On the Hyperboloidal Means and the Wave Equation

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Introduction. In a recent paper [1] hyperboloidal means HM were used to obtain representations of solutions of the wave equation $\square^2 w(X, t) = 0$ in the exterior of C_0 . In forming $HM[\zeta]$, the function ζ is integrated over $n - 1$ dimensional half-hyperboloids Hd which are the intersections of C_0 with time-like hyperplanes. In Sections 1 and 2 of this paper we introduce a class $\mathfrak{R}$ of n dimensional surfaces (Hb) in the inside and on C_0 and construct on them measures which are invariant with respect to all proper Lorentz transformations. This gives an extension of the notion of HM , and we shall denote the new hyperboloidal means of functions Φ over Hb by $HS[\Phi]$.

Let $C_1(X, t)$ be the characteristic double cone with a vertex (X, t) which lies in the exterior of C_0 . Let (X, t) be fixed. Denote the upper cones of C_0 and C_1 respectively by C_0^+ and C_1^+ , and the corresponding closed solid cones by K_0 and K_1 . Set $\Omega = K_0 \cap K_1$. Inscribed in Ω there is a one parametric family $\mathfrak{F}(X, t) = \{Hb(\lambda), 0 < \lambda < \infty\}$ of half-hyperboloids $Hb(\lambda)$ of dimension n . Let $HS[\Phi; \lambda]$ be the mean of $\Phi(Y, \tau)$ over $Hb(\lambda)$. Suppose Φ is a function of class $C^{(2)}$ in K_0 with the origin deleted, and furthermore Φ vanishes properly at infinity. It is shown in Section 3 that

$$\frac{\partial}{\partial \lambda} HS[\Phi; \lambda] = \frac{1}{2\omega_n} (\tfrac{1}{4}(|X|^2 + \lambda^2 - t^2))^{-n/2} \int \square^2 \Phi d\tau dY$$

holds, where the volume integral extends over the interior of $Hb(\lambda)$, (Theorem 1).

If $\Phi = w$ is a solution of the wave equation in K_0 , then $HS[w; \lambda]$ is independent of λ , and one obtains a mean value theorem. Its applicability is restricted however to solutions which behave properly at infinity.

Let K'_0 denote the exterior of C_0 . In Section 4 of this paper we consider for each $(X, t) \in K'_0$ the n dimensional surface $Hb(X, t)$ consisting of that portion of C_0^+ which extends to infinity and is bounded below by $Hd(X, t) = C_0^+ \cap C_1^+(X, t)$. $Hb(X, t) \in \mathfrak{R}$. Let $\zeta(Y)$ be a smooth function on C_0^+ and denote by $HS[\zeta; X, t]$ the mean of ζ over $Hb(X, t)$. Then $w(X, t) = HS[\zeta; X, t]$ is a solution

of the wave equation in K'_0 . When the space dimension n is even and smooth data $\varphi(Y)$ is assigned on C_0^+ we set

$$\zeta(Y) = \text{const. } \rho^{-k} \left(\frac{\partial}{\partial \rho} \right)^{n/2} \rho^{n-3/2} \varphi(Y)$$

with $k = (n - 3)/2$. Then the solution $w(X, t)$ given by $HS[\zeta]$ takes on the prescribed values φ as (X, t) approaches C_0^+ , (Theorem 2). This statement extends a part of a previous result for n odd (Theorem 1.3 [1]). There is however a significant difference between the two results: in the present case the solution $w(X, t)$ no longer tends to zero as (X, t) approaches the lower part C_0^- of C_0 . We give explicitly the boundary values of w on C_0^- , when φ is assumed to have compact support.

When n is even, the question remains as to how to construct a solution in K'_0 which will take on prescribed smooth data on both parts of C_0 . Also the related and important question of uniqueness is not discussed in this paper.

A number of mean value theorems for differential equations of second order have been derived by John, using ellipsoidal means which are invariant with respect to affine transformations, [2, Chap. V and the references given there]. For the wave equation see also Serrin [3].

1. Definitions and notations.

1.1. $P(a, b; X, t)$ and $\ell(c; X, t)$. In R^{n+1} let the time and space variables be τ and $Y = (y_1, y_2, \dots, y_n)$ with $|Y| = \rho$. By (X, t) , $X = (x_1, x_2, \dots, x_n)$ with $|X| = r$, we denote a point in K'_0 (the exterior of C_0). Hence $r^2 > t^2$. Set $s^2 = r^2 - t^2$.

The pseudo-spheres which lie respectively in K_0 and $K_1(X, t)$ are denoted by

$$(1.1) \quad P_0(a): \tau^2 - |Y|^2 = a^2, \quad \tau \geq 0,$$

and

$$(1.2) \quad P_1(b): (\tau - t)^2 - |Y - X|^2 = b^2, \quad \tau \geq t.$$

We shall insist on taking $a \geq 0$ and $b \geq 0$. $P_0(0)$ and $P_1(0)$ are the same as C_0^+ and C_1^+ . Let $P(a, b; X, t)$ or simply $P(a, b)$ denote the following

$$(1.3) \quad P(a, b) = P_0(a) \cap P_1(b).$$

The time-like hyperplanes $\ell(c; X, t)$, or $\ell(c)$, are given by

$$(1.4) \quad \ell(c): Y \cdot X - \tau t - \frac{1}{2}s^2 = \frac{1}{2}c.$$

Evidently one has

$$(1.5) \quad P(a, b) = P_0(a) \cap \ell(b^2 - a^2).$$

Each $P(a, b)$ is an $n - 1$ dimensional half-hyperboloid in K_0 . In particular $P(0, 0; X, t) = C_0^+ \cap C_1^+ = Hd(X, t)$ which was used in forming HM in [1].

1.2. Parametric representation of $P(a, b)$. Let $X = r\alpha$, $Y = \rho\gamma$ where α and γ are unit vectors in the space. For a given $X \neq 0$ we may set

$$(1.6) \quad Y = \rho\gamma = \rho(q\alpha + q'\alpha')$$

where unit vector $\alpha' \perp \alpha$ and

$$q = \gamma \cdot \alpha, \quad q' = (1 - q^2)^{1/2}.$$

A point $(Y, \tau) \in \ell(c)$ if and only if $q = (2\tau t + s^2 + c)/2r\rho$, and for $(Y, \tau) \in P_0(a)$ one has $\tau = (\rho^2 + a^2)^{1/2}$. Thus we may use (τ, α') as local coordinates on $P(a, b)$ by putting in (1.6),

$$(1.7) \quad \rho = (\tau^2 - a^2)^{1/2}, \quad q = (2\tau t + (b^2 - a^2) + s^2)/2r\rho.$$

We let α' vary on the spheres $|\alpha'| = 1$ of dimension $n - 2$. Instead of (τ, α') we may also use (ρ, α') .

1.3. Definition of $\mathfrak{R}$. An n dimensional surface in K_0 belongs to $\mathfrak{R}$, if it can be covered simply by a one parametric family, with parameter $\mu = \mu(a, b)$, of half-hyperboloids $P(a, b; X, t)$ for some (X, t) .

As an example consider the portion of $C_0^+(X, t)$ which extends to infinity and is bounded below by $Hd(X, t)$. This surface will be denoted by $Hb(X, t)$, and obviously it belongs to $\mathfrak{R}$ because

$$(1.8) \quad Hb(X, t) = \bigcup_{b=0}^{\infty} P(0, b).$$

1.4. The family $\mathfrak{F} = \mathfrak{F}(X, t)$ of inscribed hyperboloids in Ω . In order that a hyperboloid be circumscribed by both C_0 and $C_1(X, t)$, it must be tangent to these cones along the hyperplanes which are parallel to the plane of intersection of C_0 and C_1 . Corresponding to each $\lambda > 0$ there is one such inscribed hyperboloid, one half of it lies in Ω and will be denoted by $Hb(\lambda)$, and the family $\mathfrak{F} = \{Hb(\lambda), 0 < \lambda < \infty\}$. In (Y, τ) coordinates $Hb(\lambda)$ satisfies the following equation,

$$(1.9) \quad \tau^2 - |Y|^2 - \lambda^{-2}(\frac{1}{2}s^2 + \frac{1}{2}\lambda^2 + \tau t - Y \cdot X)^2 = 0,$$

or the same equation in a different form:

$$(1.10) \quad (\tau - t)^2 - |Y - X|^2 - \lambda^{-2}(\frac{1}{2}s^2 - \frac{1}{2}\lambda^2 + \tau t - Y \cdot X)^2 = 0.$$

Lemma 1.1. $Hb(\lambda) \in \mathfrak{R}$.

Proof. By (1.5),

$$P_0(a) \cap \ell(c) = P(a, (c + a^2)^{1/2}),$$

provided $c + a^2 \geq 0$. If one substitutes in (1.9), $\tau^2 - |Y|^2 = a^2$, then (1.9) becomes two equations representing $\ell(\lambda^2 \pm 2\lambda a)$, see (1.4). It follows that $P(a, b) \subset Hb(\lambda)$ if and only if $\lambda^2 \pm 2\lambda a = b^2 - a^2$. Hence we have

$$(1.11) \quad Hb(\lambda) = L \cup M \cup N$$

where

$$(1.12) \quad L = \bigcup_{a-b=\lambda} P(a, b), \quad M = \bigcup_{a+b=\lambda} P(a, b), \quad N = \bigcup_{b-a=\lambda} P(a, b).$$

We introduce the parameter μ , $-\infty < \mu < \infty$ as follows:

$$(1.13) \quad \begin{aligned} P(a, b) \in L: & a = \tfrac{1}{2}(\lambda - \mu), b = -\tfrac{1}{2}(\lambda + \mu), \text{ for } -\infty < \mu < -\lambda, \\ P(a, b) \in M: & a = \tfrac{1}{2}(\lambda - \mu), b = \tfrac{1}{2}(\lambda + \mu), \text{ for } -\lambda \leq \mu \leq \lambda, \\ P(a, b) \in N: & a = -\tfrac{1}{2}(\lambda - \mu), b = \tfrac{1}{2}(\lambda + \mu) \text{ for } \lambda < \mu < \infty. \end{aligned}$$

For $P(a, b) \subset Hb(\lambda)$ one has always

$$(1.14) \quad b^2 - a^2 = \lambda\mu.$$

2. The mean HS.

2.1. Measure dm on $P(a, b; X, t)$. We derive dm on $P(a, b)$ in the same way as it was derived in $Hd(X, t) = P(0, 0; X, t)$ in [1], Lemma 1.4, p. 39. First $P_0(a)$ is transformed into itself by any proper Lorentz transformation A . If we use Y as local coordinates on $P_0(a)$, then the following measure is invariant on $P_0(a)$:

$$\frac{dY}{\tau} = \frac{d\tau dY}{a da}.$$

Secondly we cover $P_0(a)$ by its intersections with the one parametric family of hyperplanes $\ell(c)$ with parameter c . By (1.4), if $(Y, \tau) \rightarrow (Y', \tau')$ and $(X, t) \rightarrow (X', t')$ by A , then c remains unchanged. Thus we may define

$$(2.1) \quad dm(a, b) = \left(\frac{dY}{\tau dc} \right)_{c=b^2-a^2}$$

as the invariant measure on $P(a, b)$.

Lemma 2.1. *Let $P(a, b; X, t)$ be represented by the local coordinates (ρ, α') in (1.6) and (1.7). On $P(a, b)$ one has explicitly the measure*

$$(2.2) \quad dm(a, b) = \frac{1}{r} \frac{1}{\tau} (1 - q^2)^k \rho^{n-2} d\rho d\omega_{\alpha'},$$

where $k = (n - 3)/2$, and $d\omega_{\alpha'}$ is the surface element of $|\alpha'| = 1$ of dimension $n - 2$. If instead of (ρ, α') we use (τ, α') , then

$$(2.3) \quad dm(a, b) = \frac{1}{r} (1 - q^2)^k \rho^{2k} d\tau d\omega_{\alpha'}.$$

Proof. The lemma is proved by observing that

$$dY = |\text{grad } c|^{-1} d\sigma dc$$

where c is the function of Y given by (1.4) with $\tau = (\rho^2 + a^2)^{1/2}$, and $d\sigma$ = euclidean surface element of $P(a, b) \in P_0(a)$ in terms of (ρ, α') .

2.2. Measure dS on surfaces belonging to $\mathfrak{F}(X, t)$.

Definition. Let $Hb \in \mathfrak{R}$ and $Hb = \bigcup_{\mu} P(a, b; X, t)$ with parameter $\mu = \mu(a, b)$. Then we define

$$(2.4) \quad dS = d\mu \cdot dm(a, b)$$

as the measure on Hb .

We apply the above definition of dS to $Hb(\lambda) \in \mathfrak{F}$ described in 1.4. Here we have $\mu = (b^2 - a^2)/\lambda$. If we use (τ, α') as local coordinates on $P(a, b)$, then dS on $Hb(\lambda)$ is expressed in terms of $d\mu \, d\tau \, d\omega_{\alpha'}$ by (2.4) and Lemma 2.1.

There is another way to represent dS on $Hb(\lambda)$. We cover Ω by the two parametric family $\{a \geq 0, b \geq 0\}$ of $P(a, b; X, t)$. A point (Y, τ) in Ω is first characterized by the parameters (a, b) namely when $(Y, \tau) \in P(a, b)$, and secondly by the local coordinates on $P(a, b)$ itself.

Lemma 2.2. One has in Ω ,

$$(2.5) \quad d\tau \, dY = \frac{1}{r} ab(1 - q^2)^k \rho^{2k} da \, db \, d\tau \, d\omega_{\alpha'}$$

where ρ and q are given by (1.7). The measure dS on $Hb(\lambda)$ is given by the following:

$$(2.6) \quad dS = 2 \frac{d\tau \, dY}{ab \, d\lambda}, \quad \text{on } M,$$

and

$$(2.7) \quad dS = -2 \frac{d\tau \, dY}{ab \, d\lambda}, \quad \text{on } L \text{ and } N,$$

with $a = (\tau^2 - |Y|^2)^{1/2}$, $b = ((\tau - t)^2 - |Y - X|^2)^{1/2}$ and $\lambda = a + b$ on M , $\lambda = \pm(a - b)$ on L and N (see 1.12).

Proof. (2.5) is obtained by a simple computation. To obtain (2.6) and (2.7) note that

$$da \, db = \frac{1}{2} d(a + b) \, d(b - a) = \pm \frac{1}{2} d\lambda \, d\mu$$

by (1.12) and (1.13).

Definition of $HS[\Phi, \lambda]$. Let $\Phi(Y, \tau)$ be a continuous function in K_0 . Then the mean of Φ on $Hb(\lambda)$ is defined by

$$(2.8) \quad HS[\Phi, \lambda] = \frac{1}{\omega_n} \left(\frac{1}{4}(s^2 + \lambda^2)\right)^{-k-1/2} \int_{Hb(\lambda)} \Phi(Y, \tau) \, dS$$

where dS is the measure on $Hb(\lambda)$. It is assumed of Φ that the integral in (2.8) converges absolutely.

Lemma 2.3. $HS[\Phi, \lambda]$ is invariant with respect to all proper Lorentz transformations.

Proof. Let A be a proper Lorentz transformation in R^{n+1} and let $(Y, \tau) \rightarrow (Y', \tau'), (X, t) \rightarrow (X', t')$ by A , then $P(a, b; X, t) \rightarrow P(a, b; X', t')$ and $\mathfrak{F}(X, t) \rightarrow \mathfrak{F}(X', t'), Hb(\lambda) \rightarrow Hb'(\lambda)$ with the same λ . Suppose we define

$$\Psi(Y', \tau') = \Phi(Y, \tau).$$

Then $HS[\Psi; \lambda] = HS[\Phi; \lambda]$.

2.3. A special case. Consider the case that (X, t) is a point on the positive y_1 axis, that is, $x_1 > 0, x_2 = x_3 = \dots = x_n = t = 0$. Then for points on $P(a, b)$ one has in (1.7), $\rho q = (x_1^2 + b^2 - a^2)/2x_1$. By using λ and μ instead of (a, b) one verifies easily the following representation of $Hb(\lambda)$ in local variables (μ, τ, α') :

$$(2.9) \quad y_1 = (x_1^2 + \lambda\mu)/2x_1, \quad Y' = (y_2, \dots, y_n) = Q\alpha'$$

where

$$Q^2 = \rho^2(1 - q^2) = \tau^2 - (x_1^2 + \lambda^2)(x_1^2 + \mu^2)/4x_1^2.$$

In the following it is convenient to set

$$(2.10) \quad s_1 = \frac{1}{2}(x_1^2 + \lambda^2)^{1/2}.$$

Equation (1.9) for $Hb(\lambda)$ becomes

$$(2.11) \quad \tau^2 - s_1^2 = |Y'|^2 + \frac{4}{\lambda^2} s_1^2 \left(y_1 - \frac{x_1}{2} \right)^2$$

Introduce on $Hb(\lambda)$ a new variable $\theta, 0 \leq \theta \leq \pi$, to replace μ , by setting in (2.9),

$$(2.12) \quad \begin{aligned} Q &= (\tau^2 - s_1^2)^{1/2} \sin \theta, & Y' &= Q\alpha' \\ \left(y_1 - \frac{x_1}{2} \right) &= \frac{\lambda}{2s_1} (\tau^2 - s_1^2)^{1/2} \cos \theta. \end{aligned}$$

This leads to a simple expression for $HS[\Phi, \lambda]$.

Lemma 2.4. For $Hb(\lambda) \ni \mathfrak{F}(x_1, 0, \dots, 0)$ one has

$$(2.13) \quad HS[\Phi, \lambda] = \frac{1}{\omega_n} s_1^{-(n-1)} \int_{s_1}^{\infty} \int_{|\beta|=1} \Phi(Y, \tau) (\tau^2 - s_1^2)^{(k+1/2)} d\tau d\omega_\beta$$

where $\beta = (\beta_1, \dots, \beta_n), |\beta| = 1$ with $\beta_1 = \cos \theta, (\beta_2, \dots, \beta_n) = (\sin \theta)\alpha'$. The (Y, τ) are given by (2.12).

Proof. Comparing (2.12) with (2.9), one has

$$(2.14) \quad \mu = 2 \frac{x_1}{\lambda} \left(y_1 - \frac{x_1}{2} \right) = \frac{x_1}{s_1} (\tau^2 - s_1^2)^{1/2} \cos \theta.$$

Hence it follows from (2.3) and (2.4) that

$$(2.15) \quad dS = \frac{1}{s_1} (\tau^2 - s_1^2)^{k+1/2} d\tau d\omega_\beta.$$

The limits of integration are easily determined.

3. The wave operator. Denote by K the solid cone K_0 with its vertex removed, and by $K(\delta)$ that part of K with $\tau \geq \delta > 0$. On a space-like hyperplane e let $\partial\Phi/\partial\nu$ denote the directional derivative of Φ along the transversal with increasing τ .

Theorem 1. Let $\Phi(Y, \tau)$ be a function of class $C^{(2)}$ in K with the following properties at $\tau = +\infty$:

E1. For some $\epsilon > 0$, $\Phi(Y, \tau) = O \tau^{-(n-1+\epsilon)}$ uniformly in Y .

E2. There is a one parametric family of parallel space-like hyperplanes $\{e_n\}$ tending to $\tau = +\infty$ such that for $n \rightarrow \infty$,

$$\int_{e_n \cap K} \left| \frac{\partial\Phi}{\partial\nu} \right| dY \rightarrow 0.$$

E3. For any $\delta > 0$,

$$\int_{K(\delta)} |\Box^2 \Phi| d\tau dY < \infty.$$

Let $HS[\Phi, \lambda]$ be the mean of Φ on $Hb(\lambda) \in \mathfrak{F}(X, t)$, $\mathfrak{F}$ being the family of inscribed hyperboloids in $\Omega = K \cap K_1(X, t)$.

Then

$$(3.1) \quad \frac{\partial}{\partial\lambda} HS[\Phi, \lambda] = \frac{1}{2\omega_n} (\tfrac{1}{4}(s^2 + \lambda^2))^{-n/2} \int \Box^2 \Phi d\tau dY$$

holds with the volume integral over the interior of $Hb(\lambda)$.

Proof. 1. Both sides of (3.1) are invariant with respect to all proper Lorentz transformations in the sense of Lemma 2.3.

2. If $\Phi(Y, \tau)$ satisfies E1 to E3, then the function $\Psi(Y', \tau')$ transformed from Φ also satisfies E1 to E3 in the variables (Y', τ') . From E1 and E3 it follows by the divergence theorem that E2 holds not only for the family $\{e_n\}$ but also for any such family of hyperplanes tending to $\tau = +\infty$.

3. It suffices to verify (3.1) for the special case of $(X, t) = (x_1, 0, \dots, 0)$, $x_1 > 0$. By using (2.13) one can differentiate under the integral sign as follows:

$$(3.2) \quad \frac{\partial}{\partial\lambda} HS[\Phi, \lambda] = \frac{1}{\omega_n} s_1^{-(n-2)} \int_{Hb(\lambda)} \frac{d\Phi}{d\lambda} dS,$$

where

$$(3.3) \quad \frac{d\Phi}{d\lambda} = \Phi_\tau \frac{\partial\tau}{\partial\lambda} + \Phi_{y_1} \frac{\partial y_1}{\partial\lambda} + \Phi_{\tau'} \frac{\partial Y'}{\partial\lambda},$$

and

$$(3.4) \quad \frac{\partial \tau}{\partial \lambda} = \frac{\lambda}{4s_1^2} \tau, \quad \frac{\partial y_1}{\partial \lambda} = \left(y_1 - \frac{x_1}{2}\right) \frac{1}{\lambda}, \quad \frac{\partial Y'}{\partial \lambda} = \frac{\lambda}{4s_1^2} Y'.$$

To prove (3.2)–(3.4) we introduce in (2.13) the variable σ by $\tau = s_1 \sigma$. One has from (2.12),

$$(3.5) \quad \tau = s_1 \sigma, \quad y_1 - \frac{x_1}{2} = \frac{\lambda}{2} (\sigma^2 - 1)^{1/2} \cos \theta, \quad Y' = (\sigma^2 - 1)^{1/2} s_1 \sin \theta \alpha'.$$

In terms of $d\sigma d\omega_{\alpha'}$, (2.13) becomes

$$(3.6) \quad HS[\Phi, \lambda] = \frac{1}{\omega_n} \int_1^\infty \int_{|\beta|=1} \Phi(Y, \tau) (\sigma^2 - 1)^{k+1/2} d\sigma d\omega_\beta.$$

The differentiation with respect to λ in (3.6) can be carried out under the integral sign. From (3.5) and (2.10) one obtains (3.4). After the differentiation one goes back to $d\tau d\omega_\beta$ and finds (3.2).

4. Recall (1.12). On M one has $\lambda = a + b$, and one may form $\text{grad } \lambda$ with respect to (Y, τ) by differentiating $a = (\tau^2 - |Y|^2)^{1/2}$ and $b = (\tau^2 - (y_1 - x_1)^2 - |Y'|^2)^{1/2}$. One forms it similarly on L and N . We are going to verify the following:

$$(3.7) \quad (\lambda_\tau, \lambda_{y_1}, \lambda_{Y'}) = \left(\frac{\partial \tau}{\partial \lambda}, -\frac{\partial y_1}{\partial \lambda}, -\frac{\partial Y'}{\partial \lambda} \right) \frac{4s_1^2}{ab} \quad \text{on } M$$

and

$$(3.8) \quad (\lambda_\tau, \lambda_{y_1}, \lambda_{Y'}) = -\left(\frac{\partial \tau}{\partial \lambda}, -\frac{\partial y_1}{\partial \lambda}, -\frac{\partial Y'}{\partial \lambda} \right) \frac{4s_1^2}{ab} \quad \text{on } L \text{ and } N,$$

where $\partial\tau/\partial\lambda$, $\partial y_1/\partial\lambda$, and $\partial Y'/\partial\lambda$ are given by (3.4).

It is immediately seen that (3.7) and (3.8) hold for λ_τ and $\lambda_{Y'}$. It requires a little computation to verify them for λ_{y_1} . One has to evaluate $by_1 \pm a(y_1 - x_1)$ separately on L , M and N . This is done by replacing (a, b) by (λ, μ) , using (1.13), and then substituting for μ the expression in the first equation of (2.14). Altogether (3.7) and (3.8) are then verified.

5. Substituting (3.7)–(3.8) into (3.2)–(3.4) and applying Lemma 2.2, (2.6)–(2.7), one finds

$$(3.9) \quad \begin{aligned} \frac{\partial}{\partial \lambda} HS[\Phi, \lambda] &= \frac{1}{2\omega_n} s_1^{-n} \int_{Hb(\lambda)} (\Phi_{\tau\lambda_\tau} - \Phi_{y_1\lambda_{y_1}} - \Phi_{Y'\lambda_{Y'}}) \frac{d\tau dY}{d\lambda} \\ &= \frac{1}{2\omega_n} s_1^{-n} \int \square^2 \Phi d\tau dY \end{aligned}$$

where the last integral is over the interior of $Hb(\lambda)$.

6. To justify the convergence one considers first that part of $HS[\Phi, \lambda]$ from $\tau = s_1$ to $\tau = s_1 B$ for some constant $B > 1$. One carries the argument through

to (3.9), with a remainder term involving the integral of $\partial\Phi/\partial\nu$ over $(\tau = s_1 B) \cap Hb(\lambda)$. Conditions E2 and E3 allow the passage of $B \rightarrow \infty$.

Theorem 1 is proved.

Corollary. Suppose $w(Y, t)$ is a solution of the wave equation with the properties of $\Phi = w$ in Theorem 1. Then $HS[\omega, \lambda]$ is independent of λ .

Remark. The growth condition required of w is much too strong for general application to the homogeneous wave equation. One could also discuss the mean over $Hb(0)$ which is the plane portion in Ω of $\ell(0)$ bounded by $Hd(X, t)$.

4. Solutions in K'_0 of the wave equation with even space dimensions. Let $\varphi(Y)$ be a smooth function with compact support in R^n_y . For $Y = \rho\gamma$ let

$$(4.1) \quad \zeta(\rho\gamma) = (-1)^{n/2} \frac{2}{\left(\frac{n}{2} - 1\right)!} \rho^{-k} \left(\frac{\partial}{\partial\rho}\right)^{n/2} \rho^{n-3/2} \varphi(\rho\gamma).$$

Over the surface $Hb(X, t)$ of (1.8) form the mean

$$(4.2) \quad HS[\zeta; X, t] = \frac{1}{\omega_n} s^{-(n-2)} \int_{Hb(X, t)} \zeta(Y) dS$$

with dS given by (2.4). We have the following

Theorem 2. In K'_0 the function

$$(4.3) \quad w(X, t) = HS[\zeta(Y); X, t]$$

has the following properties:

1. It satisfies the wave equation.
2. On C_0^+ it takes on the boundary values

$$(4.4) \quad w(r\alpha, r) = \varphi(r\alpha).$$

3. On C_0^- its boundary values are given by

$$(4.5) \quad w(r\alpha, -r) = (-1)^{k+1/2} \frac{1}{\pi} r^{-n+3/2} \int_0^\infty \frac{1}{\rho + r} \rho^{n-3/2} \varphi(-\rho\alpha) d\rho.$$

Remark. To write dS explicitly, note that $a = 0$ in (2.2). Hence $\tau = \rho$ and $q = (t\rho + (1/2)(b^2 + s^2))/r\rho$ by (1.7). Thus from (2.4),

$$(4.6) \quad dS = r^{-1} Q^{2k} db d\rho d\omega_\alpha,$$

where $Q^2 = \rho^2(1 - q^2)$. In $HS[\zeta]$ of (4.2) one has the limits of integrations as follows: $|\alpha'| = 1$ over the sphere of dimension $n - 2$ and because of $|q| \leq 1$,

$$(4.7) \quad \frac{1}{2} \frac{s^2 + b^2}{r - t} \leq \rho < +\infty, \quad 0 \leq b < +\infty,$$

or

$$(4.8) \quad 0 \leq b^2 \leq (2(r-t)\rho - s^2), \quad \frac{1}{2}(r+t) \leq \rho < +\infty.$$

Proof. 1. Let A be a proper Lorentz transformation in R^{n+1} leaving the origin fixed, and suppose A' is the induced map of A in R^n through projection ([1], p. 34). Then because of the invariance of the measure dS one has

$$(4.9) \quad HS[\zeta(Y); X', t'] = HS[\zeta'(Y); X, t],$$

where $(X', t') = A(X, t)$, $\zeta'(Y) = \zeta(A'Y)$. Furthermore the above holds true also when A and A' are dilations, namely, $Y \rightarrow \lambda Y$, $(X, t) \rightarrow \lambda(X, t)$ with $\lambda > 0$. For then $b \rightarrow \lambda b$ by (1.2), and $s \rightarrow \lambda s$. Thus $HS[\zeta]$ has the same invariance property as that possessed by $HM[\zeta]$ of Theorem 1.1 in [1]. It follows that $w(X, t)$ of (4.3) is a solution of the wave equation in K'_0 . This statement is valid independent of the parity of n , provided ζ is a function in $C^{(2)}$ vanishing together with its derivatives at infinity.

2. In discussing the boundary values it is convenient to set

$$(4.10) \quad b = s\lambda.$$

Then in (4.6) one has

$$(4.11) \quad r^{-1}s^{-(n-2)}Q^{2k}db \\ = r^{-2k-1}(\rho - \frac{1}{2}(r+t)(1+\lambda^2))^k(\rho + \frac{1}{2}(r-t)(1+\lambda^2))^k d\lambda.$$

When $t \rightarrow r$ one finds, because of $q \rightarrow 1$,

$$(4.12) \quad w(r\alpha, r) = \frac{\omega_{n-1}}{\omega_n} r^{-2k-1} \int_0^\infty d\lambda \int_{r(1+\lambda^2)}^\infty \rho^k (\rho - r(1+\lambda^2))^k \zeta(\rho\alpha) d\rho.$$

Interchange the order of integrations in (4.12) and evaluate first $d\lambda$ over $0 \leq \lambda \leq (\rho - r)^{1/2}r^{-1/2}$. After that substitute (4.1) for ζ and integrate by parts $n/2$ times. This leads to (4.4).

3. For proving (4.5) one again starts from (4.11). Because of the limits of ρ in (4.7) it is convenient to set

$$(4.13) \quad \rho = (1 + \lambda^2)Z.$$

One finds for $t \rightarrow -r$, since $q \rightarrow -1$,

$$(4.14) \quad w(r\alpha, -r) \\ = \frac{\omega_{n-1}}{\omega_n} r^{-2k-1} \int_0^\infty (1 + \lambda^2)^{2k+1} d\lambda \int_0^\infty Z^k (Z + r)^k \zeta(-(1 + \lambda^2)Z\alpha) dZ.$$

Now write (4.1) in the following way,

$$(4.15) \quad \zeta(-(1 + \lambda^2)Z\alpha) = \text{const. } Z^{-k} \left(\frac{\partial}{\partial Z} \right)^{n/2} Z^{n-3/2} \varphi(-(1 + \lambda^2)Z\alpha).$$

Carry out integration by parts with respect to Z , $n/2$ times. This leads to the integral

$$\text{const.} \int_0^\infty (Z+r)^{-3/2} Z^{n-3/2} \varphi(-(1+\lambda^2)Z\alpha) dZ.$$

Use ρ again instead of Z by (4.13). Integrating first with respect to λ one finds

$$\int_0^\infty (\rho+r+r\lambda^2)^{-3/2} d\lambda = r^{-1/2} \frac{1}{\rho+r}.$$

Altogether (4.5) is then verified.

Remark. One could treat the even dimensional problem by the method of descent, using one additional space dimension x_{n+1} , and data on the cone independent of x_{n+1} . The representation formula for the odd dimension leads to an integral over $Hb(\lambda)$ or $Hb(0)$ of (1.9). Convergence difficulties arise since one is now integrating in the interior of C_0 in the even dimensional space.

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