Some Banach Algebras Associated With Quotients of $L^1(R)$

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Communicated by H. LEWY

1. Introduction. In this paper we continue the discussion begun in [11] of the algebras $B(E)$ associated with quotients $A(E)$.

Let A denote the Banach algebra of Fourier transforms g of elements $\hat{g}$ of $L^1 = L^1(R)$:

$$g(x) = \int_{-\infty}^{\infty} \hat{g}(y) e^{-ixy} dy \quad \text{for } x \in R.$$

In A we have pointwise multiplication and the norm $\|g\|_A = \|\hat{g}\|_{L^1}$. Let E always denote a compact subset of R . Let $I(E)$ be the closed ideal in A made up of the functions in A that vanish on E ; let $A(E)$ denote the quotient algebra $A/I(E)$. Its elements may be thought of as the restrictions to E of functions in A . The norm is given by:

$$\|f\|_{A(E)} = \inf \{ \|g\|_A : g \in A \text{ and } g|_E = f \}.$$

This algebra is semisimple, and its maximal ideal space is E . Let $C(E)$ denote the algebra of all continuous functions on E , with the supremum norm. If $A(E) = C(E)$, then E is called a *Helson set*.

The Banach algebra $B(E)$ consists of the uniform limits on E of bounded sequences in $A(E)$, with norm

$$\|f\|_{B(E)} = \inf \{ \sup_n \|f_n\|_{A(E)} : \{f_n\}_{n=1}^{\infty} \subset A(E) \text{ and } \|f - f_n\|_{C(E)} \rightarrow 0 \}.$$

Evidently, for every E ,

$$A(E) \subseteq B(E) \subseteq C(E),$$

and each of the two inclusion maps is norm-decreasing. For a Helson set, the three algebras are the same and the three norms are equivalent. Various conditions on E are known that imply $A(E) = B(E)$; for example, if every portion of E supports a nonzero measure μ such that $\lim_{|y| \rightarrow \infty} \hat{\mu}(y) = 0$, then $A(E) =$

$B(E)$. (A *portion* of E is a non-empty closed set of the form $E \cap J$ where J is an open interval.)

In [11] we gave the first example of a set E for which $A(E) \neq B(E)$ and proved the following two theorems.

Theorem I. *Let E be a countable non-Helson set with a finite number of accumulation points. Then $A(E) \neq B(E)$.*

Theorem II. *There exists a perfect set E such that for every portion G of E , $A(G) \neq B(G)$.*

In Section 2 of the present paper we show that there exist sets E and functions

$$f = \sum_{n=-\infty}^{\infty} c_n \chi_n$$

where $\sum |c_n| < \infty$ and the χ_n are characters (not necessarily continuous), such that $f \in B(E)$ but $f \notin A(E)$. We also show that there exist countable non-Helson sets E for which every such f is in $A(E)$.

In Section 3 we discuss the functions that operate in algebras $A(E)$ and $B(E)$ (range transformations).

Varopoulos discovered cases in which $A(E)$ is not closed in $B(E)$ ([18, Theorem 3] and [14, Chapters 10 and 11]). In Section 4 we present a variation on this result.

In Section 5 we list some open questions.

First we need to introduce some more notation and basic facts. Let PM denote the conjugate space of A . Its elements are called *pseudomeasures*, S ; for each $\hat{S} \in L^\infty = (L^1)^*$, S is the functional defined on A by:

$$(g, S) = \int_{-\infty}^{\infty} g(y) \overline{\hat{S}(y)} dy \quad \text{for } g \in A.$$

Thus $\|S\|_{PM} = \|\hat{S}\|_\infty = \text{ess sup}_{y \in \mathbb{R}} |\hat{S}(y)|$.

The conjugate space of $A(E)$ is the subspace of PM that annihilates $I(E)$; call it $N(E)$. The conjugate space of $C(E)$ is $M(E)$, the space of Borel measures μ with support contained in E , with the total variation norm $\|\mu\|_M$. For $f \in C(E)$ and $\mu \in M(E)$, we have

$$(f, \mu) = \int f(x) \overline{d\mu(x)}; \quad \hat{\mu}(y) = \int e^{ixy} d\mu(x) \quad \text{for } y \in \mathbb{R}.$$

We have natural norm-decreasing inclusions, $A(E) \subseteq C(E)$ and $M(E) \subseteq N(E)$. Evidently, E is a Helson set if and only if $M(E) = N(E)$; or, if and only if $h(E) > 0$, where

$$\begin{aligned} h(E) &= \inf \left\{ \frac{\|f\|_{C(E)}}{\|f\|_{A(E)}} : f \in A(E), f \neq 0 \right\} \\ &= \inf \left\{ \frac{\|\mu\|_{PM}}{\|\mu\|_M} : \mu \in M(E), \mu \neq 0 \right\}. \end{aligned}$$

Also, $B(E)$ may be defined as the class of functions $f \in C(E)$ such that the quantity

$$\|f\|_{B(E)} = \sup \left\{ \frac{|(f, \mu)|}{\|\mu\|_{PM}} : \mu \in M(E), \mu \neq 0 \right\}$$

is finite; and this definition of the $B(E)$ norm agrees with the earlier one.

Let T denote the circle group (the real numbers modulo 2π), and let Z denote its dual, the integer group. Let A' denote $A(T)$, the algebra of absolutely convergent Fourier series with period 2π :

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int},$$

with norm $\|f\|_{A'} = \sum |\hat{f}(n)|$. We shall make use of the fact that A and A' are locally the same, in the following sense. When $\delta > 0$ and $E \subseteq [-\pi + \delta, \pi - \delta]$, let $A'(E)$ be the algebra of restrictions to E of functions in A' . Then $A(E)$ and $A'(E)$ are identical classes of functions, and they are isomorphic as algebras:

$$\|f\|_{A(E)} \leq \|f\|_{A'(E)} \leq k \|f\|_{A(E)} \quad \text{for } f \in A(E),$$

where k depends on δ (this follows from [17, 2.7.6] or [8, 8.6.2]). The dual fact is that

$$\|S\|_{PM'} \leq \|S\|_{PM} \leq k \|S\|_{PM'} \quad \text{for } S \in N(E)$$

where

$$\|S\|_{PM'} = \sup_{n \in Z} |\hat{S}(n)|.$$

Let R_d denote R with the discrete topology. Let R_0 denote the dual group of R_d , that is, the Bohr compactification of R . Let $M(R_0)$ be the measure algebra of R_0 , $M_d(R_0)$ the subalgebra consisting of the discrete measures. Let $D(E)$, $D_d(E)$ denote the classes of restrictions to E of the Fourier transforms $\hat{\nu}$ of measures ν in $M(R_0)$, $M_d(R_0)$ respectively.

Note that $A(E)$ consists precisely of the restrictions to E of transforms of discrete measures *with support in R* (whereas $D_d(E)$ is a larger class). In fact, if $m > 0$ is chosen so that $E \subset (-m\pi, m\pi)$, then every $f \in A(E)$ may be represented as an absolutely convergent Fourier series of period $2\pi m$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx/m} \quad \text{for } x \in E$$

where $\sum |c_n| < \infty$. Thus f agrees on E with $\hat{\nu}$, where $\nu = \sum_n c_n \delta_{n/m}$, where δ_{ν} denotes the unit mass placed at y .

Now let us point out why, for countable E , $B(E) = C(E) \cap D(E)$. Let AP be the closed subspace of L^∞ consisting of the continuous almost periodic functions. It may be regarded as $C(R_0)$, and its conjugate space as $M(R_0)$. Now AP contains $\hat{\mu}$ for every discrete measure $\mu \in M(R)$. Therefore if E is countable and $\nu \in M(R_0)$, the rule

$$\mu \rightarrow \int_{R_0} \hat{\mu} d\nu$$

defines a linear functional on $M(E)$. For $t \in E$, the value of this functional at δ_t is

$$\int_{R_0} \hat{\delta}_t d\nu = \int_{R_0} \chi(t) d\nu(\chi) = \hat{\nu}(t) = \int_E \hat{\nu}(x) d\delta_t(x).$$

By linearity and continuity,

$$\int_{R_0} \hat{\mu} d\nu = \int_E \hat{\nu} d\mu \quad \text{for all } \mu \in M(E).$$

If $\hat{\nu}$ is continuous on E , then clearly $\hat{\nu} \in B(E)$. We have proved that $D(E) \cap C(E) \subseteq B(E)$. Now suppose that $f \in B(E)$. Then f defines a linear functional of norm $\|f\|_{B(E)}$ on the linear submanifold $\{\hat{\mu} : \mu \in M(E)\}$ of AP . By the Hahn-Banach theorem, there is a measure $\nu \in M(R_0)$ such that

$$\|\nu\|_M = \|f\|_{B(E)} \quad \text{and}$$

$$\int_E f d\mu = \int_{R_0} \hat{\mu} d\nu \quad \text{for } \mu \in M(E),$$

and it follows that f agrees with $\hat{\nu}$ on E . Therefore $B(E) \subseteq D(E) \cap C(E)$, and we have proved that $B(E) = D(E) \cap C(E)$ when E is countable.

2. Restrictions of Fourier transforms of discrete measures on the Bohr group.

Theorem III. *There exists a set E such that $A(E) \neq D_d(E) \cap C(E)$.*

Proof. We may restate the theorem as follows: There exists a set E , a sequence $\{c_n\}_{n=-\infty}^{\infty}$ such that $\sum |c_n| < \infty$, and a sequence $\{\chi_n\}_{n=-\infty}^{\infty}$ of elements of R_0 (characters on R_d) such that the function

$$f(x) = \sum c_n \chi_n(x)$$

is continuous on E (and hence is in $B(E)$) but is not in $A(E)$. Note that then f is the Fourier transform of the measure $\sum c_n \delta_{\chi_n}$ in $M_d(R_0)$.

Let $\{r_k, s_k\}_{k=1}^{\infty}$ be a set of numbers that is independent over the rationals, such that

$$\frac{1}{k+1} < r_k < r_k + ks_k < \frac{1}{k} \quad \text{for } k = 1, 2, \dots$$

Let F be a compact subset of $(0, 1)$ that disobeys harmonic synthesis (see [7, Chapter IX]; [8, VIII.10]; or [5, V.6]). For $h > 0$ let F_h denote the set of all points whose distance from F is no greater than h . Since F disobeys synthesis, there exists a function $g \in I(F)$ such that

$$\lim_{h \rightarrow 0} \|g\|_{A(F_h)} > 0$$

—even though, of course,

$$\lim_{h \rightarrow 0} \|g\|_{C(F_n)} = 0.$$

We may require that, furthermore, $g(0) = 0$. Since $g \in A$, on each closed sub-interval of $(-\pi, \pi)$, g may be represented as the sum of an absolutely convergent Fourier series. Thus

$$g(x) = \sum c_n e^{inx} \quad \text{for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

where $\sum |c_n| < \infty$.

For each k , let H_k be the set of numbers j/k , where j is an integer, whose distance from F is no greater than $1/k$. By the procedure of Herz ([7, IX.8] or [5, V.3]),

$$\lim_{k \rightarrow \infty} \|g\|_{A(H_k)} > 0.$$

Let $\phi_k(x) = r_k + ks_k x$ for $x \in R$. Then ϕ_k , being an affine-linear map of R onto R , induces an isometry of A onto A by the rule $f \rightarrow f \circ \phi_k^{-1}$. In particular, if we let $E_k = \phi_k(H_k)$, we have

$$\|g \circ \phi_k^{-1}\|_{A(E_k)} = \|g\|_{A(H_k)}.$$

Note that

$$E_k \subset \{r_k + js_k : j = 0, 1, \dots, k\} \subset \left(\frac{1}{k+1}, \frac{1}{k}\right).$$

Let

$$E = \{0\} \cup \bigcup_{k=1}^{\infty} E_k.$$

Let n be an integer. Because the set $\{r_k, s_k\}$ is independent, there is a character $\chi_n \in R_0$ such that

$$\chi_n(r_k) = 1 \quad \text{and} \quad \chi_n(s_k) = e^{in/k} \quad \text{for all } k.$$

Necessarily,

$$\chi_n(r_k + js_k) = e^{inj/k},$$

and in particular,

$$\chi_n(x) = e^{[in\phi_k^{-1}(x)]} \quad \text{for } x \in E_k, \quad \text{for all } k.$$

Let $f = \sum c_n \chi_n$. Then $f(0) = \sum c_n = g(0) = 0$ and for $x \in E_k$, $f(x) = g \circ \phi_k^{-1}(x)$. Therefore

$$\lim_{k \rightarrow \infty} \|f\|_{C(E_k)} = 0,$$

and $f \in C(E) \cap D_d(E)$. But

$$\lim_{k \rightarrow \infty} \|f\|_{A(E_k)} > 0.$$

If f were in $A(E)$, then there would be an $f_1 \in A$ such that

$$f_1|_E = f|_E;$$

and since $f_1(0) = 0$, and since a one-point set obeys synthesis, we would have

$$\lim_{\epsilon \rightarrow 0} \|f_1\|_{A((-\epsilon, \epsilon))} = 0.$$

But this limit cannot be zero because

$$\|f_1\|_{A((-\epsilon, \epsilon))} \geq \|f\|_{A(E_k)} \quad \text{for all } k \geq 1/\epsilon.$$

Therefore $f \notin A(E)$. The theorem is proved.

Remark. By using techniques like those of Körner ([13]), we may modify the proof to obtain a perfect set P such that $E \subseteq P$ and $A(P) \neq D_d(P) \cap C(P)$. Here is a sketch of the method. We choose $\epsilon_k > 0$ sufficiently small so that the sets $E_k + (-\epsilon_k, \epsilon_k)$; $k = 1, 2, \dots$ are pairwise disjoint and so that

$$\lim_{k \rightarrow \infty} \|g \circ \phi_k^{-1}\|_{C(E_k + (-\epsilon_k, \epsilon_k))} = 0.$$

Then we construct sets P_k such that $E_k \subseteq P_k \subseteq E_k + (-\epsilon_k, \epsilon_k)$; such that $P = \{0\} \cup \bigcup_{k=1}^{\infty} P_k$ is a perfect set; such that for some constant C ,

$$\sum \|S_k\|_{PM} \leq C \|S\|_{PM}$$

whenever $S \in N(P)$ and $S_k = S|_{P_k}$; and such that if $x_i \in P_i$ and u_i is an integer and $\sum_{i=1}^k u_i x_i = 0$, then $u_1 = u_2 = \dots = u_k = 0$. Then we define f just as before.

The set E in Theorem III consists of *proper subsets* of finite arithmetic progressions whose lengths tend to infinity. It will be proved as a corollary of Theorem IV that if E contained *all* the points of these progressions, then $D_d(E) \cap C(E)$ would coincide with $A(E)$, even though, of course, $A(E) \neq B(E) = D(E) \cap C(E)$.

Theorem IV. Let $f = \sum_{n=1}^{\infty} c_n \chi_n$ where $\sum |c_n| < \infty$ and $\chi_n \in R_0$. Let

$$E_k = \{r_k + js_k : j = 0, 1, \dots, N_k\} \text{ for } k = 1, 2, \dots$$

where $s_k \neq 0$ and $N_k \rightarrow \infty$. Let $\lim_{k \rightarrow \infty} \|f\|_{C(E_k)} = 0$. Then $\lim_{k \rightarrow \infty} \|f\|_{A(E_k)} = 0$.

Proof. If $\chi \in R_0$ and E is a finite set, then $\|\chi\|_{A(E)} = 1$. Therefore for every k , $\|f\|_{A(E_k)} \leq \sum_n |c_n|$. Let $L = \limsup_{k \rightarrow \infty} \|f\|_{A(E_k)}$, and let m be an arbitrary integer. We shall prove that in the definition of f , a subset of the coefficients $\{c_n\}$, including c_m , may be replaced by zeros without changing the value of L . By repeating this process, we may make $\sum_n |c_n|$ arbitrarily small without changing the value of L . The theorem follows, since L must then be zero.

For each k , consider a finite partition of the interval $[0, \pi]$:

$$F_k = \left\{ \frac{j\pi}{N_k} : j = 0, 1, \dots, N_k \right\},$$

and an affine-linear map from R onto R :

$$\Psi_k(x) = r_k + \frac{s_k N_k}{\pi},$$

which induces an isometry of A onto A . In particular, Ψ_k maps F_k onto E_k and

$$\|f \circ \Psi_k\|_{A(F_k)} = \|f\|_{A(E_k)}.$$

Evidently,

$$\|f \circ \Psi_k\|_{C(F_k)} = \|f\|_{C(E_k)} \rightarrow 0.$$

For each pair n, k , let $\theta(n, k)$ and $\mu(n, k)$ be defined by these conditions:

$$\begin{aligned} e^{i\theta(n, k)} &= \chi_n(r_k) = \chi_n(\Psi_k(0)); & |\theta(n, k)| &\leq \pi; \\ e^{i\mu(n, k)\pi/N_k} &= \chi_n(s_k) = \chi_n\left(\Psi_k\left(\frac{\pi}{N_k}\right)\right); & |\mu(n, k)| &\leq N_k. \end{aligned}$$

For each pair n, k , let $\lambda(n, k) = \mu(n, k) - \mu(m, k) + 2N_k \epsilon_{n, k}$ where $\epsilon_{n, k}$ is an integer chosen so that

$$-N_k \leq \lambda(n, k) \leq N_k.$$

Let $g_k(x) = \sum_n c_n e^{i[\lambda(n, k)x + \theta(n, k)]}$ for $x \in R$. Then for $x \in F_k$, $g_k(x) = f(\psi_k(x))e^{-i\mu(n, k)x}$, so that $\|g_k\|_{A(F_k)} = \|f\|_{A(E_k)}$ and $\|g_k\|_{C(F_k)} = \|f\|_{C(E_k)}$. Thus

$$L = \limsup_{k \rightarrow \infty} \|g_k\|_{A(F_k)}.$$

Since we may replace $\{g_k\}$ by a subsequence without changing L , we may assume (a) that for every n , either $|\lambda(n, k)| \rightarrow \infty$, or else $\lambda(n, k)$ converges to a finite limit $\lambda(n)$, and (b) that for every n , $\theta(n, k)$ converges to a finite limit $\theta(n)$. Let

$$X_1 = \{n: |\lambda(n, k)| \rightarrow \infty\},$$

$$X_2 = \{n: \lambda(n, k) \rightarrow \lambda(n)\}.$$

Note that $m \in X_2$. We shall show that if c_n is replaced by zero for every $n \in X_2$, then L is unchanged.

For $j = 1$ and 2 , let

$$g_{jk}(x) = \sum_{n \in X_j} c_n e^{i[\lambda(n, k)x + \theta(n, k)]}.$$

Then $g_k = g_{1k} + g_{2k}$. Let

$$h(x) = \sum_{n \in X_1} c_n e^{i[\lambda(n)x + \theta(n)]}.$$

It is an elementary fact that when $y_k \rightarrow y$, K is a compact set, and $p_k(x) = e^{iy_k x} - e^{iyx}$, then $\|p_k\|_{A(K)} \rightarrow 0$. It follows that $\|g_{2k} - h\|_{A([0, \pi])} \rightarrow 0$, and in particular that

$$\|g_{2k} - h\|_{C(F_k)} \rightarrow 0.$$

We shall show that $h = 0$ on $[0, \pi]$ by showing that whenever $0 \leq a < b \leq \pi$, $\int_a^b h = 0$. Using Riemann sums, we obtain that

$$\int_a^b h = \lim_{k \rightarrow \infty} \frac{\pi}{N_k} \sum \{h(x) : x \in F_k \cap [a, b]\}.$$

Now

$$\begin{aligned} \|\bar{h} + g_{1k}\|_{C(F_k)} &= \|h - g_{2k} + g_k\|_{C(F_k)} \\ &\leq \|h - g_{2k}\|_{C(F_k)} + \|g_k\|_{C(F_k)} \rightarrow 0. \end{aligned}$$

Therefore

$$\int_a^b h = -\lim_{k \rightarrow \infty} \frac{\pi}{N_k} \sum \{g_{1k}(x) : x \in F_k \cap [a, b]\}.$$

Because $|\lambda(n, k)| \leq N_k$ for every pair n, k ,

$$\begin{aligned} \left| \sum \{e^{i\lambda(n, k)x} : x \in F_k \cap [a, b]\} \right| &\leq \frac{2}{|1 - e^{i\lambda(n, k)\pi/N_k}|} \\ &= \frac{1}{|\sin(\pi\lambda(n, k)/2N_k)|} \\ &\leq \frac{N_k}{|\lambda(n, k)|}. \end{aligned}$$

Therefore

$$\frac{\pi}{N_k} \left| \sum \{g_{1k}(x) : x \in F_k \cap [a, b]\} \right| \leq \pi \sum_{n \in X_1} |c_n| \min \{|\lambda(n, k)|^{-1}, 2(b - a)\},$$

which evidently tends to zero as $k \rightarrow \infty$. It follows that $\int_a^b h = 0$.

Therefore $h = 0$ on $[0, \pi]$, so that $\|g_{2k}\|_{A([0, \pi])} \rightarrow 0$ and in particular $\|g_{2k}\|_{C(F_k)} \rightarrow 0$. Thus L remains the same for the sequence $\{g_{1k}\}$, which was obtained from $\{g_k\}$ by changing c_n to 0 for every $n \in X_2$. The theorem is proved.

Corollary. Let $E = \{0\} \cup \bigcup_{k=1}^{\infty} E_k$, where

$$E_k = \{r_k + js_k : j = 0, 1, \dots, N_k\},$$

$N_k \rightarrow \infty$; the set $\{r_k, s_k\}_{k=1}^{\infty}$ is independent over the rationals; and $E_k \subset (1/(k+1), (1/k))$, so that E has just one accumulation point. Then $D_d(E) \cap C(E) = A(E)$, even though $A(E) \neq B(E)$.

Proof. It follows as in [11, Section 2] that $A(E) \neq B(E)$. It remains to show that if $f \in D_d(E) \cap C(E)$, then $f \in A(E)$. It suffices to show this in the case $f(0) = 0$. In this case, by Theorem IV, $\|f\|_{A(E_k)} \rightarrow 0$. Since the set $\{r_k, s_k\}_{k=1}^{\infty}$ is independent, $\|g\|_{B(E)} = \sup_k \|g\|_{A(E_k)}$ for $g \in C(E)$. For $n = 1, 2, \dots$ define g_n to equal f on $\bigcup_{k=1}^n E_k$ and 0 on $\bigcup_{k>n} E_k$. Evidently $\{g_n\}$ is a Cauchy sequence in $A(E)$, so that it has a limit in $A(E)$. This limit is f .

3. The algebras $B(E)$ and the dichotomy conjecture for countable sets. For background material used in this Section, see [5, Chapter VI], [17, Chapter 6], and [8, Section VIII.8]. For other results on the dichotomy problem, see [6], [9], and [10]. For some similar work in another setting, see [4].

Let $\mathcal{A}$ be an algebra of functions. Let Φ be a function defined on an interval $I \subseteq \mathbb{R}$. We say that Φ *operates in* $\mathcal{A}$ if for every $f \in \mathcal{A}$ whose range is contained in I , $\Phi \circ f$ is also an element of $\mathcal{A}$. We say that $\mathcal{A}$ is *analytic* (respectively, *entire*) if for every such Φ and I , Φ is the restriction to I of a function that is analytic on a neighborhood of I (respectively, of an entire function).

It is known that if E has positive measure, or if E is a set of multiplicity in the strict sense, or if E contains arbitrarily long arithmetic progressions, then $A(E)$ is analytic. If E is a Helson set, on the other hand, then every continuous Φ operates in $A(E)$ because $A(E) = C(E)$. The *dichotomy conjecture* states that for every E , either E is a Helson set or $A(E)$ is analytic.

For $u > 0$, let

$$(1) \quad N_E(u) = \sup \{ \|e^{iuf}\|_{A(E)} : f \in A(E), f \text{ is real-valued, and } \|f\|_{A(E)} \leq 1 \}.$$

It is known that $A(E)$ is analytic if and only if there exists an $a > 0$ such that $N_E(u) \geq e^{au}$ for all $u > 0$.

Note that for countable E , since the measures in $M(E)$ that have finite support are dense in $A(E)^*$,

$$(2) \quad \|g\|_{A(E)} = \sup \{ \|g\|_{A(F)} : F \text{ is a finite subset of } E \} \text{ for } g \in A(E).$$

It follows that

$$(3) \quad N_E(u) = \sup \{ N_F(u) : F \text{ is a finite subset of } E \}.$$

If Φ operates in $A(E)$, and $F \subseteq E$, then Φ operates in $A(F)$; and every countable non-Helson set E (indeed, every non-Helson E that satisfies (2)) contains a non-Helson set F that consists of a sequence with just one accumulation point. Therefore, in order to prove the dichotomy conjecture for all countable E , it would suffice to prove it for the case when E is a sequence with just one accumulation point. Our partial results in this case may shed some light on the dichotomy conjecture, and they raise other interesting questions.

Let E be a countable set whose only accumulation point is 0. Let X denote the maximal ideal space of $B(E)$. Evidently, E is a closed subset of X . We know of no example where $E = X$, except when E is a Helson set. For $x \in X$, let $\pi(x)$ be the point of E such that $f(\pi(x)) = \hat{f}(x)$ for $f \in A(E)$. For $t \in E$, $\pi^{-1}(t)$ is called the *fiber over* t . Evidently $\pi^{-1}(t) = \{t\}$ for $t \neq 0$; that is, $\pi^{-1}(0)$ is the only fiber that can be non-trivial. If $f, g \in B(E)$ and f agrees with g on $E \cap (-\epsilon, \epsilon)$ for some $\epsilon > 0$, then $\hat{f}(x) = \hat{g}(x)$ for all $x \in X - E$. It is easy to show also that $\pi^{-1}(0)$ is connected, by the Sylow idempotent theorem.

We shall need the following definition. For $u > 0$ and $0 < \eta \leq 1$, let

$$(4) \quad N_{E,\eta}(u) = \sup \{ \|e^{iuf}\|_{A(E)} : f \in A(E), f \text{ is real-valued, } \|f\|_{A(E)} \leq 1, \text{ and } \|f\|_{C(E)} \leq \eta \}.$$

Note that if in (1) and (4), $B(E)$ is substituted for $A(E)$, the values of $N_E(u)$ and $N_{E,\eta}(u)$ remain exactly the same. Note also that $N_{E,1} = N_E$ and that $N_{E,1/n}(u) \geq (N_E(u))^{1/n}$ for $n = 1, 2, \dots$. Thus if N_E grows exponentially, then so does $N_{E,\eta}$ for every $\eta > 0$; but it is not clear that necessarily

$$\inf_{\eta} \liminf_{u \rightarrow \infty} \frac{\log N_{E,\eta}(u)}{u} > 0.$$

Proposition. *If E contains arbitrarily long arithmetic progressions, then there is a constant $c > 0$ such that*

$$N_{E,\eta}(u) > ce^u \text{ for all } u > 0, \text{ all } \eta \in (0, 1].$$

Proof. Let $\eta > 0$. There exists a real-valued function $h \in A' = A(T)$ such that $\|h\|_{C(T)} \leq \eta$, $\|h\|_{A'} = 1$. Then

$$\|e^{i\eta h}\|_{A'} = 1 + \eta + O(\eta^2) \quad \text{as } \eta \rightarrow 0.$$

Fix $u > 0$. For each N we may pick integers $n_1 < n_2 < \dots < n_N$ increasing fast enough so that

$$\begin{aligned} \left\| \exp \left(\frac{i u}{N} \sum_{j=1}^N h(n_j x) \right) \right\|_{A'} &> \left(1 - \frac{1}{N} \right) \prod_{j=1}^N \left\| \exp \left(\frac{i u}{N} h(n_j x) \right) \right\|_{A'} \\ &> \left(1 - \frac{1}{N} \right) \left(1 + \frac{u}{N} + O\left(\frac{u^2}{N^2}\right) \right)^N \end{aligned}$$

(see [8, p. 241]). Let

$$f = \frac{1}{N} \sum_{j=1}^N h(n_j x).$$

Then

$$\|f\|_{C(T)} \leq \eta, \quad \|f\|_{A'} \leq 1.$$

For N sufficiently large, we have

$$\|e^{i u f}\|_{A'} > \frac{1}{2} e^u.$$

If

$$P = \left\{ \frac{2\pi j}{n} : -n < j \leq n \right\},$$

then for n sufficiently large,

$$\|e^{i u f}\|_{A'(P)} > \frac{1}{2} e^u.$$

There exists $\tau \in A'$ such that $\|\tau\|_{A'} = \|1 - \tau\|_{A'} \leq 3$, $\tau(t) + \tau(t + \pi) = 1$ for all t , and

$$\tau(t) = 0 \quad \text{for } t \notin \left[-\frac{2\pi}{3}, \frac{2\pi}{3} \right].$$

Then $e^{iuf} = \tau e^{iuf} + (1 - \tau)e^{iuf}$, so that, if

$$P_1 = P \cap \left[-\frac{2\pi}{3}, \frac{2\pi}{3} \right] \quad \text{and} \quad P_2 = P \cap \left[\frac{\pi}{3}, \frac{5\pi}{3} \right],$$

$$\|e^{iuf}\|_{A'(P)} \leq 3 \|e^{iuf}\|_{A'(P_1)} + 3 \|e^{iuf}\|_{A'(P_2)}.$$

Evidently we may choose Q , an arithmetic progression of length about $2n/3$, such that

$$\|e^{iuf}\|_{A'(Q)} \geq \frac{1}{6} \|e^{iuf}\|_{A'(P)} > \frac{1}{12} e^u,$$

and Q is contained in an interval of length $4\pi/3$. As explained in Section 1, there is a constant k such that

$$\|e^{iuf}\|_{A(Q)} > \frac{1}{12k} e^u.$$

We have proved that for every $u > 0$ there is an n such that if E contains an arithmetic progression of length at least n , then $N_{E,\eta}(u) > (1/12k)e^u$. The proposition follows.

Theorem V. *Let E be a countable non-Helson set whose only accumulation point is 0.*

a. The following three conditions are equivalent.

- (i) *there exists $a > 0$ such that $N_{E,\eta}(u) > e^{au}$ for all $u > 0$, and all $\eta \in (0, 1]$.*
- (ii) *$B(E)$ is entire.*
- (iii) *$B(E)$ is not self-adjoint.*

b. If E is a proper subset of the maximal ideal space X of $B(E)$, then the functions that operate in $A(E)$ are contained in a quasi-analytic class.

Proof. Part a. (i) $\Rightarrow$ (ii). It suffices to show that if Φ is a continuous function on R with period 2π , and $\Phi(0) = 0$, and $\Phi \circ f \in B(E)$ for every real-valued $f \in B(E)$, then $\hat{\Phi}(n) = O(e^{-b|n|})$ for every $b > 0$.

For every $d > 0$ there is a $\delta > 0$ such that

$$(5) \quad \sup \{ \|\Phi \circ f\|_{B(E)} : f \in B(E), \quad f \text{ is real-valued},$$

$$\|f\|_{B(E)} \leq d, \|f\|_{C(E)} \leq \delta \} < \infty.$$

Otherwise, the methods of [11, Section 3] would allow the construction of an $f \in B(E)$ such that $f(0) = 0$ and $\Phi \circ f \notin B(E)$.

Let k be a positive integer. Let δ correspond to $d = k + \pi$, and let K be the supremum (5). Then for $-\pi \leq t \leq \pi$, and each real-valued $f \in B(E)$ with

$$\|f\|_{B(E)} \leq k \quad \text{and} \quad \|f\|_{C(E)} \leq \delta,$$

we have that

$$\Phi \circ (f + t) \in B(E) \quad \text{and} \quad \|\Phi \circ (f + t)\|_{B(E)} \leq K.$$

Then for each integer n , the Bochner integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi \circ (f + t)e^{-int} dt = \hat{\Phi}(n)e^{inf}$$

is in $B(E)$, and $\|\hat{\Phi}(n)e^{inf}\|_{B(E)} \leq K$. Therefore

$$|\hat{\Phi}(n)| N_{E, \delta/k}(|n|k) \leq K \quad \text{for all } n.$$

Therefore

$$|\hat{\Phi}(n)| \leq Ke^{-ak|n|} \quad \text{for all } n.$$

Hence $\hat{\Phi}(n) = O(e^{-ak|n|})$ for every positive integer k .

(ii) $\Rightarrow$ (iii). Suppose $B(E)$ is self-adjoint. If $\Phi(x) = \sum_{n=-\infty}^{\infty} e^{-|n|} e^{inx}$ for real x , then Φ extends to an analytic function on an open horizontal strip. If f is a real-valued function in $B(E)$, then $\hat{f}(x)$ is real for all $x \in X$; and therefore $\Phi \circ f \in B(E)$. Thus Φ operates in $B(E)$ even though it is not entire; hence $B(E)$ is not entire.

(iii) $\Rightarrow$ (i). Suppose $B(E)$ is not self-adjoint. Then there is an f in $B(E)$ that is real-valued (on E) such that $\hat{f}(x_0) = ia$ for some $x_0 \in X$ and some nonzero real a . We may suppose that $f(0) = 0$ and $\|f\|_{B(E)} = 1/3$. For each $\epsilon > 0$, there is a function $V_\epsilon \in A$ such that

$$V_\epsilon(x) = 1 \quad \text{for } |x| \leq \epsilon;$$

$$V_\epsilon(x) = 0 \quad \text{for } |x| > 2\epsilon;$$

$$\|V_\epsilon\|_A \leq 3.$$

Then $(V_\epsilon f)^\wedge(x) = \hat{f}(x)$ for $x \in X - E$; $\|V_\epsilon f\|_{B(E)} \leq 3 \|f\|_{B(E)} = 1$; and $\|V_\epsilon f\|_{C(E)} \rightarrow 0$ as $\epsilon \rightarrow 0$.

Let $\eta > 0$. For ϵ sufficiently small, if we take $g = V_\epsilon f$, then $\|g\|_{B(E)} \leq 1$, $\|g\|_{C(E)} \leq \eta$, and g is real-valued (on E), but $\hat{g}(x_0) = ia$. For $u > 0$, $\|e^{iu\hat{g}}\|_{B(E)} \geq \sup_{x \in X} |e^{iu\hat{g}(x)}| \geq e^{au}$; therefore $N_{E, \eta}(u) \geq e^{au}$ for all $u > 0$.

Part b. For basic facts about quasi-analyticity, see [8, Chapter V].

If $B(E)$ is not self-adjoint, then Part b of the theorem follows from Part a. Therefore we shall assume that $B(E)$ is self-adjoint, so that if $f \in B(E)$ and f is real-valued on E , then $\hat{f}$ is real-valued on X .

Suppose we can prove that

$$(6) \quad \sum_{n=1}^{\infty} \frac{\log N_E(n)}{n^2} = \infty.$$

Consider a function Φ that operates in $A(E)$. By standard arguments (see [8, p. 240]), it suffices to deal with the case when Φ is a 2π -periodic continuous function on R such that $\sup \{\|\Phi \circ f\|_{A(E)} : f \in A(E), f \text{ real-valued}, \|f\|_{A(E)} \leq 1 + \pi\}$ is finite. Then it follows, as in the proof above that (i) $\Rightarrow$ (ii), that

$$|\hat{\Phi}(n)| = O((N_E(|n|))^{-1}).$$

Hence there exists a sequence $\{\omega_n\}$, depending on Φ , such that

$$\sum |\hat{\Phi}(n)| \omega_{|n|} < \infty \quad \text{and} \quad \sum \frac{\log \omega_n}{n^2} = \infty.$$

Therefore the functions that operate in $A(E)$ are in a quasi-analytic class.

It remains to prove (6). To do so it suffices to show that if Φ is any 2π -periodic continuous function on R such that

$$(7) \quad \sum_n |\hat{\Phi}(n)| N_E(|n|) < \infty,$$

then Φ cannot vanish near zero without vanishing identically. If $f \in B(E)$, $\|f\|_{B(E)} \leq 1$, and f is real-valued, then (7) implies that the series $\sum \hat{\Phi}(n)e^{in f}$ converges in norm, so that

$$(8) \quad (\Phi \circ f)^\wedge(x) = \Phi(\hat{f}(x)) \quad \text{for all } x \in X.$$

If X is strictly larger than E , there is a real-valued $f \in B(E)$ such that the range of $\hat{f}$ contains properly the range of $f = \hat{f}|_E$; and for any $\delta > 0$ we may find such an f with $\|f\|_{B(E)} \leq 1$, $\|f\|_{C(E)} \leq \delta$, and such that there is an $x_0 \in X$ where $\hat{f}(x_0) > \delta$. If Φ vanishes on $[-\delta, \delta]$, then $\Phi \circ f = 0$ and hence by (8) Φ vanishes on the range of $\hat{f}$. Since $\pi^{-1}(0)$ is connected, Φ in fact vanishes at least on $[-\epsilon, \hat{f}(x_0)]$. It follows that if Φ satisfies (7), if $\delta > 0$, and if Φ vanishes on $[-\delta, \delta]$, then Φ vanishes identically. Thus (6) is established, and the proof of Part b is complete.

Remark. It remains an open question whether $B(E)$ is non-self-adjoint whenever X is strictly larger than E . The corresponding statement in a broader Banach-algebra context is false, as the following example shows. Let Q be the class

$$\{g \in C(T) : \sum_n |\hat{g}(n)| \omega_n < \infty\}.$$

If $\omega_n = \omega_{-n}$, $\omega_{n+m} \leq \omega_n \omega_m$, then Q is a Banach algebra. If the growth of ω_n is subexponential, then the maximal ideal space is T . If

$$\sum \frac{\log \omega_n}{n^2} = \infty,$$

the class is quasi-analytic; thus if E is any infinite proper closed subset of T , then an element $g \in Q$ is determined by its values on E . Thus Q is a self-adjoint Banach algebra of functions on a set E , with maximal ideal space containing E properly. Furthermore, there are non-analytic functions that operate in Q ([1, Theorem III]).

4. A variation on a result of Varopoulos.

Theorem VI. *Let F be a perfect set of measure zero that disobeys synthesis.*

Then there is a countable set H whose accumulation points are all in F , such that $A(F \cup H)$ is not closed in $B(F \cup H)$.

Remarks. $A(E)$ is closed in $B(E)$ if and only if $b(E) > 0$, where

$$b(E) = \inf \left\{ \frac{\|f\|_{B(E)}}{\|f\|_{A(E)}} : f \in A(E), f \neq 0 \right\}.$$

This number is positive if and only if $M(E)$ is weak* sequentially dense in $N(E)$ (see [3]; by the “weak* topology” we mean $\sigma(N(E), A(E))$). The number is characterized by the condition that the ball $\{\mu \in M(E) : \|\mu\|_{PM} \leq 1\}$ is weak* dense in $\{S \in N(E) : \|S\|_{PM} \leq b(E)\}$ but not in any larger ball of $N(E)$. Note that when E is countable (and in some other cases, as in [11, Section 4]), $M(E)$ is norm dense in $N(E)$, so that $b(E) = 1$.

Lemma. Let F be as in the theorem. Let $\epsilon > 0$. Then there is a countable set H whose accumulation points are all in F , such that $b(F \cup H) < \epsilon$.

How the lemma implies the theorem. Since F disobey synthesis, F contains an infinite number of portions F_n , $n = 1, 2, \dots$, that are pairwise disjoint and disobey synthesis. For each n , let τ_n be a function in A that equals one on a neighborhood of F_n and vanishes on the rest of F . The lemma implies that there is a suitable set $H_n \subset \tau_n^{-1}(1)$ and a function $g_n \in A$ such that

$$\|g_n\|_{A(F_n \cup H_n)} = 1$$

but

$$\|g_n\|_{B(F_n \cup H_n)} < 1/(n\|\tau_n\|_A).$$

Let $H = \bigcup_{n=1}^{\infty} H_n$. Then for each n ,

$$\|g_n \tau_n\|_{A(F \cup H)} \geq 1,$$

whereas

$$\|g_n \tau_n\|_{B(F \cup H)} < 1/n.$$

Therefore $b(F \cup H) = 0$, and the theorem is proved.

Proof of the lemma. Since the rule $f(x) \rightarrow f(ax)$ ($a \neq 0$) gives an isometric isomorphism of A , we may assume without loss of generality that $F \subseteq [-1, 1]$ and that F contains no rational multiples of π .

Let $\epsilon' > 0$ be so small that $\epsilon'(2 + C)/(1 - \epsilon') < \epsilon$, where C is an absolute constant to be identified later.

Since F disobey synthesis, there is a function $f \in I(F)$ and a pseudomeasure S with support contained in F such that $\|S\|_{PM} \leq 1$ and $(f, S) > 1$. It follows that if E is a set of synthesis and $E \supseteq F$, then $S \in N(E) = A(E)^*$ and consequently $\|f\|_{A(E)} > 1$.

Let A_F denote the class of functions $g \in A$ such that the restriction of g to some neighborhood of F has finite range. Since F has measure zero, A_F is dense in A

(this fact is easy to prove; see [12, Section 1.3]). Therefore there are a g in A and a finite number of pairwise disjoint open intervals $I_1, I_2, \dots, I_J$ that cover F , such that g is constant on each I_i and

$$\|f - g\|_A < \epsilon'.$$

It follows that

$$\sup \left\{ |g(x)| : x \in \bigcup_{i=1}^J I_i \right\} < \epsilon'.$$

Let $P_1, \dots, P_J$ be distinct primes. For $q = 1, 2, \dots$ and $j = 1, \dots, J$ let $H(j, q)$ denote the set of numbers $2\pi k p_j^{-q}$ with $k \in \mathbb{Z}$ whose distance from $F \cap I_j$ is no greater than $2\pi p_j^{-q}$. Let Q be sufficiently large so that

$$H(j, q) \subseteq I_j \quad \text{for } q \geq Q, \quad \text{for } j = 1, \dots, J.$$

Let

$$H(j) = \bigcup_{q=Q}^{\infty} H(j, q); \quad H = \bigcup_{j=1}^J H(j).$$

Since the disjoint sets $(F \cap I_j) \cup H(j)$ are each of synthesis ([7, p. 124]), the set $F \cup H$ is of synthesis. Therefore $\|f\|_{A(F \cup H)} > 1$, so that

$$\|g\|_{A(F \cup H)} > 1 - \epsilon'.$$

We shall show that $\|g\|_{B(F \cup H)}$ is small by estimating $\int g d\mu$ for $\mu \in M(F \cup H)$. If ν is the restriction of μ to F , then $\mu - \nu$ is a discrete measure assigning mass only to points in H . Let $\eta > 0$. We may choose a finite number of the sets $H(j, q)$ such that if ρ is the restriction of $\mu - \nu$ to their union, and if $\alpha = \mu - \nu - \rho$, then $\|\alpha\|_M < \eta$. The group generated by the finite set that is the support of ρ is disjoint from the rest of $F \cup H$; therefore

$$\|\rho\|_{PM} \leq \|\mu\|_{PM}$$

(see the methods of [11, Section 3]), so that $\|\nu\|_{PM} \leq 2 \|\mu\|_{PM} + \eta$. Since ν is independent of the choice of η , evidently

$$\|\nu\|_{PM} \leq 2 \|\mu\|_{PM}.$$

Now $\rho = \sum_{i=1}^J \rho_i$, where $\rho_i \in M(H(j))$. We claim that there is an absolute constant C such that

$$(1) \quad \sum_{i=1}^J \|\rho_i\|_{PM} \leq C \|\rho\|_{PM}.$$

Evidently $F \cup H \subset [-3, 3]$, and, as explained in Section 1, there is a constant k such that

$$\|S\|_{PM} \leq k \|S\|_{PM}, \quad \text{for } S \in N(F \cup H).$$

Therefore to establish (1) it suffices to show that

$$(2) \quad \sum_{j=1}^J \|\rho_j\|_{PM'} \leq 12 \|\rho\|_{PM'} .$$

For each j , the support of ρ_j is contained in the set

$$\left\{ \frac{2\pi k}{m_j} : k \in Z \right\}$$

where m_j is some power of p_j . Therefore $\hat{\rho}_j(n)$ depends only on n modulo m_j . It follows by the Chinese Remainder Theorem that the range of $\hat{\rho}$ (restricted to Z) is the sum of the ranges of the $\hat{\rho}_j$'s; it is an easy exercise to show that this implies (2), and then (1) is established.

On the support of ρ_j , g is constant and $|g| < \epsilon'$, so that

$$\begin{aligned} \left| \int g \, d\rho_j \right| &= |g(I_j) \hat{\rho}_j(0)| \\ &\leq \epsilon' \|\rho_j\|_{PM} . \end{aligned}$$

Thus

$$\begin{aligned} \left| \int g \, d\mu \right| &\leq \left| \int g \, d\nu \right| + \left| \int g \, d\rho \right| + \left| \int g \, d\alpha \right| \\ &= \left| \int (g - f) \, d\nu \right| + \left| \sum_{j=1}^J \int g \, d\rho_j \right| + \left| \int g \, d\alpha \right| \\ &\leq \|g - f\|_{A(F)} \|\nu\|_{PM} + \epsilon' \sum_{j=1}^J \|\rho_j\|_{PM} + \epsilon' \|\alpha\|_M \\ &\leq \epsilon'(2 + C) \|\mu\|_{PM} + \epsilon'\eta \end{aligned}$$

for arbitrary $\eta > 0$. Therefore

$$\begin{aligned} b(F \cup H) &\leq \frac{\|g\|_{B(F \cup H)}}{\|g\|_{A(F \cup H)}} \\ &\leq \frac{\epsilon'(2 + C)}{1 - \epsilon'} \\ &< \epsilon . \end{aligned}$$

The lemma is proved, and Theorem VI follows.

5. Some Questions.

- (1) Is there an E such that $B(E) = D_d(E) \cap C(E) \neq A(E)$?
- (2) What properties characterize the sets E such that $A(E) = D_d(E) \cap C(E)$?
- (3) Does condition (i) of Theorem V hold whenever $A(E)$ is analytic?
- (4) Is there a countable non-Helson set E such that the maximal ideal space X of $B(E)$ is precisely E ?

- (5) Is there a countable non-Helson set E such that $X \not\cong E$ and $B(E)$ is self-adjoint?
- (6) If E is countable and $A(E)$ is analytic, must $B(E)$ be entire?
- (7) When Y is a separable Banach space and Γ is a linear submanifold of Y^* , let Γ_1 denote the set of weak* limits of sequences in Γ . For ordinal numbers $\xi > 1$, define Γ_ξ inductively by

$$\Gamma_\xi = \left(\bigcup_{\eta < \xi} \Gamma_\eta \right)_1.$$

There is always a countable ordinal ξ such that Γ_ξ is weak* closed. The first such ξ is called the *order of Γ in Y^** .

Question: Is there an E such that the order of $M(E)$ in $N(E)$ is greater than 2?

Addendum. Professor Sadahiro Saeki has kindly pointed out to us the following results. Let K and L be disjoint perfect sets whose union is a Kronecker set, so that by Varopoulos's well-known arguments, the algebras $A(K + L)$ and $V(K, L) = C(K) \hat{\otimes} C(L)$ are isomorphic. Theorem 4.3 of Saeki's paper, *The ranges of certain isometries of tensor products of Banach spaces*, J. Math. Soc. Japan 23 (1971), 27–39, states that every continuous function f on $K \times L$ that can be written

$$f(x, y) = \sum_{n=1}^{\infty} g_n(x)h_n(y)$$

where the g 's and h 's are bounded functions on K and L , respectively, such that $\sum \|g_n\|_\infty \|h_n\|_\infty < \infty$, is in fact an element of $V(K, L)$. It follows that for every portion E of $K + L$, $A(E) = D_d(E) \cap C(E)$. On the other hand, as Varopoulos has shown, $A(E) \not\cong B(E)$.

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