If the Essential Range of $f \in C + H^\infty$ is Countable, then f is Constant

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For all notations and definitions, see Gamelin's book [1]. In his thesis [3] Klein asks whether a non-constant function of the form

$$f = g + h \quad f \in C(\mathbf{T}), \quad h \in H^\infty(\mathbf{T})$$

can have a countable essential range on the circle $\mathbf{T}$. We prove:

Theorem. *Let $f = g + h$, where $g \in C(\mathbf{T})$ and $h \in H^\infty(\mathbf{T})$. If the essential range of f is countable, then f is (essentially) constant.*

Matthew Lee has given an operator-theoretic proof of this result. Our proof uses only facts about uniform algebras.

Proof. Suppose $f \in C + H^\infty$ has a countable essential range. It is well-known (and easy to see) that the essential range of f equals the spectrum σ of f in L^∞ . Since σ is countable, σ is polynomially convex [1, III 1.3], and σ "is" the maximal ideal space of the uniform subalgebra $\mathcal{A}$ of L^∞ which f generates. Since [4, p. 191] $C + H^\infty$ is an algebra, $\mathcal{A} \subseteq C + H^\infty$. Obviously, the spectrum of f in $\mathcal{A}$ contains the spectrum of f in $C + H^\infty$, which contains the spectrum of f in L^∞ . Hence σ is the spectrum of f in $C + H^\infty$.

If σ is countable, then there exists an isolated point $z_0 \in \sigma$. Let F be any function which is one at z_0 , and zero on $\sigma \setminus \{z_0\}$, and analytic in a neighborhood of σ . By Gelfand's theorem [1, I5.1], $F \bullet f \in \mathcal{A}$. But $(F \bullet f)^2 = F \bullet f$. From an exercise (p. 207) and a corollary (p. 188) in Hoffman's book [2] and Sarason's remark [4, p. 191] that $C + H^\infty$ equals the uniform algebra generated (in L^∞) by $\bar{z}$ and H^∞ , we conclude that the maximal ideal space of $C + H^\infty$ is connected. Therefore $F \bullet f$ is constant, so $\sigma = \{z_0\}$; that is, f is constant. Q.E.D.

REFERENCES

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