On the Total Variation of Solutions of Parabolic Equations

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Introduction. We shall be concerned with solutions of

$$(1) \quad u_t + a(t, x)u_x + \frac{1}{2}\sigma^2(t, x)u_{xx} = b(t, x)u + c(t, x)$$

defined in an open set U . The main result of this paper (Theorem 2) is an upper bound on the variation of u with respect to the variable x . The upper bound depends upon the boundary values, the functions b and c , and certain geometrical properties of the boundary itself, *i.e.*, it is an *a priori* estimate.

Explicit consideration of the variation of solutions is rare; other functionals such as the oscillation and various norms of the derivatives have been studied much more thoroughly. However, in some contexts the total variation arises naturally and bounds on the derivatives just won't do. A striking example is afforded by Oleinik's study [12] of a singular perturbation problem for a quasi-linear equation in which she proves convergence by a compactness argument. Since the limit function was discontinuous, a bound on the variation and not on the sup or on the L_2 norm of derivatives was needed. Another example would be the study of degenerate parabolic equations with bad coefficients.

Recently Sattinger [13] studied the variation of solutions of (1) when $b \equiv c \equiv 0$ and when σ^2 is bounded uniformly away from zero, *i.e.*, when the equation is uniformly parabolic. He considered separately the Cauchy Problem and the mixed initial-boundary value problem. In each case he showed that the variation was bounded by that of the data.

Even before Sattinger, however, Karlin and McGregor [7] had considered the equation $u_t + Au = 0$ where A is the generator of a one-dimensional diffusion process (cf. Ito and McKean, [5]). This equation includes as a special case the equation (1) when $b = c = 0$ and a and σ are independent of t . They proved that the fundamental solution for the pure Cauchy problem or mixed problem with constant boundary values is *totally positive*. But it is a basic property of totally positive kernels that the associated integral operator maps functions of bounded variation onto functions of bounded variation without

increasing the total variation (cf. [8] Chapter 6, Theorem 6.2). Hence, solutions of the above mentioned problems have total variation not exceeding that of their initial values.

Our main result, Theorem 2 and its corollary, is proved in Section 1. Its chief feature is the generality of the region considered; both the Cauchy Problem and the mixed problem with time varying boundaries are included as special cases. Secondly, the full inhomogeneous equation is studied. Finally, we allow degeneracies, *i.e.*, σ^2 is allowed to vanish. This is natural since we are studying not *decay* of variation but rather its *growth* and *propagation*, and these are governed by the same factors present in the first order case, $\sigma^2 \equiv 0$.

The proof makes use of the underlying Markov process. This method makes possible the full exploitation of the topological properties unique to the plane. This is seen especially in Lemmas 1 and 2. Probabilistic methods also seem to be essential if fairly general regions are to be considered; condition 2° is the only way I know of expressing the complex interrelation between the region U and the coefficients a and σ . Condition 5° is the most novel from the analytic point of view, but it is perfectly natural from the stochastic point of view. That some such *a priori* restriction must be imposed upon the solution is well known from analytic discussions of uniqueness in unbounded regions.

The work of Karlin and McGregor [7], and that of McKean [10] on which it is based, are both analytic. However, they are both clearly motivated by probabilistic considerations. Total positivity of the transition density function seems to be an analytic expression of the fact that the underlying stationary process is *one-dimensional, strong Markov* and has *continuous paths* [6]. Now, our proof rests directly upon just these properties of the process. It thus is tempting to guess that in the non-stationary case the transition function possesses a property which is a generalization of the notion of the total positivity.

In Section 2 we give various special cases of Theorem 2 having purely analytic (non-probabilistic) hypotheses. Included in this section are explicit results for the classical Cauchy and mixed problems.

In these first two sections we prove our results under the assumption that the solution, the boundary data, and the coefficients satisfy certain regularity assumptions that in many cases are unduly restrictive. This is apparent since the estimate itself involves only bounds on the variation and modulus of the coefficients and data. These assumptions can often be relaxed by the familiar process of approximating (1) by a sequence of smooth, non-degenerate equations and then passing to the limit. An example of this procedure for the Cauchy Problem is given in Section 3.

§1. We shall study (1) in a *simply connected open set* U contained in $I \times \mathbb{R}$ where $I = (t, T)$ is an open interval. t is a fixed real number while T can be real or $+\infty$. The boundary of U is $L \cup \bar{J}$ where J is an open Jordan arc (homeomorphic to $(0, 1)$) and L is an open line segment contained in the line $\{(t, x): x \text{ real}\}$. The bar superscript denotes closure.

With (1) we associate the characteristic stochastic equation,

$$(2) \quad dx(t) = a(t, x)dt + \sigma(t, x)d\beta(t)$$

where β denotes a Wiener process. Our chief references for these equations and the associated stochastic calculus are the books [11] and [14]. A good account of the interplay between (1) and (2) is given in [4]. We use the term "characteristic" since when $\sigma \equiv 0$ (deterministic case) (2) is precisely the characteristic ordinary differential equation for the resulting first order equation (1). For an account of this classical theory see [2].

About a and σ we assume:

- 1°. The functions a and σ are continuous in U and satisfy a Lipschitz condition with respect to x in every compact subset of U .

It is the fundamental result of the theory of stochastic equations that if condition 1° is satisfied, then for any point (s', x') in U there is a unique solution $\xi(s; s', x')$ of (2) such that

$$(3) \quad \xi(s'; s', x') = x' \quad \text{a.s.}$$

$\xi(s)$ is defined for s between s' and a random stopping time. For such values of s , $(s, \xi(s))$ is in U . We refer to this solution as the *characteristic process* through (s', x') . We can now express the fundamental restriction imposed on J .

- 2°. $P[\tau(t, x) < \infty] = 1$ where $\tau(t, x)$ is the first time the characteristic process through (t, x) hits J , i.e., $\tau(t, x) = \inf \{s: \xi(s; t, x) \in J, t \leq s\}$ where we operate under the convention that the infimum of the empty set is $+\infty$. We also assume:

- 3°. the function u is a solution of (1) in U which is continuous in $\bar{U}$. The partial derivatives u_t , u_x and u_{xx} are continuous in U .

- 4°. The functions b and c are continuous in U , and

$$|b(s, x)| \leq b_0(s), \quad |c(s, x)| \leq c_0(s)$$

where b_0 is continuous and integrable on I , and c_0 is continuous and square integrable on I .

Consider the process $\bar{u}(s) = u(s, \xi(s; t, x))$, $t \leq s < \tau = \tau(t, x)$, defined by restricting u to the paths of the characteristic process. Ito's Lemma ([11], p. 32) tells us that $\bar{u}$ has a stochastic differential given by

$$d\bar{u} = (\bar{u}_t + \bar{a}\bar{u}_x + \frac{1}{2}\overline{\sigma^2 u_{xx}}) ds + \bar{\sigma} \bar{u}_x d\beta(s)$$

where the bar denotes the restriction of the function in question to the paths of ξ . Since u is a solution of (1) in U , we see that

$$(4) \quad d\bar{u}(s) = (\bar{b}\bar{u} + \bar{c})ds + \bar{\sigma}\bar{u}_x d\beta(s), \quad t \leq s < \tau.$$

This equation describes the propagation of values of u along the paths of the

characteristic process. As in the deterministic case $\sigma \equiv 0$ in U , it enables us to derive a functional representation for u .

We integrate (4) by the same trick used in the deterministic case, *i.e.*, an integrating factor. The only change is that we must use Ito's Lemma to compute the differential of a product ([11], p. 34):

$$d\left(\bar{u}(s) \exp \left\{ -\int_t^s \bar{b}(\theta) d\theta \right\}\right) = \exp \left\{ -\int_t^s \bar{b} \right\} (-\bar{b}\bar{u} ds + d\bar{u}).$$

Integrating this differential from t to τ we derive

$$\begin{aligned} (5) \quad \bar{u}(\tau) \exp \left\{ -\int_t^\tau \bar{b} \right\} - u(t, x) \\ = \int_t^\tau \bar{c}(s) \exp \left\{ -\int_t^s \bar{b} \right\} ds + \int_t^\tau \bar{\sigma}(s) \bar{u}_x(s) \exp \left\{ -\int_t^s \bar{b} \right\} d\beta(s) \end{aligned}$$

where we have used (3) and (4). We now impose our fifth basic condition.

$$5^\circ \quad E \int_t^\tau I_\tau(s) |\bar{\sigma}(s) \bar{u}_x(s)|^2 \exp \left\{ -2 \int_t^s \bar{b} \right\} ds < \infty$$

where $I_\tau(s)$ is the indicator random variable of the event $[s < \tau]$ and E denotes expected value.

Under condition 5° the stochastic integral in (5) has mean zero ([11], p. 25) while the other term on the right hand side of (5) has finite mean because of condition 4° . We can therefore take expectations in (5) to obtain

$$(6) \quad u(t, x) = E \left[\bar{u}(\tau) \exp \left\{ -\int_t^\tau \bar{b}(\theta) d\theta \right\} - \int_t^\tau \bar{c}(s) \exp \left\{ -\int_t^s \bar{b}(\theta) d\theta \right\} ds \right]$$

where $\bar{c}(s) = c(s, \xi(s; t, x))$, etc. In summary we have:

Theorem 1. *Under conditions 1° – 5° , the solution u of (1) satisfies the representation (6).*

If for each s in I the function $x \rightarrow b(s, x)$ is of bounded variation over the set $\{x: (s, x) \in U\}$, we shall denote its variation by $V_*(b)$. If the function $s \rightarrow V_*(b)$ is integrable on I , then we shall say that b is of *integrable variation* in U . This implies that the distributional derivative of b with respect to x is no worse than a signed measure which is locally finite in U ([9], Theorem 5.2).

6° . The functions b and c are of integrable variation in U .

The Jordan arc J is the homeomorphic image of the unit interval. Let the parametric equations for J be $s = \varphi(\alpha)$, $x = \psi(\alpha)$ where $0 < \alpha < 1$. φ is a continuous map of $(0, 1)$ into I .

7° . φ is locally of bounded variation in $(0, 1)$, *i.e.*, φ is of bounded variation on any compact subinterval of $(0, 1)$.

For f continuous and real-valued on $(0, 1)$ we can consider its improper Stieltjes integral [15] with respect to the variation of φ :

$$\int_0^1 f(\alpha) d|\varphi(\alpha)| = \lim_{\epsilon \rightarrow 0} \int_\epsilon^{1-\epsilon} f(\alpha) d|\varphi(\alpha)|.$$

$$8^\circ. \quad \int_0^1 b_0(\varphi(\alpha)) d|\varphi(\alpha)| \quad \text{and} \quad \int_0^1 c_0(\varphi(\alpha)) d|\varphi(\alpha)| \quad \text{are finite.}$$

Theorem 2. Under conditions 1°–8° if u is of bounded variation along J , then u is of bounded variation along L and

$$\begin{aligned} V_L(u) \leq & \exp \left(\int_I b_0 \right) \left(V^J(u) + m \int_I V_s(b) ds + \int_0^1 [mb_0(\varphi(\alpha)) + c_0(\varphi(\alpha))] d|\varphi(\alpha)| \right) \\ & + \int_I V_s(b) \exp \left(\int_t^s b_0 \right) ds + \int_I c_0(s) \exp \left(\int_t^s b_0 \right) \int_t^s V_0(b) d\theta ds \end{aligned}$$

where $V^J(u)$ denotes the total variation along J and m denotes the supremum of $|u|$ along J .

Proof. Let $x_1 < x_2 < \dots < x_{N+1}$ and (t, x_i) be on L for $i = 1, \dots, N+1$. Let $\xi(s; t, x_i) \equiv \xi_i(s)$ for $t \leq s < \tau(t, x_i) \equiv \tau_i$. Let $Q_i = (\tau_i, \xi_i(\tau_i))$ be the (random) point where the characteristic through (t, x_i) first meets J . We shall need two lemmas.

Lemma 1. $P[\xi_i(s) \leq \xi_{i+1}(s) \text{ for } t \leq s < \tau_i \wedge \tau_{i+1}] = 1$.

This is well known and easily proved ([11], p. 58). Now the mapping of the unit interval onto J induces an order relation with respect to which we have

Lemma 2. The points $Q_1, \dots, Q_N$ are almost surely linearly ordered along J , i.e., with probability one Q_{i+1} is between Q_i and Q_{i+2} .

Proof. Because of Lemma 1 we see that with probability one, the path of the characteristic from (t, x_i) to Q_i , the arc of J from Q_i to Q_{i+2} , the characteristic path from Q_{i+2} to (t, x_{i+2}) and the segment of L from (t, x_{i+2}) to (t, x_i) form a closed Jordan curve. Since the characteristic path issuing from (t, x_{i+1}) enters the interior and with probability one cannot meet the paths from (t, x_i) and (t, x_{i+2}) , we see that with probability one Q_{i+1} must lie on the arc of J joining Q_i and Q_{i+2} . This proves the lemma.

We shall let $e_i(s)$ denote

$$\exp \left\{ - \int_t^s b(\theta, \xi_i(\theta)) d\theta \right\}.$$

From (6) we see that

$$\sum_{i=1}^N |u(t, x_{i+1}) - u(t, x_i)| \leq A + B$$

where

$$A = \sum_{j=1}^N |E[u(Q_{j+1})e_{j+1}(\tau_{j+1}) - u(Q_j)e_j(\tau_j)]|$$

$$B = \sum_{j=1}^N \left| E \left[\int_t^{\tau_{j+1}} c(s, \xi_{j+1}(s)) e_{j+1}(s) ds - \int_t^{\tau_j} c(s, \xi_j(s)) e_j(s) ds \right] \right|.$$

Now from 4° we see that

$$A \leq \exp \left\{ \int_I b_0 \right\} E \left[\sum_{j=1}^N |u(Q_{j+1}) - u(Q_j)| \right] + \sum_{j=1}^N |E(u(Q_j)|e_{j+1}(\tau_{j+1}) - e_j(\tau_j))|.$$

From Lemma 2 we see that $\sum |u(Q_{j+1}) - u(Q_j)|$ is almost surely bounded by $V^J(u)$. Using this and the definition of m we see that

$$(7) \quad A \leq V^J(u) \exp \left\{ \int_I b_0 \right\} + m \sum_{j=1}^N |E[e_{j+1}(\tau_{j+1}) - e_j(\tau_j)]|.$$

Let $\tilde{j}$ be the random variable equal to j and on the event $[\tau_j > \tau_{j+1}]$ and equal to $j + 1$ on $[\tau_j \leq \tau_{j+1}]$. Then using the mean value theorem and 4° we obtain

$$(8) \quad \sum_{j=1}^N |e_{j+1}(\tau_{j+1}) - e_j(\tau_j)|$$

$$\leq \exp \left\{ \int_I b_0 \right\} \sum_{j=1}^N \left| \int_t^{\tau_{j+1}} b(s, \xi_{j+1}(s)) ds - \int_t^{\tau_j} b(s, \xi_j(s)) ds \right|$$

$$\leq \exp \left\{ \int_I b_0 \right\} \sum_{j=1}^N \int_t^{\tau_j \wedge \tau_{j+1}} |b(s, \xi_{j+1}(s)) - b(s, \xi_j(s))| ds$$

$$+ \exp \left\{ \int_I b_0 \right\} \sum_{j=1}^N \left| \int_{\tau_j}^{\tau_{j+1}} |b(s, \xi_j^\vee(s))| ds \right|$$

$$\leq k \int_I \sum_{j=1}^N Z_j(s) |b(s, \xi_{j+1}(s)) - b(s, \xi_j(s))| ds + k \sum_{j=1}^N \left| \int_{\tau_j}^{\tau_{j+1}} b_0(s) ds \right|$$

where we have set $k = \exp \left\{ \int_I b_0 \right\}$ and $Z_j(s)$ is the indicator function of the event $[s \leq \tau_j \wedge \tau_{j+1}]$. But from Lemma 1 we see that with probability one,

$$\sum_{j=1}^N Z_j(s) |b(s, \xi_{j+1}(s)) - b(s, \xi_j(s))| \leq V_*(b).$$

We also notice that

$$\sum_{j=1}^N \left| \int_{\tau_j}^{\tau_{j+1}} b_0(s) ds \right| \leq \int_0^1 b_0(\varphi(\alpha)) d|\varphi(\alpha)|,$$

so that from (8) and (7) we obtain

$$A \leq \left[V^J(u) + m \int_I V_*(b) ds + m \int_0^1 b_0(\varphi(\alpha)) d|\varphi(\alpha)| \right] \exp \left\{ \int_I b_0 \right\}.$$

We now turn to B . We have the inequalities

$$\begin{aligned}
B &\leq E \sum_i \int_t^{\tau_i \wedge \tau_{i+1}} |c(s, \xi_{i+1}(s)) - c(s, \xi_i(s))| e_{i+1}(s) ds \\
&\quad + E \sum_i \int_t^{\tau_i \wedge \tau_{i+1}} |e_{i+1}(s) - e_i(s)| |c(s, \xi_i(s))| ds \\
&\quad + E \sum_i \left| \int_{\tau_i}^{\tau_{i+1}} |c(s, \xi_i(s))| e_i^\vee(s) ds \right| = B_1 + B_2 + B_3, \\
B_1 &\leq E \int_I \exp \left\{ \int_t^* b_0 \right\} \sum_i Z_i(s) |c(s, \xi_{i+1}(s)) - c(s, \xi_i(s))| ds \\
&\leq \int_I V_*(c) \exp \left\{ \int_t^* b_0 \right\} ds, \\
B_2 &\leq E \sum_i \int_t^{\tau_i \wedge \tau_{i+1}} c_0(s) \exp \left\{ \int_t^* b_0 \right\} \int_t^* |b(\theta, \xi_{i+1}(\theta)) - b(\theta, \xi_i(\theta))| d\theta ds \\
&= E \int_I c_0(s) \exp \left\{ \int_t^* b_0 \right\} \sum_i Z_i(s) \int_t^* |b(\theta, \xi_{i+1}(\theta)) - b(\theta, \xi_i(\theta))| d\theta ds \\
&\leq E \int_I c_0(s) \exp \left\{ \int_t^* b_0 \right\} \int_t^* \sum_i Z_i(\theta) |b(\theta, \xi_{i+1}(\theta)) - b(\theta, \xi_i(\theta))| d\theta ds \\
&\leq \int_I c_0(s) \exp \left\{ \int_t^* b_0 \right\} \int_t^* V_\theta(b) d\theta ds, \\
B_3 &\leq \sum_i \left| \int_{\tau_i}^{\tau_{i+1}} c_0(s) \exp \left\{ \int_t^* b_0 \right\} ds \right| \leq \exp \left\{ \int_I b_0 \right\} \int_0^1 c_0(\varphi(\alpha)) d|\varphi(\alpha)|.
\end{aligned}$$

The resulting inequality for B together with that for A completes the proof of Theorem 2 since the partition $x_1 < \dots < x_{N+1}$ of L was arbitrary.

We wish to mention explicitly one corollary of the proof of Theorem 2. It concerns the special case when $b \equiv c \equiv 0$ i.e., we consider solutions in U of the homogeneous equation,

$$(1h) \quad u_t + au_x + \frac{1}{2}\sigma^2 u_{xx} = 0.$$

Then of course conditions 4°, 6° and 8° are automatically satisfied. However, a specialization of the proof of Theorem 2 reveals that 7° is also superfluous so that we have

Corollary. *In U let u be a solution of (1h) and let conditions 1°, 2°, 3° and 5° be satisfied. Then if u is of bounded variation along J , it must also be of bounded variation along L and $V_t(u) \leq V^J(u)$.*

§2. If U is bounded, then condition 2° is automatically satisfied. In this

case we can assume T to be finite so that the following non-probabilistic condition assures that 5° will be satisfied.

$B5^\circ$. $|\sigma u_x|$ is uniformly bounded in U .

We therefore have:

Theorems B1-B2. *Theorems 1 and 2 are valid if condition 2° is replaced by the assumption that U is bounded and condition 5° is replaced by $B5^\circ$.*

If U is unbounded, then we can obtain non-probabilistic theorems by assuming that a and σ are defined throughout $I \times R$ and, for example, that the following is satisfied.

$A2^\circ$. For all (t, x) in U , $|a(t, x)| + |\sigma(t, x)| \leq k_0(t)(1 + |x|)$ where $k_0 \in L_p(I)$ for some $p \geq 2$.

That this assures 2° is well known (cf. [11], p. 56 for $p = 2$ and [17] for $p \geq 2$. The proofs are given for k_0 constant, but standard differential inequalities transform these proofs to the more general situation; cf. [1] for an example of this.).

As is shown in [17], $E |\xi(t)|^p$ is uniformly bounded in I so that condition 5° will be satisfied if we assume the following.

$A5^\circ$. For all $(t, x) \in U$, $|\sigma(t, x)u_x(t, x)|^2 \leq \text{constant} \cdot (1 + |x|^p)$.

We therefore have

Theorems A1-A2. *Theorems 1 and 2 are valid if 2° is replaced by $A2^\circ$ and 5° is replaced by $A5^\circ$.*

When Theorem A2 is applied to the (backward) *Cauchy Problem* $U = \{(s, x): t < s < 0\}$, J is the x -axis and L is the line $s = t$. In this case φ is a constant so that 7° and 8° are automatically satisfied.

Theorem C2. *For the Cauchy Problem, if Conditions 1° , $A2^\circ$, 3° , 4° , $A5^\circ$ and 6° are satisfied and if u is of bounded variation on the line $s = 0$, then it is of bounded variation on the line $s = t$ and*

$$\begin{aligned} V_t(u) \leq \exp \left(\int_t^0 b_0 \right) & \left[V_0(u) + m \int_t^0 V_s(b) ds \right] + \int_t^0 V_s(b) \exp \left(\int_t^s b_0 \right) ds \\ & + \int_t^0 c_0(s) \exp \left(\int_t^s b_0 \right) \int_t^s V_\theta(b) d\theta ds. \end{aligned}$$

Again $m = \sup_x |u(0, x)|$.

For the *Mixed Initial-Boundary Value Problem* we may take $U = \{(x, s): t < s < 0, h_1(s) < x < h_2(s)\}$ where h_1 and h_2 are continuous on $\bar{I} = [t, 0]$. $J = \partial U \cap \{s > t\}$ consists of the two lateral sides and the terminal segment. In this case 7° is automatically satisfied and 8° follows from 6° since the total variation is $2|t|$.

Theorem M2. *For the Mixed Problem, if Conditions 1°, 3°, 4°, B5° and 6° are satisfied and if u is of bounded variation along J , then $u(t, \cdot)$ is of bounded variation along $h_1(t) \leq x \leq h_2(t)$ and*

$$V_t(u) \leq \exp \left(\int_t^0 b_0 \right) \left[V^J(u) + \int_t^0 (mV_s(b) + 2mb_0(s) + 2c_0(s)) ds \right] \\ + \int_t^0 \left[V_s(b) + c_0(s) \int_t^s V_\theta(b) d\theta \right] \exp \left(\int_t^s b_0 \right) ds,$$

where m is the supremum of $|u|$ along J and $V^J(u)$ denotes the total variation along J .

§3. Let us consider a terminal Cauchy Problem:

$$(9) \quad u_t + au_x + \frac{1}{2}\sigma^2 u_{xx} = bu + c \quad \text{in } U = \{(t, x) : T < t < 0\}$$

$$u(0, x) = \varphi(x).$$

If the data, φ , or the coefficients a , σ , b or c are allowed to be irregular in some sense, then it may be necessary to consider solutions in some generalized sense. Now the particular definition may vary but it seems that at very least a generalized solution should be *stable under regularization*. To explain what we mean by this, consider the sequence of regularized problems:

$$(10) \quad u_t^{(n)} + a_n u_x^{(n)} + \frac{1}{2}\sigma_n^2 u_{xx}^{(n)} = b_n u^{(n)} + c_n u^{(n)}(0, x) = \varphi_n.$$

Here, $a_n = a * \omega_{1/n}$ where ω_ρ is the usual two dimensional smoothing kernel of support radius ρ (cf. "regularization" in [16]). The functions b_n , c_n and φ_n are defined in the same way except that φ_n is regularized in one dimension. The function σ_n is defined by

$$\sigma_n = \sigma * \omega_{1/n} + 1/n$$

which in addition to smoothing also insures *uniform* parabolicity. If b and c are bounded by functions integrable on $(T, 0)$, then there is a unique bounded solution $u^{(n)}$ to each problem (10), [3]. We say that a generalized solution (no matter how defined) is *stable under regularization* if it is the limit almost everywhere in U of a subsequence of $\{u^{(n)}\}$, $n \geq 1$.

Theorem 3. *Let a and σ be bounded. Let b and c be of integrable variation in U with $|b(t, x)| \leq b_0(t)$, $|c(t, x)| \leq c_0(t)$ where b_0 and c_0 are locally bounded on $(T, 0]$. Let φ be of bounded variation on $\mathbf{R}$ and denote its total variation by m_1 and $\|\varphi\|_\infty$ by m_0 . If u is any generalized solution of (10) which is stable under regularization, then it is equal almost everywhere in U to a function v such that for all $t \in (T, 0)$, $v(t, \cdot)$ is of bounded variation and*

$$(11) \quad V_t(u) \leq \left[m_1 + m_0 \int_t^0 V_s(b) ds \right] \exp \left(\int_t^0 b_0 \right) \\ + \int_t^0 V_s(b) \exp \left(\int_t^s b_0 \right) ds + \int_t^0 c_0(s) \exp \left(\int_t^s b_0 \right) \int_t^s V_\alpha(b) d\alpha ds.$$

Proof. We shall apply Theorem C2 to the sequence of problems (10). Each of the regularized functions is smooth, a_n , σ_n and φ_n are bounded and $|b_n(t, x)| \leq b_0(t)$, $|c_n(t, x)| \leq c_0(t)$. Hence there is one and only one bounded solution, $u^{(n)}$, to (10). It also follows that conditions 1°, A2°, and 4° are satisfied for (10), and it is easy to see that 3° is also satisfied.

Now by elementary properties of convolutions we see that

$$\left| \frac{\partial a_n}{\partial x} \right| = \left| a_n * \frac{\partial \omega_{1/n}}{\partial x} \right| \leq \text{constant} \cdot n \cdot \|a\|_\infty.$$

Similarly for fixed n the derivatives of all regularized functions in (10) are bounded in U . Hence $u_x^{(n)}$ is bounded in U , and thus condition A5° is trivially satisfied. Finally, we see that $V_t(b_n) \leq V_t(b)$ and $V_t(c_n) \leq V_t(c)$ and that the total variation of φ_n is bounded by m_1 . Hence we may apply Theorem C2 to (10) to conclude that $V_t(u^{(n)})$ is bounded by the right hand side of (11).

Now there is a subsequence, S of the positive integers such that $u^{(n)}$ converges almost everywhere to u as $n \rightarrow \infty$, $n \in S$. Define v by

$$v(t, x) = \limsup_{n \in S} u^{(n)}(t, x).$$

Since the $u^{(n)}$'s are bounded in U independently of n , this defines a finite, measurable function such that $v = u$ almost everywhere. But since $\{u^{(n)}\}$, $n \in S$, converges everywhere to v , we have $V_t(u) \leq \limsup V_t(u^{(n)})$ and the theorem is proved.

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