, y, t_1)) g(y) dy \right| < WBe^{-M}$$

by the non-negative factor $\sum_n q(n, x, s_1)$ and integrating both sides over D_1 with respect to x yields (3.2) with $A = B_{s_1}$. The sum $\sum_n q(n, x, s_1)$ is a non-negative function of x by (2.5). An induction process is established by noticing that we get (3.2) with $A = B_{s_2} B_{s_1}$ if we premultiply (3.6) by $\sum_{n_1} q(n_1, x_1 - x_2, s_1 - s_2)$, a non-negative function of $x_1 - x_2$, integrate both sides over D_1 with respect to x_1 , premultiply this resulting inequality on both sides by $\sum_{n_2} q(n_2, x_2, s_2)$, a non-negative function of x_2 , and integrate both sides over D_2 with respect to x_2 . The procedure for a general index is now clear, and (3.2) holds for arbitrary elementary events.

This proves the sequences $\{x_N : N = 1, 2, \dots\}$ are mixing according to (1.3) with $\psi(M) = e^{-M}$.

4. The approximating subclasses. In this section we define subclasses $\mathfrak{A}_i \subset \mathfrak{A}$ and show that conditions i)–vi) of Theorem P are satisfied. $\mathfrak{A}$ is defined by (2.3). For fixed n , and $0 \leq \alpha < \beta \leq 1$, every random variable $x_n(f; \alpha, \beta)$ must be approximated by a sequence from $\mathfrak{A}_i$ according to (1.6) and (1.7).

Any number ω in the unit interval can be diadically expanded as

$$\omega = \sum_{i=1}^{\infty} \frac{\epsilon_i(\omega)}{2^i}$$

where $\epsilon_i(\omega)$ is 0 or 1 for each i . We call a number x a diadic rational of order n if

$$x = \sum_{i=1}^n \frac{\epsilon_i}{2^i} \quad \text{or} \quad x = \sum_{i=1}^n \frac{\epsilon_i}{2^i} + \frac{1}{2^n},$$

where $\epsilon_i = 0$ or 1 for $i = 1, 2, \dots, n$. For any ω

$$\sum_{i=1}^n \frac{\epsilon_i(\omega)}{2^i} \leq \omega < \sum_{i=1}^n \frac{\epsilon_i(\omega)}{2^i} + \frac{1}{2^n}.$$

For each non-negative integer t we define

$$(4.1) \quad \Gamma_{2t} = \Gamma_{2t+1} = \{(\alpha, \beta) \in \Gamma : \alpha \text{ and } \beta \text{ are diadic rationals of order } t\},$$

and for each positive integer t we define the classes

$$(4.2) \quad \mathfrak{A}_t = \{\{x_n(f; (\alpha, \beta)); n = 1, 2, \dots\}, \\ \{-x_n(f; (\alpha, \beta)); n = 1, 2, \dots\} \quad \text{with} \quad (\alpha, \beta) \in \Gamma_t\}.$$

Now $\mathfrak{A}_t \subset \mathfrak{A}$, and a quick estimate gives

$$(4.3) \quad 1 \leq \text{card } \mathfrak{A}_t \leq e^{4t} \quad \text{for} \quad t = 1, 2, \dots.$$

For fixed $0 \leq \alpha < \beta \leq 1$, let

$$k_0 = \min \{\text{integer } k \geq 0 : \alpha < i/2^k \leq \beta \text{ for some integer } i\}.$$

In the rest of this section we suppress the f and write $x_n(\alpha, \beta)$ for $x_n(f; \alpha, \beta)$. Define

$$x_n(t) = 0 \quad \text{for} \quad t = 0, 1, \dots, 2k_0;$$

$$x_n(2k_0 + 1) = x_n\left(\sum_{k=0}^{k_0} \frac{\epsilon_k(\alpha)}{2^k} + \frac{1}{2^{k_0}}, \sum_{k=0}^{k_0} \frac{\epsilon_k(\beta)}{2^k}\right);$$

and

$$x_n(2t) = x_n\left(\sum_{k=1}^t \frac{\epsilon_k(\alpha)}{2^k} + \frac{1}{2^t}, \sum_{k=1}^{t-1} \frac{\epsilon_k(\alpha)}{2^k} + \frac{1}{2^{t-1}}\right);$$

$$x_n(2t + 1) = x_n\left(\sum_{k=1}^{t-1} \frac{\epsilon_k(\beta)}{2^k}, \sum_{k=1}^t \frac{\epsilon_k(\beta)}{2^k}\right);$$

for $t > k_0$. Also, define

$$\bar{x}_n(1) = 1;$$

$$\bar{x}_n(2) = -x_n\left(0, \frac{\epsilon_1(\alpha)}{2}\right);$$

$$\bar{x}_n(3) = -x_n\left(\frac{\epsilon_1(\beta)}{2} + \frac{1}{2}, 1\right)$$

and

$$\begin{aligned}\bar{x}_n(2t) &= -x_n\left(\sum_{k=1}^{t-1} \frac{\epsilon_k(\alpha)}{2^k}, \sum_{k=1}^t \frac{\epsilon_k(\alpha)}{2^k}\right); \\ \bar{x}_n(2t+1) &= -x_n\left(\sum_{k=1}^t \frac{\epsilon_k(\beta)}{2^k} + \frac{1}{2^t}, \sum_{k=1}^{t-1} \frac{\epsilon_k(\beta)}{2^k} + \frac{1}{2^{t-1}}\right)\end{aligned}$$

for $t > 1$. For each t , the sequences $\{x_n(t); n = 1, 2, \dots\}$ and $\{\bar{x}_n(t); n = 1, 2, \dots\}$ are members of $\mathfrak{A}_t$. Next we define $y_n(s)$ and $\bar{y}_n(s)$ for s a positive integer by

$$y_n(s) = \sum_{t=1}^s x_n(t) \leq x(\alpha, \beta) \leq \sum_{t=1}^s \bar{x}_n(t) = \bar{y}_n(s).$$

By the additive property (2.2) we see that the sequences $\{y_n(s); n = 1, 2, \dots\}$ and $\{\bar{y}_n(s); n = 1, 2, \dots\}$ are themselves members of the family $\mathfrak{A}$. Furthermore,

$$\bar{y}_n(s) - y_n(s) = x_n(\omega_1, \omega_2) + x_n(\omega_3, \omega_4)$$

with

$$\omega_2 - \omega_1 \leq 2^{-\lfloor s/2 \rfloor} \quad \text{and} \quad \omega_4 - \omega_3 \leq 2^{-\lfloor s/2 \rfloor + 1},$$

and $0 \leq \omega_1 \leq \omega_2 \leq \omega_3 \leq \omega_4 \leq 1$. Now

$$0 \leq E(\bar{y}_n(s) - y_n(s)) \leq 1 \quad \text{for all } s \text{ and } n,$$

and straightforward computations using (2.6) yield

$$E(\bar{y}_n(s) - y(s)) < e^{-cs} \quad \text{for } s > 10 \text{ and all } n.$$

Here $c = \log 2/4$. Finally, recalling the notation (1.4), we have from (2.23) and (2.25) that

$$(4.4) \quad S_N^2(x_n(t)) = N\sigma^2(t) + O(1) = O(N2^{-t/2}) \text{ as } N \rightarrow \infty,$$

which implies condition iv) of Theorem P is satisfied.

5. Proof of the Theorem. In section 2. we defined the class Γ and the family $\mathfrak{A}$ of uniformly bounded sequences (2.1). We computed the first and second moments, and (2.25) shows (1.5) is satisfied uniformly over Γ . In section 4. we constructed the approximating subfamilies $\mathfrak{A}_t$ depending on subclasses Γ_t ((4.1) and (4.2)) and saw that (4.3) satisfies condition i) of Theorem P. Condition ii) follows from the fact that all sequences in $\mathfrak{A}$ satisfy the mixing condition (1.3) with exponential rate of mixing (see section 3.). The approximating sequences $\bar{x}_n$, x_n , $\bar{y}_n$ and y_n are defined in section 4., and satisfy condition iii) (modified in paragraph after the statement of Theorem P), iv) (see (4.4) and v) (since y_n and $\bar{y}_n$ are members of $\mathfrak{A}$). We have already mentioned in

Section 1 that vi) follows from the hypothesis (1.5). Therefore all the conditions of Theorem P are satisfied and conclusions (1.8) and (1.9) follow for our family of sequences.

For T an integer we have

$$\int_0^T \chi(t) dt = \sum_{k=1}^T x_k$$

and from (2.7) we get

$$(5.1) \quad \int_0^T (\chi(t) - (\beta - \alpha)) dt = \sum_{k=1}^T (x_k - Ex_k) + O(1)$$

and it follows that (1.9) implies (1.2).

To establish (1.1), we write

$$(5.2) \quad \int_0^T (\chi(t) - (\beta - \alpha)) dt = \left(\int_0^{\{T\}} + \int_{\{T\}}^T \right) (\chi(t) - (\beta - \alpha)) dt \\ = \sum_{k=1}^{\{T\}} (x_k - Ex_k) + O(1)$$

by (5.1) and the fact that the absolute value of the last integral in (5.2) is bounded by 1. Now, given $\epsilon > 0$, there exists a $T_0(\epsilon)$ such that

$$\frac{\left| \int_0^T (\chi(t) - (\beta - \alpha)) dt \right|}{(2T \log \log T)^{1/2}} < \frac{\left| \sum_{k=1}^{\{T\}} (x_k - Ex_k) \right|}{(2\{T\} \log \log \{T\})^{1/2}} + \epsilon, \quad T > T_0(\epsilon)$$

and hence (1.8) implies (1.1). This proves the theorem.

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Duke University

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