Invariant Subspaces of Abstract Multiplication Operators

HERMANN FLASCHKA

Communicated by P. R. HALMOS

§1. We want to describe a class of operators on a Banach space $\mathfrak{B}$ whose members behave, in a sense, like multiplication operators, and consequently leave invariant a proper closed subspace of $\mathfrak{B}$ —that is, they are *intransitive*. One of the sufficient conditions for an operator to be such an “abstract multiplication” is similar to an assumption made by Wermer [1] (and later by Ljubic and Macaev [2], Wolf [3]) who approached the invariant-subspace problem from a different (and rather more sophisticated) point of view. The approach indicated here is, perhaps, more intuitive, and we hope that the more general pattern which appears to emerge may be more than superficial.

Wermer considered operators whose deviation from being an isometry is limited. More precisely, he assumed that T and T^{-1} are both bounded, and that either

(A) $\|T^n\| = O(e^{|\alpha|n})$, $n = 0, \pm 1, \dots$, $0 < \alpha < 1$, and the spectrum of T contains at least two points, or

(B) $\|T^n\| = O(|n|^k)$, for some fixed $k < \infty$.

Such a T is *intransitive*. If $\alpha = 0$ in (A), or $k = 0$ in (B), then $\|T^n\| \leq K$ for all n ; the space may then be renormed to make T an isometry, and intransitivity follows from a theorem of Godement [4].

The crucial estimate is that in (A); it is a reformulation of the requirement that

$$(1) \quad \sum_{n=-\infty}^{\infty} \frac{\log \|T^n\|}{1 + n^2} < \infty$$

which in turn plays an essential role in Wermer’s delicate and highly analytic proof.

We take as model of an intransitive operator not an isometry, but a normal operator on a Hilbert space. Because of the spectral representation theorem, such an operator is basically a multiplication by an L^∞ -function, acting on an L^2 -space. By controlling the deviation of a more general operator from

normality, we are able to retain at least those properties of multiplication operators which have a bearing on intransitivity.

The hypotheses are formulated in the language of Banach algebras. Let T be a bounded linear operator on the (complex, separable) Banach space $\mathfrak{B}$, let $\mathfrak{A}$ be the Banach algebra of operators generated by T ; denote by $\mathfrak{M}$ the maximal ideal space of $\mathfrak{A}$, and by $A \rightarrow \tilde{A}$ the Gel'fand homomorphism of $\mathfrak{A}$ to $C(\mathfrak{M})$.

Assume that:

- (I) $\mathfrak{A}$ is semi-simple, i.e., $\tilde{A} \equiv 0$ if and only if $A = 0$;
- (II) $\mathfrak{A}$ is regular, i.e., if $F \subset \mathfrak{M}$ is closed and $M_0 \notin \mathfrak{M} - F$, then there is an $A \in \mathfrak{A}$ such that $\tilde{A}(M_0) \neq 0$ and $\tilde{A}(F) = \{0\}$; (this may be replaced by the less restrictive condition
- (II)' $\mathfrak{A}$ is contained in a regular commutative Banach Algebra $\mathfrak{A}'$;
- (III) for each $A \in \mathfrak{A}$, there is a sequence $\{B_n\} \subset \mathfrak{A}$ converging to A strongly, with the support of each $\tilde{B}_n$ contained in the interior of the support of $\tilde{A}$.

Theorem. *If T satisfies the above hypotheses, then T is intransitive.*

§2. These assumptions are quite severe, and we hope to motivate them in a moment. First we give an example of a non-normal operator satisfying (I)–(III) (this illustration actually suggested the present development).

Let H be the Hilbert space obtained by completing $C_0^\infty([0, 1])$ in the norm

$$\|u\|^2 = \int_0^1 (|u(x)|^2 + |u'(x)|^2) dx.$$

Thus, H is the first Sobolev space on $[0, 1]$. Now let $\varphi \in C_0^\infty([0, 1])$; the map $M_\varphi: u \rightarrow \varphi \cdot u$ is a bounded operator which, despite its analytic simplicity, is not normal. This can be seen from the following explicit formula for the adjoint M_φ^* :

$$M_\varphi^* = \left(1 - \frac{d^2}{dx^2}\right)^{-1} M_{\bar{\varphi}} \left(1 - \frac{d^2}{dx^2}\right).$$

On the right, the last factor maps H onto its distribution-dual space, which is a Sobolev space of negative index (usually written H_{-1}). In H_{-1} , the adjoint of M_φ is simply $M_{\bar{\varphi}}$, and finally, the first factor maps H_{-1} back onto H (the relevant properties of Sobolev spaces are described, for instance, by P. D. Lax, Comm. Pure Appl. Math. **VIII**(1955), 615–633). Otherwise, M_φ is well behaved. Its spectrum is the range of φ ; its norm satisfies $\|M_\varphi\| \leq \sup_{[0,1]} (|\varphi(x)| + |\varphi'(x)|)$. If we now take φ_0 real-valued and $T = M_{\varphi_0}$, then the assumptions of the theorem are satisfied: every operator in $\mathfrak{A}$ is of the form M_φ , and such an algebra must be semi-simple (note, however, that the generating function φ of M_φ need no longer be C^∞); regularity of $\mathfrak{A}$ is a consequence of Shilov's criterion described below, and condition (III) can be verified by expressing suitable B_n 's as limits of polynomials in φ_0 . Finally, since the multiplications in $\mathfrak{A}$ will be defined by functions subject at least to mean smoothness properties, $\mathfrak{A}$ will not contain idempotents, and renorming of H cannot convert T into a normal operator.

Of course, any of the numerous other Sobolev spaces, Hilbert or Banach, would do just as well; more generally, multiplications by suitable bounded functions acting on Hilbert spaces obtained by equipping the domains of certain unbounded operators in $L^2(X, \mu)$ with the graph norm may be expected to illustrate conditions (I)–(III). I suspect that the converse is also true: operators (on Hilbert space) satisfying conditions similar to (I)–(III) can be represented as multiplications of this form. Such a realization (if possible) might be thought of as the analogue for T of the finite-dimensional concept of *diagonalizability*. Assumption (I) then becomes reasonable; it is necessary and sufficient in order that a finite matrix be diagonalizable.

Property (II)—regularity of $\mathfrak{A}$ —appears to be common to most multiplication algebras of non-quasianalytic functions. There is a sufficient condition for regularity, due to Shilov [5], which establishes the promised connection with Wermer's theorem:

If $\text{sp } T$ is real, then $\mathfrak{A}$ is regular if

$$(2) \quad \int_0^\infty \frac{\log \|e^{isT}\|}{1+s^2} ds < \infty.$$

For self-adjoint T , $\log \|e^{isT}\| \equiv 0$, so (2) is trivially true. In the example above with φ_0 real, $e^{isM_{\varphi_0}} = M_{\exp(is\varphi_0)}$, and since

$$\|M_{\exp(is\varphi_0)}\| \leq \sup_{[0,1]} (1 + s |\varphi'_0(x)|) \leq 1 + s \|M_{\varphi_0}\|,$$

(2) again holds. In this sense, Shilov's criterion restricts the degree of "non-self-adjointness" of T . Should φ_0 be complex-valued (so that (2) would not, in general, be true), we note that $\mathfrak{A}$ is contained in the algebra $\mathfrak{A}'$ generated by $T_1 \equiv M_{\text{Re } \varphi_0}$ and $T_2 \equiv M_{\text{Im } \varphi_0}$; since (2) holds for T_1 and T_2 , $\mathfrak{A}'$ is regular ([5]), and assumption (II)' is verified.

An attempt to estimate $\log \|e^{isT}\|$ via the series expansion of e^{isT} will quickly lead to the imposition of Wermer's growth condition on $\|T^n\|$; however, (1) and (2) are not quite equivalent since (1) requires $\text{sp } T$ to be located on the unit circle, while (2) is still satisfied by the operator M_{φ_0} of our example.

Condition (III) is the least satisfactory, since its consequences for T are unexplored. It is needed in the proof of our theorem to guarantee that certain closed ideals in $\mathfrak{A}$ are intersections of maximal ideals; examples from harmonic analysis indicate that this latter property cannot be expected to follow from (I) and (II) alone. It is conceivable, however, that (III), being an assertion about an algebra of operators, is derivable from the other two assumptions.

§3. Proof of the theorem. In what follows, the letters u, v, w will denote elements of the Banach space $\mathfrak{B}$, and A, B will denote operators from $\mathfrak{A}$. We assume that (II) holds; in case (II)', we would throughout replace $\mathfrak{A}$ by $\mathfrak{A}'$ (we exclude the trivial case $T = \lambda 1$).

Definition. Let $I_u = \{A \in \mathfrak{A}; Au = 0\}$. (I_u is a closed ideal.) The support of u is the set

$$[u] = \{M \in \mathfrak{M}; I_u \subset M\}.$$

The support of $A \in \mathfrak{A}$ is the set

$$[A] = \text{closure } \{M \in \mathfrak{M}; \tilde{A}(M) \neq 0\}.$$

$[A]^0$ is the interior of $[A]$.

(The motivation for these definitions is simple: if $u \in \mathfrak{B}$ were a function on $\mathfrak{M}$, and $\tilde{A}$ were a multiplication operator, we would want $u(M) \neq 0$ whenever $\tilde{A}(M) \neq 0 \Rightarrow Au \neq 0$).

The following propositions express those properties of $[u]$ and $[A]$ which are to be expected of multiplication operators.

$$(\alpha) \quad [u] = \emptyset \text{ iff } u = 0.$$

Proof. If $u = 0$, then $I_u = \mathfrak{A}$, so I_u is contained in no maximal ideal, and $[u] = \emptyset$. Conversely, if I_u is contained in no maximal ideal, then—being itself an ideal—it must coincide with $\mathfrak{A}$. In particular, $u = 1u = 0$.

$$(\beta) \quad [Au] \subseteq [u].$$

Proof. Let $M \in [Au]$. If $Bu = 0$, then $0 = ABu = B(Au)$, whence $B \in M$. So $I_u \subset M$, and $M \in [u]$.

$$(\gamma) \quad [Au] \subseteq [A].$$

Proof. Suppose to the contrary that there is an $M_0 \in [Au]$ such that $\tilde{A}(M) = 0$ on a neighborhood $\mathcal{O}$ of M_0 . Since $\mathfrak{A}$ is regular, there is a $B \in \mathfrak{A}$ such that $\tilde{B}(M_0) \neq 0$, but $\tilde{B} = 0$ off $\mathcal{O}$. Then $\tilde{A}\tilde{B} = 0$, and by semi-simplicity, $AB = 0$. Hence $BAu = 0$, and $B \in I_{Au} \subset M_0$. But this is impossible, since $\tilde{B}(M_0) \neq 0$.

$$(\delta) \quad Au = 0 \text{ if and only if } [u] \subseteq \{M; \tilde{A}(M) = 0\}.$$

(Remark: The “if” part of (δ) is equivalent to the assertion: $I_u = \bigcap_{M \in [u]} M$, meaning that I_u is the intersection of the maximal ideals containing it. See §2 above.)

Proof. By (III), there is a sequence $\{B_n\} \subset \mathfrak{A}$, such that $B_n u \rightarrow Au$ and $[B_n] \subset [A]^0$. For each n , $B_n u = 0$, since on the one hand $[B_n u] \subseteq [u]$ (by (β)), and on the other $[B_n u] \subseteq [B_n]$ (by (γ)); but $[B_n] \cap [u] = \emptyset$ by hypothesis, so $[B_n u] = \emptyset$, and by (α) , $B_n u = 0$. Hence $Au = 0$.

Conversely, let $M \in [u]$; since $Au = 0$, there is a $B \in \mathfrak{A}$ such that $BAu = 0$ and $B \notin M$ (for otherwise $\mathfrak{A} \subset M$). From $BAu = 0$ follows $BA \in M$. So $0 = \tilde{B}\tilde{A}(M) = \tilde{B}(M)\tilde{A}(M)$, and $\tilde{B}(M) \neq 0$ requires that $\tilde{A}(M) = 0$. Q.E.D.

$$(\epsilon) \quad \text{If } [v] \subseteq [u], \text{ and } Au = 0, \text{ then } Av = 0.$$

Proof. By (δ) , $[u] \subseteq \{M; \tilde{A}(M) = 0\}$. Since $[v] \subseteq [u]$, the direct part of (δ) gives $Av = 0$.

The theorem can now be stated in somewhat greater generality. Let $u \in \mathfrak{B}$. The set $\mathfrak{B}_u = \{v \in \mathfrak{B}; [v] \subseteq [u]\}$ is a closed linear subspace of $\mathfrak{B}$ and is invariant under $\mathfrak{A}$.

Proof. Invariance is immediate from (β) . To see that $\mathfrak{B}_u$ is a subspace, note first that obviously $[v] = [\alpha v]$ for any complex α . Next, let $v, w \in \mathfrak{B}_u$, and (if $v + w \neq 0$) let $M \in [v + w]$. For any $A \in I_u$ we have, by (ϵ) , $Av = Aw = 0$. Hence $A(v + w) = 0$, and so $A \in M$. Therefore $M \in [u]$, and $[v + w] \subseteq [u]$, which by definition means $v + w \in \mathfrak{B}_u$. To show that $\mathfrak{B}_u$ is closed, let $w_n \in \mathfrak{B}_u$, $w_n \rightarrow v$. Let $M \in [v]$ (again excluding the trivial case $v = 0$), and let $A \in I_u$. By (ϵ) , $Aw_n = 0$, and so $Av = 0$. As above, we conclude that $[v] \subseteq [u]$, so that $v \in \mathfrak{B}_u$. Q.E.D.

The only remaining question is whether at least one of these subspaces $\mathfrak{B}_u$ is proper. There are two possibilities. If there are distinct points $M_1, M_2 \in \mathfrak{M}$, contained in $[u_1], [u_2]$, respectively ($u_1 = u_2$ is allowed), then there is an $A \in \mathfrak{A}$ such that $\tilde{A}(M_1) \neq 0$, while $\tilde{A} \equiv 0$ on a neighborhood of M_2 . Put $u = Au_1$; then $u \neq 0$, while $u_2 \notin \mathfrak{B}_u$, so that $\mathfrak{B}_u$ is proper. The other possibility is that there is a single $M_0 \in \mathfrak{M}$ such that $[u] = \{M_0\}$ for all $u \in \mathfrak{B} - \{0\}$. This actually cannot happen: because $\mathfrak{A}$ is semi-simple and $T \neq \lambda 1$, $\mathfrak{M}$ contains at least two points. There is then an $A \in \mathfrak{A}$ such that $\tilde{A} \neq 0$ but $\tilde{A} \equiv 0$ on a neighborhood of M_0 . So we would have $Au = 0$ for all $u \in \mathfrak{B}$, and yet $A \neq 0$. This contradiction proves the assertion, and with it, the theorem.

Remarks. The results obtained in [2] and [3] are more precise, and the hypotheses more pleasant in that they are expressed in terms of the operator T and not the algebra $\mathfrak{A}$. This paper differs from the above in emphasis, which is directed here towards a spectral *representation* rather than a spectral *integral*. (According to [2], p. 110, M. G. Krein has also noticed the relevance of Shilov's criterion for the study of invariant subspaces. There is, however, no elaboration of this idea.) There is also a difference, to an extent, in hypothesis: the spectrum of T is not required to be real or to lie on the unit circle. This is one advantage of condition (II)' over (II), for the latter is false even for a non-selfadjoint normal operator. In view of the need for (II)', it might be useful to abstract the crucial property in the example of Sec. 2: when can an operator T be written as $T_1 + iT_2$, with T_1, T_2 commuting, having real spectrum, and satisfying condition (2)?

Acknowledgement. I want to thank Professor J. J. Schäffer for some comments which improved the exposition, and the referee for calling to my attention references [2] and [3].

REFERENCES

1. J. WERMER, *The existence of invariant subspaces*, Duke Math. J. 19(1952), 615–622.
2. J. I. LJUBIC & V. I. MACAEV, *Operators with separable spectrum*, Amer. Math. Soc. Transl., Ser. 2, 47(1965), 89–129.

3. F. Wolf, *Operators in Banach space which admit a generalized spectral decomposition*, Indag. Math. **19**(1957), 302–311.
4. R. GODEMENT, *Theoremes tauberians et theorie spectrale*, Ann. Sci. Ecole Norm. Sup. (3) **64** (1947), 119–138.
5. G. SHILOV, *On regular normed rings*, Trudy Matem. In-ta. im. V. A. Steklova **21**(1947), 1–118.

The work was supported by NSF Grant GU-2056.

Carnegie-Mellon University
Pittsburgh, Pennsylvania

Date communicated: JUNE 22, 1971