Differential Operators of Infinite Order on a Lie Group, II

ROE GOODMAN

Communicated by B. KOSTANT

Introduction. Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and $\mathfrak{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. In this paper, which is a continuation of [6], we construct certain complete topological algebras, each containing $\mathfrak{U}(\mathfrak{g})$ as a dense subalgebra, and study the representations of these algebras obtained from Banach-space representations of G_0 (G_0 being a simply-connected real Lie group whose Lie algebra $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$). Automorphisms of $\mathfrak{g}$ extend continuously to automorphisms of these algebras, and we obtain results on the structure of these algebras as $\text{Aut}(\mathfrak{g})$ -modules, generalizing the Poincaré–Birkhoff–Witt Theorem. We illustrate some applications of these algebras by constructing an operational calculus for elements of $\mathfrak{U}(\mathfrak{g})$ and obtaining existence and uniqueness theorems for an abstract Cauchy problem.

In more detail, the contents of the paper are as follows: In §1 we construct a one-parameter family of Fréchet algebras $\mathfrak{A}_\lambda(\mathfrak{g})$, $0 < \lambda < \infty$, each containing $\mathfrak{U}(\mathfrak{g})$ as a dense subalgebra. The assignment $\mathfrak{g} \rightarrow \mathfrak{A}_\lambda(\mathfrak{g})$ is functorial, and in §2 we show that when $\lambda \geq 1$, the space $\mathfrak{A}_\lambda(\mathfrak{g})$ is isomorphic as an $\text{Aut}(\mathfrak{g})$ -module to the space of entire functions on $\mathfrak{g}'$ of exponential order $1/\lambda$ and minimal type. In particular, the algebra $\mathfrak{A}_1(\mathfrak{g})$ coincides with the “associative hyper-envelope” $\mathfrak{F}(\mathfrak{g})$ constructed somewhat differently by P. K. Raševskii in [17] (cf. also [12]).

If π is a strongly continuous representation of the group G_0 (G_0 as above) on a Banach space $\mathcal{H}(\pi)$, and if $S_\lambda(\pi)$ is the subspace of $\mathcal{H}(\pi)$ defined in [6], then π defines a representation of the algebra $\mathfrak{A}_\lambda(\mathfrak{g})$ on $S_\lambda(\pi)$, which is treated in §3. When π is the regular representation, either on $L_p(G_0)$, $1 \leq p < \infty$, or on $C_0(G_0)$, and when $\lambda > 1$, then we obtain in this way a faithful representation of $\mathfrak{A}_\lambda(\mathfrak{g})$ as “differential operators of infinite order” acting on $S_\lambda(\pi)$. In the case $\lambda = 1$, the algebra $\mathfrak{A}_1(\mathfrak{g})$ can be faithfully represented acting on the space of germs of analytic functions at the identity of G_0 [17]. (We do not know if the representation of $\mathfrak{A}_1(\mathfrak{g})$ on the space of analytic vectors for the regular representation is faithful, in general.)

When $\mathfrak{g}$ is nilpotent, there exists a number $\lambda_0 < 1$ such that the space $S_{\lambda_0}(\pi)$ is dense in $\mathcal{H}(\pi)$ for any unitary representation π of G_0 . This is proved in §4, and is used in §5 to obtain a generalized Poincaré–Birkhoff–Witt Theorem for an algebra $\mathcal{E}(\mathfrak{g})$, which is a projective limit, as $\lambda \nearrow 1$, of the algebras $\mathcal{A}_\lambda(\mathfrak{g})$. The left regular representation L defines a faithful representation of $\mathcal{E}(\mathfrak{g})$ as “differential operators of infinite order” on the space $S_{\lambda_0}(L)$.

Let G be a complex simply-connected nilpotent group with Lie algebra $\mathfrak{g}$. In §6 we show that the convolution algebra $\mathcal{A}(G)$ of analytic functionals on G , which has been studied by G. L. Litvinov ([12], [13]), is naturally embedded in the algebra $\mathcal{E}(\mathfrak{g})$ as a proper subalgebra. We also prove that $\mathcal{E}(\mathfrak{g})$ is semi-simple relative to irreducible unitary representations of G_0 (G_0 being a simply-connected real form of G).

We conclude in §7 with the previously mentioned applications.

§1. Algebras $\mathcal{A}_\lambda(\mathfrak{g})$. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbf{C}$, and let $\mathfrak{J}(\mathfrak{g})$ be the tensor algebra over $\mathfrak{g}$. Thus

$$\mathfrak{J}(\mathfrak{g}) = \sum_{n=0}^{\infty} \bigoplus \mathfrak{J}^n(\mathfrak{g}),$$

where $\mathfrak{J}^0(\mathfrak{g}) = \mathbf{C}$, $\mathfrak{J}^n(\mathfrak{g}) = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (n times). Fix a basis $X_1, \dots, X_d$ for $\mathfrak{g}$. Then the monomials $X_\alpha = X_{j_1} \cdots X_{j_n}$ form a basis for $\mathfrak{J}^n(\mathfrak{g})$, as α ranges over all multi-indices $\{j_1, \dots, j_n\}$, where $1 \leq j_k \leq d$ (Set $|\alpha| = n$; if $n = 0$, let $X_\phi = 1$). We shall study certain topological completions of $\mathfrak{J}(\mathfrak{g})$; for this purpose it is convenient to introduce the complete tensor algebra

$$\mathfrak{J}_\infty(\mathfrak{g}) = \prod_{n=0}^{\infty} \mathfrak{J}^n(\mathfrak{g})$$

(the algebra of non-commutative formal power series in elements of $\mathfrak{g}$). We have $\mathfrak{J}(\mathfrak{g})$ naturally embedded in $\mathfrak{J}_\infty(\mathfrak{g})$, and we shall write elements of $\mathfrak{J}_\infty(\mathfrak{g})$ as $T = \sum_\alpha c_\alpha X_\alpha$ (where $0 \leq |\alpha| < \infty$ and $\sum_{|\alpha|=n} c_\alpha X_\alpha$ is the component of T in $\mathfrak{J}^n(\mathfrak{g})$).

Definition. If $0 < \lambda < \infty$, let $\mathfrak{J}_\lambda(\mathfrak{g})$ be the linear subspace of $\mathfrak{J}_\infty(\mathfrak{g})$ consisting of all elements $T = \sum c_\alpha X_\alpha$ such that

$$(1.1) \quad \|T\|_{\lambda, r} \equiv \sum_\alpha |c_\alpha| r^{|\alpha|} |\alpha|^{\lambda|\alpha|} < \infty$$

for all $r > 0$, and give $\mathfrak{J}_\lambda(\mathfrak{g})$ the locally convex topology determined by the family of norms $\{\|\cdot\|_{\lambda, r}\}_{r>0}$.

Lemma 1.1. $\mathfrak{J}_\lambda(\mathfrak{g})$ is independent of the choice of basis for $\mathfrak{g}$ and is a Fréchet algebra (complete, metrizable, locally convex topological algebra) containing $\mathfrak{J}(\mathfrak{g})$ as a dense subalgebra. If $\lambda > \mu > 0$, then $\mathfrak{J}_\lambda(\mathfrak{g}) \subset \mathfrak{J}_\mu(\mathfrak{g})$, with continuous inclusion.

(The proof of Lemma 1.1 is completely elementary and is omitted.)

If $\mathfrak{g}_i$, $i = 1, 2$, are finite-dimensional complex Lie algebras, and $\varphi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a linear map, then φ induces an algebra homomorphism $\tilde{\varphi}: \mathfrak{J}_\infty(\mathfrak{g}_1) \rightarrow \mathfrak{J}_\infty(\mathfrak{g}_2)$, such that $\tilde{\varphi}(1) = 1$ and $\tilde{\varphi}(X) = \varphi(X)$ for $X \in \mathfrak{g}_1$. Elementary estimates show that, relative to a choice of basis for $\mathfrak{g}_i$, one has

$$\|\tilde{\varphi}(T)\|_{\lambda, r} \leq \|T\|_{\lambda, Mr}$$

for some $M > 0$ and all $T \in \mathfrak{J}_\infty(\mathfrak{g}_1)$. Hence $\tilde{\varphi}: \mathfrak{J}_\lambda(\mathfrak{g}_1) \rightarrow \mathfrak{J}_\lambda(\mathfrak{g}_2)$ continuously.

Let $\mathfrak{R}(\mathfrak{g})$ be the two-sided ideal in $\mathfrak{J}(\mathfrak{g})$ generated by the elements $XY - YX - [X, Y]$, with $X, Y \in \mathfrak{g}$. Let $\mathfrak{R}_\lambda(\mathfrak{g})$ be the closure of $\mathfrak{R}(\mathfrak{g})$ in $\mathfrak{J}_\lambda(\mathfrak{g})$. Then $\mathfrak{R}_\lambda(\mathfrak{g})$ is a two-sided ideal in $\mathfrak{J}_\lambda(\mathfrak{g})$, and we define

$$\mathfrak{Q}_\lambda(\mathfrak{g}) = \mathfrak{J}_\lambda(\mathfrak{g}) / \mathfrak{R}_\lambda(\mathfrak{g}).$$

Then $\mathfrak{Q}_\lambda(\mathfrak{g})$ is again a Fréchet algebra with respect to the quotient topology. If $\varphi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a Lie algebra homomorphism, then $\tilde{\varphi}(\mathfrak{R}(\mathfrak{g}_1)) \subseteq \mathfrak{R}(\mathfrak{g}_2)$. Hence by continuity $\tilde{\varphi}(\mathfrak{R}_\lambda(\mathfrak{g}_1)) \subseteq \mathfrak{R}_\lambda(\mathfrak{g}_2)$, so that $\tilde{\varphi}$ induces an algebra homomorphism from $\mathfrak{Q}_\lambda(\mathfrak{g}_1)$ to $\mathfrak{Q}_\lambda(\mathfrak{g}_2)$.

Since the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is canonically isomorphic to $\mathfrak{J}(\mathfrak{g}) / \mathfrak{R}(\mathfrak{g})$, it follows that the inclusion maps $\mathfrak{J}(\mathfrak{g}) \subseteq \mathfrak{J}_\lambda(\mathfrak{g})$, $\mathfrak{R}(\mathfrak{g}) \subseteq \mathfrak{R}_\lambda(\mathfrak{g})$ give rise to a homomorphism

$$J_\lambda: \mathfrak{U}(\mathfrak{g}) \rightarrow \mathfrak{Q}_\lambda(\mathfrak{g}).$$

Proposition 1.1. J_λ is injective, with dense range.

Proof. Since $\mathfrak{J}(\mathfrak{g})$ is dense in $\mathfrak{J}_\lambda(\mathfrak{g})$ (Lemma 1.1), we have $\mathfrak{J}(\mathfrak{g}) / \mathfrak{R}_\lambda(\mathfrak{g}) = J_\lambda(\mathfrak{U}(\mathfrak{g}))$ dense in $\mathfrak{Q}_\lambda(\mathfrak{g})$.

To show that J_λ is injective, we use the following theorem of Harish-Chandra [9]: Given $T \in \mathfrak{U}(\mathfrak{g})$, $T \neq 0$, there exists a representation π of $\mathfrak{U}(\mathfrak{g})$ on a finite-dimensional complex vector space V , such that $\pi(T) \neq 0$. Let $\|\cdot\|$ be any norm on the space $\text{End}(V)$ of linear transformations of V , satisfying $\|AB\| \leq \|A\| \|B\|$. Since $\pi: \mathfrak{g} \rightarrow \text{End}(V)$ is a Lie algebra homomorphism, it extends uniquely to an associative algebra homomorphism $\varphi: \mathfrak{J}(\mathfrak{g}) \rightarrow \text{End}(V)$, with $\varphi(\mathfrak{R}(\mathfrak{g})) = 0$. For any multi-index α we have the estimate

$$(1.2) \quad \|\varphi(X_\alpha)\| \leq M^{|\alpha|},$$

M a constant independent of α . It follows from (1.2) and the definition of $\mathfrak{J}_\lambda(\mathfrak{g})$ that φ extends continuously to an algebra homomorphism $\varphi_\lambda: \mathfrak{J}_\lambda(\mathfrak{g}) \rightarrow \text{End}(V)$, such that $\varphi_\lambda(\mathfrak{R}_\lambda(\mathfrak{g})) = 0$. If

$$\Phi_\lambda: \mathfrak{Q}_\lambda(\mathfrak{g}) \rightarrow \text{End}(V)$$

is the induced map on the quotient, then π can be factored as $\pi = \Phi_\lambda \circ J_\lambda$. Since $\pi(T) \neq 0$, we must have $J_\lambda(T) \neq 0$. Q.E.D.

We shall henceforth identify $\mathfrak{U}(\mathfrak{g})$ with its image in $\mathfrak{Q}_\lambda(\mathfrak{g})$, so that each algebra $\mathfrak{Q}_\lambda(\mathfrak{g})$ may be viewed as a completion of $\mathfrak{U}(\mathfrak{g})$.

§2. **Generalized P.-B.-W. theorem for $\mathfrak{G}_\lambda(\mathfrak{g})$, $\lambda \geq 1$.** Let $\mathfrak{g}$, $\mathfrak{I}(\mathfrak{g})$, and $\mathfrak{R}(\mathfrak{g})$ be as in §1. Define the symmetrization operator σ on $\mathfrak{I}(\mathfrak{g})$ as usual by

$$\sigma(Y_1 \cdots Y_n) = \frac{1}{n!} \sum_{\tau \in S_n} Y_{\tau(1)} \cdots Y_{\tau(n)},$$

where S_n is the group of permutation of $\{1, \dots, n\}$, and $Y_i \in \mathfrak{g}$. Let $\mathfrak{S}(\mathfrak{g})$ be the range of σ . One version of the Poincaré–Birkhoff–Witt Theorem [1] is the assertion that

$$(2.1) \quad \mathfrak{I}(\mathfrak{g}) = \mathfrak{S}(\mathfrak{g}) + \mathfrak{R}(\mathfrak{g}) \quad (\text{direct sum}).$$

In this section we prove the following generalization of (2.1):

Theorem 2.1. *Assume $\lambda \geq 1$. Let $\mathfrak{S}_\lambda(\mathfrak{g})$ and $\mathfrak{R}_\lambda(\mathfrak{g})$ be the closures in $\mathfrak{I}_\lambda(\mathfrak{g})$ of $\mathfrak{S}(\mathfrak{g})$ and $\mathfrak{R}(\mathfrak{g})$, respectively. Then*

$$(2.2) \quad \mathfrak{I}_\lambda(\mathfrak{g}) = \mathfrak{S}_\lambda(\mathfrak{g}) + \mathfrak{R}_\lambda(\mathfrak{g}) \quad (\text{direct sum}).$$

Proof of Theorem 2.1. We shall use the following notation: Greek letters will denote ordered multi-indices $\Gamma = (j_1, \dots, j_n)$, with $1 \leq j_k \leq d = \dim \mathfrak{g}$, and $X_\Gamma = X_{i_1} \cdots X_{i_n}$ (product in $\mathfrak{I}(\mathfrak{g})$), where $\{X_i\}$ is a fixed basis for $\mathfrak{g}$. Latin letters will denote ordered multi-exponents $K = (k_1, \dots, k_d)$, with $k_i \in \mathbf{N}$, and $X^K = X_1^{k_1} \cdots X_d^{k_d}$. By the Poincaré–Birkhoff–Witt theorem there exist unique scalars c_K^Γ such that

$$(2.3) \quad (|\Gamma|!)^{-1} X_\Gamma = \sum_K (K!)^{-1} c_K^\Gamma \sigma(X^K) \mod \mathfrak{R}(\mathfrak{g}).$$

Furthermore, $c_K^\Gamma = 0$ when $|K| > |\Gamma|$. Theorem 2.1 is a consequence of the following estimate for the $\{c_K^\Gamma\}$:

$$(2.4) \quad |c_K^\Gamma| \leq M^{|\Gamma|}$$

for some constant $M > 0$.

Indeed, assume that estimate (2.4) holds, and let $T = \sum (|\Gamma|!)^{-1} a_\Gamma X_\Gamma$ belong to $\mathfrak{I}_\lambda(\mathfrak{g})$ for some $\lambda \geq 1$. Then for every $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ such that $|a_\Gamma| \leq c_\epsilon \epsilon^{|\Gamma|} |\Gamma|^{(1-\lambda)|\Gamma|}$. Using estimate (2.4), we have

$$(2.5) \quad \sum_\Gamma |a_\Gamma c_K^\Gamma| \leq c_\epsilon (1 - \epsilon dM)^{-1} (\epsilon dM)^{|K|} |K|^{(1-\lambda)|K|},$$

provided $\epsilon < (dM)^{-1}$. Thus if we let $a^K = \sum_\Gamma a_\Gamma c_K^\Gamma$, and set $\tilde{T} = \sum_K (K!)^{-1} a^K \sigma(X^K)$, then by estimate (2.5) we have $\tilde{T} \in \mathfrak{S}_\lambda(\mathfrak{g})$. Furthermore, by the absolute convergence, the series for $\tilde{T}$ may be rearranged, so that

$$\begin{aligned} \tilde{T} &= \lim_{n \rightarrow \infty} \sum_{|\Gamma| \leq n} a_\Gamma \cdot \sum_K (K!)^{-1} c_K^\Gamma \sigma(X^K) \\ &= \lim_{n \rightarrow \infty} \left\{ \sum_{|\Gamma| \leq n} (|\Gamma|!)^{-1} a_\Gamma X_\Gamma + R_n \right\}, \end{aligned}$$

where $R_n \in \mathfrak{R}(\mathfrak{g})$. Since the first term in braces converges to T in $\mathfrak{I}_\lambda(\mathfrak{g})$, it follows

that $\lim_{n \rightarrow \infty} R_n \equiv R$ exists in $\mathfrak{J}_\lambda(\mathfrak{g})$, and $\tilde{T} = T + R$. This proves that $\mathfrak{J}_\lambda(\mathfrak{g}) = \mathfrak{S}_\lambda(\mathfrak{g}) + \mathfrak{R}_\lambda(\mathfrak{g})$.

Finally, to show that $\mathfrak{S}_\lambda(\mathfrak{g}) \cap \mathfrak{R}_\lambda(\mathfrak{g}) = 0$, we let G be the simply-connected group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}$ act on germs of holomorphic functions at the identity of G :

$$X \cdot f(g) = \frac{d}{dt} f(\exp(-tX)g) \big|_{t=0},$$

for $X \in \mathfrak{g}$ and f holomorphic on a neighborhood of 1. This is a Lie algebra representation, and thus extends to a representation of $\mathfrak{J}(\mathfrak{g})$, with the ideal $\mathfrak{R}(\mathfrak{g})$ acting trivially. The symmetrized elements $\sigma(X^K)$ act on f by the formula

$$(2.6) \quad \sigma(X^K) \cdot f(g) = (-1)^{|K|} \left(\frac{\partial}{\partial t} \right)^K f(\exp X(t) \cdot g) \big|_{t=0}.$$

(Here $t \in \mathbb{C}^d$, $X(t) = t_1 X_1 + \cdots + t_d X_d$, and $(\partial/\partial t)^K = (\partial/\partial t_1)^{k_1} \cdots (\partial/\partial t_d)^{k_d}$. Formula (2.6) is a consequence of the formula (2.12) given below and "Taylor's formula"; cf. [10], Ch. II, §1.) Hence if f is holomorphic on a neighborhood of 1, then there exists an $r > 0$ such that

$$|\sigma(X^K) \cdot f(g)| \leq r^{|K|} K!$$

for g near 1. It follows from estimate (2.4) that

$$|X_\Gamma \cdot f(g)| \leq c^{|\Gamma|} |\Gamma|!$$

for some constant c . This last estimate shows that the representation of $\mathfrak{J}(\mathfrak{g})$ extends to a representation of $\mathfrak{J}_\lambda(\mathfrak{g})$, $\lambda \geq 1$, and that $T \rightarrow T \cdot f(1)$ is a continuous functional on $\mathfrak{J}_\lambda(\mathfrak{g})$. Hence $T \cdot f(1) = 0$ if $T \in \mathfrak{R}_\lambda(\mathfrak{g})$, while taking $f(\exp X(t))$ to be a monomial in t shows that the representation is faithful on $\mathfrak{S}_\lambda(\mathfrak{g})$.

It remains now to establish estimate (2.4). For this purpose we use the following basic identity from [17]. For completeness, we include a proof.

Lemma 2.1. *If $X \in \mathfrak{g}$, let D_X be the derivation of $\mathfrak{J}(\mathfrak{g})$ such that $D_X(Y) = [X, Y]$ when $Y \in \mathfrak{g}$. Then for $X, Y \in \mathfrak{g}$ one has the identity*

$$(2.7) \quad \frac{1}{n!} Y X^n = \sum_{k+m=n} \frac{b_k}{m!} \sigma(X^m D_X^k(Y)) \mod \mathfrak{R}(\mathfrak{g}),$$

where

$$\sum_{k=0}^{\infty} b_k z^k = z(e^z - 1)^{-1}.$$

Proof of Lemma 2.1. We shall obtain (2.7) by the "generating function" technique, i.e., we shall obtain a single identity in the algebra $\mathfrak{J}_\infty(\mathfrak{g})$, defined in §1, which, when expressed in terms of homogeneous components, yields (2.7) for all n .

Starting with the formula

$$\sigma(YX^m) = \frac{1}{m+1} \sum_{k=0}^m X^k YX^{m-k},$$

we obtain the following identity in $\mathfrak{J}_\infty(\mathfrak{g})$:

$$(2.8) \quad \sigma(Ye^X) = \sum_{m,k=0}^{\infty} \frac{1}{(m+k+1)!} X^k YX^m,$$

where e^X is defined in $\mathfrak{J}_\infty(\mathfrak{g})$ by the power series, and the symmetrization operator σ acts on $\mathfrak{J}_\infty(\mathfrak{g})$ via its action on each homogeneous component. Now from the well-known formula for the Beta-function, we have

$$\frac{m!k!}{(m+k+1)!} = \int_0^1 t^k(1-t)^m dt.$$

Expanding the integrand by the binomial formula and integrating term-by-term, we obtain the expansion

$$\frac{1}{(m+k+1)!} = \frac{1}{k!} \sum_{j=0}^m \frac{1}{j!} \frac{(-1)^j}{(m-j)!(j+k+1)}.$$

Substituting this in the right side of (2.8) and re-arranging the resulting series, we obtain the identity

$$(2.9) \quad \sigma(Ye^X) = \left\{ \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \nabla^m Y \right\} e^X,$$

where ∇ is the linear operator on $\mathfrak{J}_\infty(\mathfrak{g})$

$$\nabla T = XT - TX, \quad T \in \mathfrak{J}_\infty(\mathfrak{g}).$$

Now $\nabla: \mathfrak{J}^n(\mathfrak{g}) \rightarrow \mathfrak{J}^{n+1}(\mathfrak{g})$, so that the operator $f(\nabla)$ is defined on $\mathfrak{J}_\infty(\mathfrak{g})$ for any formal power series $f(z)$ in one variable. Furthermore, the map $f(z) \rightarrow f(\nabla)$ is a homomorphism from the ring of formal power series to endomorphisms of $\mathfrak{J}_\infty(\mathfrak{g})$, and the operator $f(\nabla)$ commutes with multiplication by $g(X)$, for any two formal power series f and g . Taking $f(z) = z^{-1}(e^z - 1)$, we may write (2.9) as

$$(2.9)' \quad \sigma(Ye^X) = \left(\frac{e^\nabla - 1}{\nabla} \right) (Ye^X).$$

Now apply the operator $\nabla/(e^\nabla - 1)$ to (2.9)' to obtain the identity

$$(2.10) \quad \left(\frac{\nabla}{e^\nabla - 1} \right) \sigma(Ye^X) = Ye^X.$$

Equating the homogeneous components of degree $n+1$ on each side of (2.10), we obtain the equivalent set of identities

$$(2.10)' \quad \sum_{k+m=n} \frac{b_k}{m!} \nabla^k [\sigma(YX^m)] = \frac{1}{n!} YX^n.$$

To obtain the desired identities (2.7), it suffices by (2.10)' to show that

$$(2.11) \quad \nabla^k[\sigma(YX^m)] = \sigma(X^m D^k(Y)) \mod \mathfrak{R}(\mathfrak{g}),$$

where $D = D_x$. With the convention that $\nabla^0 = D^0 =$ identity operator, this is true for $k = 0$. Assume (2.11) holds for k , and all $Y \in \mathfrak{g}$. Set $Z = D^k(Y)$. Now $\nabla(Y) = D(Y) \mod \mathfrak{R}(\mathfrak{g})$ if $Y \in \mathfrak{g}$; by the Jacobi identity and the fact that ∇ is a derivation of $\mathfrak{J}(\mathfrak{g})$ one easily verifies that $\nabla(\mathfrak{R}(\mathfrak{g})) \subseteq \mathfrak{R}(\mathfrak{g})$. Using this, we have by the induction hypothesis

$$\begin{aligned} \nabla^{k+1}[\sigma(X^m Y)] &= \nabla[\sigma(X^m Z)] \\ &= \frac{1}{m+1} \sum_{i=0}^m (X^{i+1} Z X^{m-i} - X^i Z X^{m+1-i}) \\ &= \frac{1}{m+1} \sum_{i=0}^m X^i D(Z) X^{m-i} \\ &= \sigma(X^m D^{k+1}(Y)), \end{aligned}$$

all equalities holding mod $\mathfrak{R}(\mathfrak{g})$.

Q.E.D.

Completion of the proof of Theorem 2.1. Let $\{X_i\}$ be the fixed basis for $\mathfrak{g}$, and let $X(t) = t_1 X_1 + \cdots + t_d X_d$, $t \in \mathbb{C}^d$, as before. Then we have the expansion

$$(2.12) \quad (m!)^{-1} X(t)^m = \sum_{|M|=m} (M!)^{-1} t^M \sigma(X^M).$$

Indeed, for $m = 1$ this trivially holds. Assume it holds for m , and write

$$\begin{aligned} [(m+1)!]^{-1} X(t)^{m+1} &= [(m+1)!]^{-1} (m+1)^{-1} \sum_{k=0}^m X(t)^k X(t) X(t)^{m-k} \\ &= [(m+1)!]^{-1} \sum_{i=1}^d t_i \sigma(X_i X(t)^m). \end{aligned}$$

By the induction hypothesis and the relation

$$(2.13) \quad \sigma(T\sigma(S)) = \sigma(ST), \quad S, T \in \mathfrak{J}(\mathfrak{g}),$$

(cf. [8], eqn. 4.31), we have

$$[(m+1)!]^{-1} X(t)^{m+1} = \frac{1}{m+1} \sum_{i=1}^d \sum_{|M|=m} (M!)^{-1} t_i t^M \sigma(X_i X^M).$$

But for any multi-exponent N with $|N| = m+1$, one has

$$(2.14) \quad \frac{1}{m+1} \sum_{M,i} (M!)^{-1} = (N!)^{-1},$$

where the sum in (2.14) is over all M, i with $M + e_i = N$ ($e_i = (0, \dots, 1, \dots, 0)$ with 1 in the i^{th} coordinate, 0 elsewhere). The use of relation (2.14) completes the inductive proof of (2.12).

Let the operator D_X , $X \in \mathfrak{g}$, be defined as in Lemma 2.1, and define the operators $D(M)$, M a multi-exponent, by the identity

$$(2.15) \quad (m!)^{-1} [D_{X(t)}]^m = \sum_{|M|=m} (M!)^{-1} t^M D(M),$$

$t \in \mathbf{C}^d$. (From (2.12) we see that the operator $D(M)$ is obtained by "symmetrizing" the operator $D_{X_1}^{m_1} \cdots D_{X_d}^{m_d}$.) Now apply Lemma 2.1 to $X = X(t)$ and $Y \in \mathfrak{g}$, and use (2.12), (2.13), and (2.15) to obtain the coefficient of t^N . The result of equating coefficients of t^N in (2.7) is the following multi-exponent version of the basic identity:

$$(2.16) \quad \frac{1}{N!} Y \sigma(X^N) = \sum_{K+M=N} \frac{b_K}{M!} \sigma(X^M D(K) Y) \mod (\mathfrak{R}(\mathfrak{g})),$$

where $b_K = b_{|K|} (K!)^{-1} |K|!$, with b_k as in (2.7).

We want to use (2.16) to obtain a recursive formula for the coefficients c_K^Γ of equation (2.3). For this purpose, let $[d_{i,j}(K)]$ be the matrix of $D(K)$ on $\mathfrak{g}$. Using (2.16), we can write

$$(N!)^{-1} X_i \sigma(X^N) = \sum_{\substack{K+M=N \\ 1 \leq i \leq d}} (M!)^{-1} b_K d_{i,j}(K) \sigma(X_j X^M) \mod \mathfrak{R}(\mathfrak{g}).$$

Now from (2.4), we have

$$(|\Gamma|!)^{-1} X_i X_\Gamma = \sum_N (N!)^{-1} c_N^\Gamma X_i \sigma(X^N) \mod \mathfrak{R}(\mathfrak{g}),$$

which by the above we can write as

$$(2.17) \quad (|\Gamma|!)^{-1} X_i X_\Gamma = \sum_{\substack{K,M \\ 1 \leq i \leq d}} (M!)^{-1} b_K c_{K+M}^\Gamma d_{i,j}(K) \sigma(X_j X^M) \mod \mathfrak{R}(\mathfrak{g}).$$

Thus from (2.17) we see that the coefficients c_N^Γ satisfy the recursion

$$(2.18) \quad c_N^{i \oplus \Gamma} = \frac{N!}{|\Gamma| + 1} \sum (M!)^{-1} b_K c_{K+M}^\Gamma d_{i,j}(K),$$

where the summation in (2.18) is over all K, M, j such that $M + e_i = N$, and $i \oplus \Gamma$ is the multi-index $\{i, j_1, \dots, j_p\}$, if $\Gamma = \{j_1, \dots, j_p\}$.

Set $\alpha_n^m = \max \{|c_N^\Gamma|; |\Gamma| = m, |N| = n\}$. Since $D(K)$ is the average of $|K|$ linear transformations $\{D_{X_i}\}$, we have an estimate $|d_{i,j}(K) b_K| \leq r^{|K|}$ holding for some $r > 0$. Using (2.14), we obtain from (2.18) the recursive inequalities

$$(2.19) \quad \alpha_n^{m+1} \leq \sum_{k=0}^{m-n+1} r^k \alpha_{k+n-1}^m,$$

holding for some $r > 0$ and all $n \geq 1$. (Recall that $c_M^\Gamma = 0$ if $M = 0$ and $|\Gamma| \neq 0$.) Also we have $\alpha_1^1 = 1$. Let $s > 0$ satisfy $rs < 1$. Then we claim that

$$(2.20) \quad \alpha_n^m \leq [s(1 - rs)]^{-m} s^n.$$

Indeed, for $m = n = 1$ this is true. Assume (2.20) holds for m and $1 \leq n \leq m$.

Then by (2.19) we have

$$\begin{aligned} a_n^{m+1} &\leq [s(1 - rs)]^{-m} s^{n-1} \sum_{k=0}^{\infty} (rs)^k \\ &\leq [s(1 - rs)]^{-m-1} s^n. \end{aligned}$$

Since $a_n^m = 0$ for $n > m$, estimate (2.5) follows from (2.20), completing the proof of Theorem 2.1.

Since $\mathfrak{I}_\lambda(\mathfrak{g}) \subseteq \mathfrak{I}_\mu(\mathfrak{g})$ and $\mathfrak{Q}_\lambda(\mathfrak{g}) \subseteq \mathfrak{Q}_\mu(\mathfrak{g})$ when $\lambda \geq \mu$, there are natural homomorphisms

$$J_{\lambda\mu}: \mathfrak{Q}_\lambda(\mathfrak{g}) \rightarrow \mathfrak{Q}_\mu(\mathfrak{g}).$$

Corollary 2.1. *If $\lambda \geq \mu \geq 1$, the homomorphism $J_{\lambda\mu}$ is injective. Furthermore, the algebra $\mathfrak{Q}_1(\mathfrak{g})$ is naturally isomorphic to the “hyperenveloping algebra” $\mathfrak{F}(\mathfrak{g})$, defined in [17].*

Proof. Let $T \in \mathfrak{I}_\lambda(\mathfrak{g})$, and suppose $T \notin \mathfrak{Q}_\lambda(\mathfrak{g})$. If we write $T = S + R$, with $S \in \mathfrak{S}_\lambda(\mathfrak{g})$, $R \in \mathfrak{Q}_\lambda(\mathfrak{g})$, then $S \neq 0$. If we let $\mathfrak{I}_\lambda(\mathfrak{g})$ act on germs of holomorphic functions at the identity of G , as in the proof of Theorem 2.1, then T acts non-trivially. Since $\mathfrak{Q}_\mu(\mathfrak{g})$ acts trivially, we conclude that $T \notin \mathfrak{Q}_\mu(\mathfrak{g})$. Hence $J_{\lambda\mu}$ is injective.

From equation (2.2), the closed graph theorem, and Proposition (1.1), we see that $\mathfrak{Q}_1(\mathfrak{g})$ is naturally identified, as a topological vector space, with the completion of $\mathfrak{U}(\mathfrak{g})$ relative to the family of norms

$$(2.21) \quad \|\sum c_K X(K)\|_r = \sum |c_K| r^{|K|} |K|!,$$

where $X(K)$ is the image in $\mathfrak{U}(\mathfrak{g})$ of the element $\sigma(X^K)$ of $\mathfrak{I}(\mathfrak{g})$. This, however, is exactly the definition of $\mathfrak{F}(\mathfrak{g})$ [17]. Q.E.D.

We shall henceforth identify $\mathfrak{Q}_\lambda(\mathfrak{g})$, $\lambda > 1$, with its image in $\mathfrak{Q}_1(\mathfrak{g})$, so that for $\lambda \geq \mu \geq 1$ we have the inclusions

$$\mathfrak{U}(\mathfrak{g}) \subseteq \mathfrak{Q}_\lambda(\mathfrak{g}) \subseteq \mathfrak{Q}_\mu(\mathfrak{g}).$$

Let $\mathcal{O}(\mathfrak{g}')$ be the space of holomorphic polynomial functions on $\mathfrak{g}'$, where $\mathfrak{g}'$ is the linear dual space to $\mathfrak{g}$. If φ is an automorphism of $\mathfrak{g}$, and f is a function on $\mathfrak{g}'$, define

$$\varphi \cdot f(\xi) = f(\varphi' \cdot \xi),$$

φ' denoting the adjoint of φ . With this action $\mathcal{O}(\mathfrak{g}')$ becomes an $\text{Aut } \mathfrak{g}$ -module, isomorphic to $\mathfrak{S}(\mathfrak{g})$. The Poincaré–Birkhoff–Witt Theorem, as expressed in formula (2.1), implies that $\mathfrak{U}(\mathfrak{g})$ is isomorphic as an $\text{Aut } (\mathfrak{g})$ -module to $\mathcal{O}(\mathfrak{g}')$. We may similarly use Theorem 2.1 to obtain the structure of $\mathfrak{Q}_\lambda(\mathfrak{g})$, $\lambda \geq 1$, as an $\text{Aut } (\mathfrak{g})$ -module.

Definition. Let $E_p(\mathfrak{g}')$, $0 < p < \infty$, be the space of entire functions on $\mathfrak{g}'$ for which

$$(2.22) \quad \|f\|_{p,r} \equiv \sup_{\xi \in g'} |f(\xi)| \exp(-r|\xi|^p) < \infty$$

for all $r > 0$. (Here $|\xi|$ is any norm on g' .) Give $E_p(g')$ the locally convex topology defined by the family of norms $\{\|f\|_{p,r}\}_{r>0}$.

The space $E_p(g')$ is obviously invariant under the action of $\text{Aut}(g)$ on functions on g' defined above, and thus becomes an $\text{Aut}(g)$ -module.

Corollary 2.2. *If $\lambda \geq 1$, then the canonical $\text{Aut}(g)$ -module isomorphism between $\mathfrak{U}(g)$ and $\mathcal{O}(g')$ extends continuously to a topological isomorphism between $\mathfrak{A}_\lambda(g)$ and $E_p(g')$, $p = 1/\lambda$.*

Proof. By Theorem 2.1 and the closed graph theorem, we only need to show that the canonical isomorphism between $\mathfrak{S}(g)$ and $\mathcal{O}(g')$ extends to a topological isomorphism between $\mathfrak{S}_\lambda(g)$ and $E_p(g')$, $p = 1/\lambda$.

Let $\hat{X}^K$, $K = (k_1, \dots, k_d)$ a multi-exponent, be the function on g' given by

$$(2.23) \quad \hat{X}^K(\xi) = \langle X_1, \xi \rangle^{k_1} \cdots \langle X_d, \xi \rangle^{k_d}.$$

Then the linear map sending $\sigma(X^K)$ to $\hat{X}^K$ is the canonical isomorphism between $\mathfrak{S}(g)$ and $\mathcal{O}(g')$. But if $T = \sum a_K \sigma(X^K) \in \mathfrak{S}_\lambda(g)$, define $\hat{T} = \sum a_K \hat{X}^K$. By well-known estimates in the theory of entire functions (cf. [11]), the series defining $\hat{T}$ converges in $E_p(g')$, and the map $T \rightarrow \hat{T}$ is a topological vector space isomorphism from $\mathfrak{S}_\lambda(g)$ to $E_p(g')$. Q.E.D.

Definition. The Laplace transform of an element $T \in \mathfrak{A}_\lambda(g)$, $\lambda \geq 1$, is the corresponding entire function $\hat{T} \in E_p(g')$, $p = 1/\lambda$, given by Corollary 2.2.

Remark. The Laplace transform $\hat{T}$ may be expressed in terms of T as follows (cf. [13], [17]): Let $\mathfrak{A}_\lambda(g)$, $\lambda \geq 1$, act on germs of analytic functions at the identity of G , as in the proof of Theorem 2.1. For $\xi \in g'$, let e_ξ be the function $e_\xi(g) = e^{\langle \log g, \xi \rangle}$, defined on a neighborhood of 1. If f_K is the function $f_K(\exp X(t)) = (K!)^{-1} t^K$, defined on a neighborhood of 1, then

$$e_\xi(g) = \sum f_K(g) \hat{X}^K(\xi).$$

Hence by formula (2.6) and the proof of Corollary 2.2 we obtain the formula

$$(2.24) \quad \hat{T}(\xi) = (T \cdot e_{-\xi})(1).$$

§3. Representations of $\mathfrak{A}_\lambda(g)$. We have already observed, in the course of proving Proposition 1.1, that every finite-dimensional representation of $\mathfrak{U}(g)$ extends to a continuous representation of $\mathfrak{A}_\lambda(g)$. In this section we consider certain infinite-dimensional representations of $\mathfrak{U}(g)$ which also extend to $\mathfrak{A}_\lambda(g)$. Namely, let g_0 be a real form of g , and let G_0 be the simply-connected Lie group with Lie algebra g_0 . If π is a strongly continuous representation of G_0 on a Banach space $\mathcal{H}(\pi)$, then π defines a representation $\partial\pi$ of $\mathfrak{U}(g)$ on the space $\mathcal{H}^\infty(\pi)$ of differentiable vectors. Let the spaces $S_{\lambda,i}(\pi) \subseteq \mathcal{H}^\infty(\pi)$ be as defined in [6], and denote by $\mathfrak{L}_i(S_{\lambda,i}(\pi))$ the algebra of continuous linear transformations

on $S_{\lambda,t}(\pi)$, with the topology of uniform convergence on bounded sets of $S_{\lambda,t}(\pi)$.

Proposition 3.1. *There exists a unique continuous homomorphism*

$$\Phi_{\lambda,t}: \mathcal{A}_\lambda(\mathfrak{g}) \rightarrow \mathcal{L}_b(S_{\lambda,t}(\pi))$$

such that when $T \in \mathfrak{U}(\mathfrak{g})$ and $v \in S_{\lambda,t}(\pi)$, then

$$(3.1) \quad \Phi_{\lambda,t}(T)v = \partial\pi(T)v.$$

Proof. Let φ be the representation of $\mathfrak{J}(\mathfrak{g})$ on $\mathcal{H}^\infty(\pi)$ obtained by composing $\partial\pi$ with the canonical quotient map from $\mathfrak{J}(\mathfrak{g})$ to $\mathfrak{U}(\mathfrak{g})$. If $v \in \mathcal{H}^\infty(\pi)$, let $\rho_n(v)$ and $\|v\|_{\lambda,r}$ be as defined in §1 of [6], relative to the same basis $\{X_i\}$ for $\mathfrak{g}_0$ used to define the norms on $\mathfrak{J}_\lambda(\mathfrak{g})$.

From the estimate $\rho_m(\varphi(X_\alpha)v) \leq \rho_{m+|\alpha|}(v)$ (cf. [5], inequality (1.2)), it follows that for any multi-index α , $|\alpha| = n$,

$$(3.2) \quad \|\varphi(X_\alpha)v\|_{\lambda,r} \leq \sup_m \left[\frac{(m+n)^{\lambda(m+n)} s^{m+n}}{m^{\lambda m} r^m} \right] \|v\|_{\lambda,s}.$$

If $v \in S_{\lambda,t}(\pi)$ and $0 < s < r < t$, then the right side of (3.2) is finite, and by elementary calculus the quantity within brackets is bounded by $r^{\lambda n} (r-s)^{-\lambda n} s^n n^{\lambda n}$. It follows that if $S = \sum c_\alpha X_\alpha \in \mathfrak{J}_\lambda(\mathfrak{g})$, then

$$(3.3) \quad \sum_\alpha |c_\alpha| \|\varphi(X_\alpha)v\|_{\lambda,r} \leq \|S\|_{\lambda,\delta} \|v\|_{\lambda,s},$$

where $\delta = r^\lambda (r-s)^{-\lambda}$. Thus the series $\sum c_\alpha \varphi(X_\alpha)v$ converges absolutely in $S_{\lambda,t}(\pi)$, and if we denote the sum by $\varphi_{\lambda,t}(S)v$, then by (3.3) we have

$$(3.4) \quad \|\varphi_{\lambda,t}(S)v\|_{\lambda,r} \leq \|S\|_{\lambda,\delta} \|v\|_{\lambda,s}.$$

The map $S \rightarrow \varphi_{\lambda,t}(S)$ is obviously an algebra homomorphism from $\mathfrak{J}_\lambda(\mathfrak{g})$ to $\mathcal{L}_b(S_{\lambda,t}(\pi))$, by the absolute convergence of the series defining $\varphi_{\lambda,t}(S)v$. From (3.4) we see that $\varphi_{\lambda,t}$ is continuous.

If $S \in \mathfrak{J}(\mathfrak{g})$, then $\varphi_{\lambda,t}(S) = \varphi(S)|_{S_{\lambda,t}(\pi)}$. Hence $\varphi_{\lambda,t}(\mathcal{A}(\mathfrak{g})) = 0$, and by continuity $\varphi_{\lambda,t}(\mathcal{A}_\lambda(\mathfrak{g})) = 0$. If $\Phi_{\lambda,t}$ is the homomorphism obtained from $\varphi_{\lambda,t}$ by passing to the quotient mod $\mathcal{A}_\lambda(\mathfrak{g})$, then $\Phi_{\lambda,t}$ satisfies the conditions of Proposition 3.1. Q.E.D.

We define Φ_λ to be the representation of $\mathcal{A}_\lambda(\mathfrak{g})$ on $S_\lambda(\pi)$ such that $\Phi_\lambda(S)v = \Phi_{\lambda,t}(S)v$ if $v \in S_{\lambda,t}(\pi)$. It is clear from the construction of $\Phi_{\lambda,t}$ above that this is a consistent definition. From the characterization of the bounded sets in $S_\lambda(\pi)$ (Lemma 1.2 of [6]), we see that Φ_λ is continuous from $\mathcal{A}_\lambda(\mathfrak{g})$ to the algebra of continuous operators on $S_\lambda(\pi)$, when the latter is given the topology of uniform convergence on bounded sets of $S_\lambda(\pi)$.

If π is non-trivial, then the representation Φ_λ in Proposition 3.1 is non-trivial when $\lambda \geq 1$, since $S_\lambda(\pi)$ is dense in $\mathcal{H}(\pi)$ in this case. For $\lambda > 1$ we can obtain faithful representations of $\mathcal{A}_\lambda(\mathfrak{g})$ by "differential operators of infinite order" in this way, as follows: Let $L_p(G_0)$, $1 \leq p < \infty$, be the usual Lebesgue space taken

with respect to a left-invariant Haar measure on G_0 , and let $C_0(G_0)$ be the space of continuous functions vanishing at ∞ on G_0 .

Theorem 3.1. *Let $\lambda > 1$ and let π be the left regular representation of G_0 , either on $L_p(G_0)$, $1 \leq p < \infty$, or on $C_0(G_0)$. Let Φ_λ be the associated representation of $\mathfrak{A}_\lambda(\mathfrak{g})$ on $S_\lambda(\pi)$. Then Φ_λ is faithful.*

Proof. Let φ_λ be the representation of $\mathfrak{A}_\lambda(\mathfrak{g})$ on $S_\lambda(\pi)$, as in the proof of Proposition 3.1. Let $S = \sum a_K \sigma(X^K)$ belong to $S_\lambda(\mathfrak{g})$, with some $a_K \neq 0$. Then by Theorem 2.1 it suffices to show that $\varphi_\lambda(S) \neq 0$.

By a Sobolev lemma ([15], Lemma 5.1), we have $\mathcal{H}^\infty(\pi) \subseteq C^\infty(G_0)$, and if $f \in \mathcal{H}^\infty(\pi)$, then

$$(3.4) \quad |f(1)| \leq C \sum_{n=0}^d \rho_n(f),$$

where $\{\rho_n\}$ are the semi-norms on $\mathcal{H}^\infty(\pi)$ as before, and $d = \dim \mathfrak{g}$. If $f \in S_\lambda(\pi)$, it follows from (2.6) and (3.4) that

$$(3.5) \quad \varphi_\lambda(S)f(1) = \sum_K (-1)^{|K|} a_K (\partial/\partial t)^K f(\exp X(t)) \big|_{t=0}$$

with the series on the right side of (3.5) absolutely convergent. Thus to show that $\varphi_\lambda(S) \neq 0$, it suffices to construct, for any multi-exponent K , a function $f \in S_\lambda(\pi)$ such that $f(\exp X(t)) = t^K$ for t near 0 in $\mathbf{R}^d$.

Let $\epsilon > 0$ be such that $\exp: \mathfrak{g}_0 \rightarrow G_0$ is an analytic manifold isomorphism when restricted to $\{X(t); |t| < 2\epsilon\}$. Let $\Psi \in C_c^\infty(\mathbf{R}^d)$ with $\text{Supp } (\Psi) \subseteq \{|t| \leq \epsilon\}$, such that $\Psi(t) = 1$ for $|t| \leq \epsilon/2$ and

$$|(\partial/\partial t)^M \Psi(t)| \leq C^{|M|} |M|^{\lambda|M|}$$

for all multi-exponents M . Since $\lambda > 1$, there exists such a function Ψ (cf. proof of Theorem 3.2 of [6]). Set $F(t) = t^K \Psi(t)$. Then $F \in C_c^\infty(\mathbf{R}^d)$, and is easily seen to satisfy the estimates

$$(3.6) \quad |(\partial/\partial t)^M F(t)| \leq C^{|M|} |M|^{\lambda|M|}$$

for some constant C . If $f \in C_c^\infty(G_0)$ is defined by $f(\exp X(t)) = F(t)$, then by (3.6) and Theorem 3.1 of [6], we have $f \in S_\lambda(\pi)$. When $|t| \leq \epsilon/2$, $f(\exp X(t)) = t^K$, as desired. Q.E.D.

In the case of unitary representations π of G_0 , a direct integral decomposition of π induces a direct integral decomposition of the associated representation of $\mathfrak{A}_\lambda(\mathfrak{g})$, for any $\lambda > 0$:

Theorem 3.2. *Let π be a unitary representation of G_0 , and suppose*

$$\begin{aligned} \pi &= \int^\oplus \pi^\xi d\mu(\xi) \\ \mathcal{H}(\pi) &= \int^\oplus \mathcal{H}(\pi^\xi) d\mu(\xi) \end{aligned}$$

is a direct integral decomposition of π . Let Φ_λ and Φ_λ^ξ be the representations of $\mathfrak{G}_\lambda(\mathfrak{g})$ associated with π and π^ξ , respectively.

(a) If $v \in S_{\lambda, \iota}(\pi)$ and $T \in \mathfrak{G}_\lambda(\mathfrak{g})$, then the components $v^\xi \in S_{\lambda, \iota}(\pi^\xi)$ a.e., and

$$(3.7) \quad (\Phi_\lambda(T)v)^\xi = \Phi_\lambda^\xi(T)v^\xi \quad \text{a.e.}$$

$(\delta = (\dim \mathfrak{g})^{1/2})$.

(b) If $v \in \mathcal{H}(\pi)$ is such that $v^\xi \in \mathcal{H}^\infty(\pi^\xi)$ a.e. and

$$(3.8) \quad \int \|v^\xi\|_{\lambda, \iota}^2 d\mu(\xi) < \infty,$$

then $v \in S_{\lambda, \iota}(\pi)$.

Proof. (a) From the proof of Lemma 3.1 of [7], we know that $v^\xi \in \mathcal{H}^\infty(\pi^\xi)$ a.e., and $(\partial\pi(W)v)^\xi = \partial\pi^\xi(W)v^\xi$ a.e., for any $v \in \mathcal{H}(\pi)$ and $W \in \mathfrak{U}(\mathfrak{g})$. If $\{X_j\}$, $1 \leq j \leq d$, is the fixed basis for $\mathfrak{g}_0$ used to define the semi-norms ρ_n and α is a multi-index, it follows that

$$\|\partial\pi(X_\alpha)v\|^2 = \int \|\partial\pi^\xi(X_\alpha)v^\xi\|^2 d\mu(\xi).$$

Hence by the monotone convergence theorem,

$$(3.9) \quad \sum_{n=0}^{\infty} r^{2n} n^{-2\lambda n} \sum_{|\alpha|=n} \|\partial\pi(X_\alpha)v\|^2 = \int \sum_{n=0}^{\infty} r^{2n} n^{-2\lambda n} \sum_{|\alpha|=n} \|\partial\pi^\xi(X_\alpha)v^\xi\|^2 d\mu(\xi).$$

If $v \in S_{\lambda, \iota}(\pi)$, then for any $s < t$,

$$\|\partial\pi(X_\alpha)v\| \leq s^{|\alpha|} |\alpha|^{\lambda|\alpha|} \|v\|_{\lambda, \iota}.$$

Since there are d^n multi-indices of length n , it follows that the left side of (3.9) is finite for any $r < (\delta t)^{-1}$. Hence the integrand on the right side of (3.9) is finite a.e. when $r < (\delta t)^{-1}$. Since the integrand dominates $\sum r^{2n} n^{-2\lambda n} \rho_n^\xi(v^\xi)^2$, we conclude that $v^\xi \in S_{\lambda, \iota}(\pi^\xi)$ a.e. (Here ρ_n^ξ are the canonical semi-norms for $\mathcal{H}^\infty(\pi^\xi)$, relative to the basis $\{X_j\}$.)

Let $T \in \mathfrak{G}_\lambda(\mathfrak{g})$, and pick a representative $\sum c_\alpha X_\alpha \in \mathfrak{H}_\lambda(\mathfrak{g})$ for T . Then for $v \in S_{\lambda, \iota}(\pi)$, we have $\Phi_\lambda(T)v = \lim_{n \rightarrow \infty} w_n$ where $w_n = \sum_{|\alpha| \leq n} c_\alpha \partial\pi(X_\alpha)v$. But

$$w_n^\xi = \sum_{|\alpha| \leq n} c_\alpha \partial\pi^\xi(X_\alpha)v^\xi \quad \text{a.e.},$$

so by the above, $\lim_{n \rightarrow \infty} w_n^\xi$ exists in $\mathcal{H}(\pi^\xi)$ and equals $\Phi_\lambda^\xi(T)v^\xi$ a.e. Since $\lim_{n \rightarrow \infty} w_n^\xi = (\Phi_\lambda(T)v)^\xi$ in $L_2(d\mu)$, we obtain (3.7).

(b) The integrand in (3.8) is μ -measurable (since $\xi \rightarrow \rho_n^\xi(v^\xi)$ is μ -measurable), and dominates the integrand on the right side of (3.9), with $r = (\delta t)^{-1}$. Hence by (3.9) we conclude that $\rho_n(v) \leq C(\delta t)^n n^{\lambda n}$, so that $v \in S_{\lambda, \iota}(\pi)$. Q.E.D.

Corollary 3.1. Let $\lambda > 1$ and $T \in \mathfrak{G}_\lambda(\mathfrak{g})$, $T \neq 0$. Then there exists an irreducible unitary representation π of G_0 such that T acts non-trivially on $S_\lambda(\pi)$.

Proof. Let L be the left regular representation of G_0 on $L_2(G_0)$. There exists a direct integral decomposition

$$L = \int^{\oplus} L^{\xi} d\mu(\xi),$$

where L^{ξ} is irreducible [2]. By Theorem 3.1 T acts as a non-zero operator on $S_{\lambda}(L)$. Hence by Theorem 3.2, T must act non-trivially on some $S_{\lambda}(L^{\xi})$. Q.E.D.

§4. Representations of $\mathcal{Q}_{\lambda}(\mathfrak{g})$; $\mathfrak{g}$ nilpotent, $\lambda < 1$. In this section we assume that $\mathfrak{g}$ is a complex nilpotent Lie algebra, and that $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$. Let G be the simply-connected group with Lie algebra $\mathfrak{g}$, and let $G_0 = \exp \mathfrak{g}_0$ be the analytic subgroup corresponding to the subalgebra $\mathfrak{g}_0$. In order to obtain non-trivial representations of $\mathcal{Q}_{\lambda}(\mathfrak{g})$, $\lambda < 1$, from unitary representations π of G_0 , we must establish the non-triviality of the space $S_{\lambda}(\pi)$ (at least for λ sufficiently near 1). This was done for the case of irreducible representations in §4 of [6]. Here we shall treat an arbitrary unitary representation of G_0 .

Recall that a basis $\{X_i\}$ for $\mathfrak{g}_0$ is an *ordered Jordan-Hölder basis* if the subspaces $\mathfrak{h}_k = \text{span } \{X_i\}_{i \leq k}$ satisfy $[\mathfrak{g}_0, \mathfrak{h}_k] \subseteq \mathfrak{h}_{k-1}$. Our basic technical tool in this section will be the following theorem (cf. Theorem 1.1 of [7] for the case $\lambda = 1$):

Theorem 4.1. *Let $\{X_i\}$ be an ordered Jordan-Hölder basis for $\mathfrak{g}_0$. Then there exist integers m_i , $0 \leq m_i \leq m_{i+1}$, with the following property:*

If π is a unitary representation of G_0 , and π_j is the restriction of π to the one-parameter subgroup $\{\exp tX_j\}$, then for $\frac{1}{2} < \nu < 1$ one has

$$(4.1) \quad \bigcap_{i=1}^d S_{\mu_i}(\pi_i) \subseteq S_{\lambda}(\pi),$$

where

$$\mu_1 = \nu, \quad \mu_{i+1} = 2\left(\frac{m_i + \nu}{m_i + 1}\right) - 1, \quad \text{and} \quad \lambda = \frac{m_d + \nu}{m_d + 1}.$$

Remark. Let the integers $n_k \geq 0$ satisfy

$$(4.2) \quad (\text{ad } X_{k+1})^{n_k+1}(\mathfrak{h}_k) = 0,$$

$\mathfrak{h}_k$ as above. Then the proof of Theorem 4.1 shows that we may take $m_1 = 0$ and

$$(4.3) \quad m_j + 1 = \prod_{k < j} (n_k + 1), \quad j > 1.$$

Proof of Theorem 4.1. For convenience in notation set $\mathfrak{h} = \mathfrak{h}_{d-1}$, $X = X_d$, $n = n_{d-1}$ ($d = \dim \mathfrak{g}_0$, n_{d-1} defined by (4.2)). Let $H = \exp \mathfrak{h}$, $T = \{\exp tX\}_{t \in \mathbf{R}}$, and for a unitary representation π of G_0 denote by π_H and π_T the restrictions of π to H and T , respectively. Suppose we can show that

$$(4.4) \quad S_{\nu}(\pi_H) \cap S_{\mu}(\pi_T) \subseteq S_{\lambda}(\pi),$$

where

$$\frac{1}{2} < \nu < 1, \quad \mu = 2\nu - 1, \quad \text{and} \quad \lambda = \frac{n + \nu}{n + 1}.$$

Then (4.1), with m_i defined by (4.3) and the stated relationship among the parameters ν , μ_i , and λ , follows easily by induction on $\dim \mathfrak{g}_0$.

We proceed to establish (4.4). By Theorem 1.1 of [5], $\mathcal{H}^\infty(\pi_H) \cap \mathcal{H}^\infty(\pi_T) = \mathcal{H}^\infty(\pi)$. Let $v \in S_\nu(\pi_H) \cap S_\mu(\pi_T)$, where $\mu = 2\nu - 1$, $\frac{1}{2} < \nu < 1$. Then $v \in \mathcal{H}^\infty(\pi)$, and if we denote by $\{\tau_m\}$ the canonical semi-norms on $\mathcal{H}^\infty(\pi_H)$ associated with the basis $\{X_i\}_{i \leq d-1}$ for $\mathfrak{h}$, then v satisfies the estimates

$$(4.5) \quad \begin{cases} \tau_m(v) \leq s^m m^{\nu m} \\ ||\partial\pi(X)^m v|| \leq r^m m^{\mu m} \end{cases}$$

for some $r, s > 0$ (cf. §4 of [6]). By the *a-priori* estimate (1.8) of [7],

$$[\tau_m(\partial\pi(X)^k v)]^2 \leq \sum_{j=0}^k \binom{k}{j} a_{2m,j} \tau_{2m}(v) ||\partial\pi(X)^{2k-j} v||.$$

Here $a_{m,i}/j!$ is the coefficient of z^i in $M^m(1 + z + \cdots + (z^n/n!))^m$, with M a constant independent of π and v , and $n = n_d$ as above. By (4.5) we obtain the estimate

$$[\tau_m(\partial\pi(X)^k v)]^2 \leq C^{m+k} k^{(\mu+1)k} m^{2\nu m} b_{2m,k},$$

where C is a constant depending on r and s , and

$$(4.6) \quad b_{m,k} = \sum_{j=0}^k (k-j)^{(\mu-1)(k-j)} [a_{m,i}/j!].$$

But by estimate (4.9) of [6], we have $b_{m,k} \leq C^{m+k}$ for some constant C . Since $\mu + 1 = 2\nu$, we conclude that estimate (4.5) implies that

$$(4.7) \quad \tau_m(\partial\pi(X)^k v) \leq C^{m+k} k^{\nu k} m^{\nu m}$$

for some C . By Lemma 4.3 of [6], we obtain $v \in S_\lambda(\pi)$, $\lambda = (n + \nu)/(n + 1)$.
Q.E.D.

Remark. The proof of Theorem 4.1 supplies a partial answer to a question raised by Gel'fand and Shilov concerning their spaces S_β^α ([3], footnote to p. 175). Namely, it shows that if $\frac{1}{2} < \nu < 1$, then

$$(4.8) \quad S_\nu \cap S^{2\nu-1} \subseteq S_\nu^\nu.$$

For this, let G_0 be the Heisenberg group, π the representation of G_0 as in §5 of [6], and X, Y, Z the basis for $\mathfrak{g}_0$ such that $\partial\pi(X) = d/dx$, $\partial\pi(Y) =$ multiplication by ix , $\partial\pi(Z) = i \cdot 1$, all acting on the Schwartz space $\mathcal{S}$. If $\mathfrak{h} = \text{span}\{Y, Z\}$, $H = \exp \mathfrak{h}$, $T = \{\exp tX\}_{t \in \mathbb{R}}$, and S_λ, S^λ are the spaces defined by Gel'fand and Shilov, then by Lemma 5.1 of [6],

$$(4.9) \quad \begin{cases} S_\lambda(\pi_H) \cap \mathcal{H}^\infty(\pi) = S_\lambda \\ S_\lambda(\pi_T) \cap \mathcal{H}^\infty(\pi) = S^\lambda. \end{cases}$$

The implication (4.5) $\Rightarrow$ (4.7), together with (4.9) and Lemma 5.1 of [6], establishes (4.8).

Corollary 4.1. *Let ν, μ_i, λ be as in Theorem 4.1. Let $v \in \mathcal{H}(\pi)$, and suppose that for $1 \leq j \leq d$ the function*

$$t \rightarrow \pi(\exp tX_j)v, \quad t \in \mathbf{R},$$

extends holomorphically to an entire function of exponential order $\leq (1 - \mu_j)^{-1}$. Then $v \in S_\lambda(\pi)$.

Proof. We must show that the assumptions on v imply $v \in S_{\mu_j}(\pi_j)$, $1 \leq j \leq d$. Let $F_j(t) = \pi(\exp tX_j)v$. We are assuming that F_j can be analytically continued to an entire function satisfying

$$(4.10) \quad \|F_j(t)\| \leq A \exp(B|t|^{p_j}),$$

$p_j = (1 - \mu_j)^{-1}$. In particular, v is an entire vector for π_j , by Proposition 4.1 of [5], and $F_j(t) = \sum [t^n/n!] \partial \pi(X_j)^n v$, for $t \in \mathbf{C}$. From the standard estimates in the theory of entire functions (cf. [11]), (4.10) implies that $v \in S_{\mu_j}(\pi_j)$. Q.E.D.

Corollary 4.2. *Let ν, μ_i, λ be as in Theorem 4.1, and let $\mathcal{F}_\nu$ denote the space of functions f in $L_1(G_0)$ such that*

$$z \rightarrow f(\exp(zX_j)g)$$

extends holomorphically to an entire function from $\mathbf{C}$ to $L_1(G_0)$ of exponential order $\leq (1 - \mu_j)^{-1}$, for $j = 1, 2, \dots, d$. If π is a unitary representation of G_0 and $f \in \mathcal{F}_\nu$, then the operator

$$\pi(f) = \int_{G_0} f(g)\pi(g) dg$$

maps $\mathcal{H}(\pi)$ continuously into $S_\lambda(\pi)$.

Proof. Let $v \in \mathcal{H}(\pi)$ and $f \in \mathcal{F}_\nu$. If $z \in \mathbf{R}$, then

$$\pi(\exp zX_j)\pi(f)v = \int_{G_0} f(\exp(-zX_j)g)\pi(g)v dg.$$

The right side extends to an entire function of z of order $\leq (1 - \mu_j)^{-1}$, so $v \in S_\lambda(\pi)$, by Corollary 4.1. Furthermore, if $z \in \mathbf{C}$,

$$(4.11) \quad e^{A|z|^p} \|\pi(\exp zX_j)\pi(f)v\| \leq e^{A|z|^p} \left[\int_{G_0} |f(\exp(zX_j)g)| dg \right] \|v\|.$$

Let π_j denote the restriction of π to the one-parameter subgroup $\{\exp tX_j\}$. Then by (4.11) and the estimates used in the proof of Corollary 4.1, we have

$\pi(f)$ mapping $\mathcal{H}(\pi)$ continuously into $S_{\mu_i}(\pi_i)$. The proof of Theorem 4.1 shows that the inclusion

$$\bigcap_{i=1}^d S_{\mu_i}(\pi_i) \subseteq S_\lambda(\pi)$$

is continuous, so we conclude that

$$\pi(f): \mathcal{H}(\pi) \rightarrow S_\lambda(\pi)$$

is continuous.

Q.E.D.

We now state and prove the main result of this section:

Theorem 4.2. *Let G_0 be a simply-connected nilpotent Lie group. There exists a number λ_0 , $0 < \lambda_0 < 1$, such that the space $S_{\lambda_0}(\pi)$ is dense in $\mathcal{H}(\pi)$ for any unitary representation π of G_0 .*

Proof. Suppose we can show that the space $\mathcal{F}_\nu$ in Corollary 4.2 is dense in $L_1(G)$ for some $\nu_0 < 1$. If λ_0 is determined by ν_0 as in Theorem 4.1, then we have

$$(4.12) \quad \bigcup_{f \in \mathcal{F}_\nu} \text{Range } \pi(f) \subseteq S_{\lambda_0}(\pi)$$

by Corollary 4.2. Assuming $\mathcal{F}_{\nu_0}$ dense in $L_1(G_0)$, we immediately obtain the density of the left side of (4.12) in $\mathcal{H}(\pi)$.

To establish the density of $\mathcal{F}_\nu$, we recall that the ordered Jordan–Hölder basis $\{X_i\}$ for $\mathfrak{g}_0$ defines global “canonical coordinates of the second kind” on G : An element g of G may be uniquely written as

$$g = \exp z_1 X_1 \cdots \exp z_d X_d \equiv g(z),$$

where $z = (z_1, \dots, z_d) \in \mathbf{C}^d$. Let F be an entire function on $\mathbf{C}^d$ which satisfies

$$(4.13) \quad |F(z_1, \dots, z_d)| \leq C \exp \left\{ -A \sum_i |x_i|^{p_i} + B \sum_i |y_i|^{p_i} \right\},$$

where $z_i = x_i + iy_i$, and A, B, C are positive constants. Assume that $p_{i+1} \geq p_i > 1$. If f is the holomorphic function on G such that $f(g(z)) = F(z)$, then by estimate (5.7) of [7],

$$(4.14) \quad \begin{aligned} & |f(\exp(\xi X_i)g(x))| \\ & \leq C_1 \exp \left\{ -A_1 \sum_i |x_i|^{p_i} + B_1 \sum_i |x_i|^{mp_i-1} + C_1 |\xi|^q \right\}, \end{aligned}$$

where $x \in \mathbf{R}^d$, $\xi \in \mathbf{C}$, $q = np_d$, $p_0 = 0$. Here m and n are integers depending on G_0 , and A_1, B_1, C_1 are positive constants depending on A, B, C , and $\{p_i\}$.

Suppose now that $\{p_i\}$ satisfies

$$(4.5) \quad p_i > mp_{i-1}, \quad 2 \leq i \leq d$$

as well as the condition $p_i > p_{i-1} > 1$. Then by estimate (4.14) and the fact that Haar measure on G_0 is given by Lebesgue measure on $\mathbf{R}^d$ in the coordinates

$g(x)$, we see that $\zeta \rightarrow f(\exp(\zeta X_i)g)$ is an entire function from $\mathbf{C}$ to $L_1(G_0)$ of exponential order $\leq q$. We may assume $q > 1$. Set $\nu = (2q - 1)/(2q)$, and let μ_i be defined in terms of ν as in Theorem 4.1. Then $\frac{1}{2} < \nu < 1$ and $\mu_i \geq 1 - 1/q$. Hence $f \in \mathcal{F}_\nu$.

To complete the proof of Theorem 4.2, it suffices to show that the space of entire functions F satisfying an estimate (4.13) (where $p_i > 1$ are fixed, and A, B, C are positive constants which may depend on F) is dense in $L_1(R^d)$. This space is the space $S_\alpha^\beta(\mathbf{R}^d)$ of Gel'fand-Shilov [3], where $\alpha = (\alpha_1, \dots, \alpha_d)$, $\beta = (\beta_1, \dots, \beta_d)$, with $\alpha_i = 1/p_i$ and $\beta_i = 1 - \alpha_i$. Since $0 < \alpha_i < 1$, we have $S_\alpha^\beta(\mathbf{R}^d)$ dense in $L_1(\mathbf{R}^d)$ by the results of §8.4 of Chapter IV of [3]. Q.E.D.

§5. Generalized P.-B.-W. Theorem for nilpotent $\mathfrak{g}$. Let $\mathfrak{g}$ be a complex nilpotent Lie algebra, $\mathfrak{g}_0$ a real form of $\mathfrak{g}$, and $\{X_i\}$ an ordered Jordan-Hölder basis for $\mathfrak{g}_0$. Let $K = (k_1, \dots, k_d)$ be a multi-exponent, and let $X^K = X_1^{k_1} \cdots X_d^{k_d}$ (product in $\mathfrak{J}(\mathfrak{g})$). If we let

$$\mathfrak{M} = \text{span } \{X^K\}_{0 \leq |K| < \infty},$$

then by the Poincaré-Birkhoff-Witt Theorem

$$(5.1) \quad \mathfrak{J}(\mathfrak{g}) = \mathfrak{M} + \mathfrak{R}(\mathfrak{g}) \quad (\text{direct sum}).$$

The next theorem gives a partial generalization of (5.1) to the algebra $\mathfrak{J}_\lambda(\mathfrak{g})$, $\lambda < 1$.

Theorem 5.1. *Suppose $\mathfrak{g}$ is nilpotent and $\{X_i\}$ is an ordered Jordan-Hölder basis for $\mathfrak{g}$. Then there exists an integer $m \geq 0$ with the following property:*

If $0 < \mu < 1$ and $\mathfrak{M}_\mu$ is the closed linear span in $\mathfrak{J}_\mu(\mathfrak{g})$ of the monomials X^K , $0 \leq |K| < \infty$, then

$$(5.2) \quad \mathfrak{J}_\lambda(\mathfrak{g}) \subseteq \mathfrak{M}_\mu + \mathfrak{R}_\mu(\mathfrak{g}),$$

where $\lambda = (m + \mu)/(m + 1)$.

Remark. The proof of Theorem 5.1 shows that we may take $m = m_d$, where the integers m_i are defined by equation (4.3). Thus $m = 0$ if $\mathfrak{g}$ is abelian, but $m \geq 1$ otherwise.

Proof of Theorem 5.1. Let $\mathfrak{h}, X, n$ be as in the proof of Theorem 4.1. We shall show that if $0 < \mu < 1$ and $\lambda = (n + \mu)/(n + 1)$, then every element $T \in \mathfrak{J}_\lambda(\mathfrak{g})$ may be written as

$$(5.3) \quad T = \sum_{k=0}^{\infty} T_k X^k \pmod{\mathfrak{R}_\mu(\mathfrak{g})},$$

where $T_k \in \mathfrak{J}_\mu(\mathfrak{h})$ and the sum (5.3) converges absolutely in $\mathfrak{J}_\mu(\mathfrak{g})$. Theorem 5.1 then follows easily by induction on $\dim \mathfrak{g}$.

To facilitate the proof of (5.3), we introduce the following notation: Set $Y_i = X_i$, $1 \leq j \leq d - 1$ ($d = \dim \mathfrak{g}$). If $A = \{\alpha_1, \dots, \alpha_p\}$ is an ordered set

of p multi-indices (each α_i a multi-index with $\alpha_i = \{r_1, \dots, r_a\}$, $1 \leq r_k \leq d-1$), define $|A| = \sum |\alpha_i|$, $\ell(A) = p$, and

$$\oplus A = \alpha_1 \oplus \dots \oplus \alpha_p \quad (\text{juxtaposition}).$$

If $B = \{\beta_1, \dots, \beta_a\}$ is another ordered set of multi-indices, define $B \sim A$ if $p = q$ and $|\alpha_i| = |\beta_i|$ for each j . If $K \in \mathbf{N}^p$ is a multi-exponent, define $\ell(K) = p$. Finally, if A is an ordered set of multi-indices and K is a multi-exponent with $\ell(A) = \ell(K) = p$, define

$$\{X; Y\}_A^K = X^{k_1} Y_{\alpha_1} \dots X^{k_p} Y_{\alpha_p}.$$

In the notation just defined we can write formula (4.6) of [6] as

$$(5.4) \quad \{X; Y\}_A^K = \sum_{J, B} \left\{ \begin{matrix} K \\ J \end{matrix} \right\} C_{A, B}^J Y_{\oplus B} X^{|K-J|} \mod \mathfrak{O}(\mathfrak{g}).$$

Here the summation is over all ordered sets B of multi-indices such that $B \sim A$, and over all multi-exponents J with $\ell(J) = \ell(K)$. The coefficients are given by:

$$\left\{ \begin{matrix} K \\ J \end{matrix} \right\} = \binom{k_1}{j_1} \binom{k_1 + k_2 - j_1}{j_2} \dots \binom{k_1 + \dots + k_p - j_1 - \dots - j_{p-1}}{j_p},$$

if $K = (k_1, \dots, k_p)$, $J = (j_1, \dots, j_p)$, where $\binom{a}{b}$ is the binomial coefficient, defined to be zero if $b > a$, and

$$C_{A, B}^J = C_{\alpha_1, \beta_1}^{j_1} \dots C_{\alpha_p, \beta_p}^{j_p},$$

where $\{C_{\alpha, \beta}^i\}$ is the matrix for D_X^i acting on $\mathfrak{V}^{|\alpha|}(\mathfrak{g})$.

Suppose now that $T \in \mathfrak{V}_\lambda(\mathfrak{g})$. Then we can write T uniquely as

$$(5.5) \quad T = T_0 + \sum d_A^K \{X; Y\}_A^K.$$

Here $T_0 \in \mathfrak{V}_\lambda(\mathfrak{h})$, and the sum is over all pairs K, A such that $K = (k_1, \dots, k_p)$ is a multi-exponent, $k_i > 0$ for each j , A is a set of multi-indices, and $1 \leq \ell(K) = \ell(A) < \infty$. If we can show that the series, obtained from (5.5) by formally substituting for $\{X; Y\}_A^K$ the expression on the right side of (5.4), converges absolutely in $\mathfrak{V}_\mu(\mathfrak{g})$, then we can sum this series according to powers of X and obtain (5.3). The series in question is

$$(5.6) \quad \sum \left\{ \begin{matrix} K \\ J \end{matrix} \right\} C_{A, B}^J d_A^K Y_{\oplus B} X^{|K-J|},$$

where the summation is over all multi-exponents J, K and sets of multi-indices A, B . (Define $\left\{ \begin{matrix} K \\ J \end{matrix} \right\} = C_{A, B}^J d_A^K = 0$ unless $\ell(K) = \ell(J) = \ell(A)$ and $B \sim A$.) Since

$$\|Y_{\oplus B} X^k\|_{\mu, r} \leq (re^\mu)^{|B|+k} |B|^{\mu|B|} k^{\mu k},$$

it suffices to establish the finiteness of the numerical series

$$(5.7) \quad \sum \left\{ \begin{matrix} K \\ J \end{matrix} \right\} |C_{A, B}^J d_A^K| r^{|B|+|K-J|} |B|^{\mu|B|} |K-J|^{\mu|K-J|}$$

for all $r > 0$, where the range of summation in (5.7) is the same as in (5.6).

To estimate (5.7), we recall from the proof of Lemma 4.2 of [6] that

$$\left\{ \begin{matrix} K \\ J \end{matrix} \right\} \leq \frac{|K|!}{|K-J|!J!}$$

and that

$$\sum_{|J|=j} (j!)^{-1} |C_{A,B}^J| \leq (j!)^{-1} a_{|A|,j},$$

where $a_{m,j}$ is as in §4. Since $|B| = |A|$ in the non-zero terms of (5.7) and there are $(\dim \mathfrak{h})^a$ multi-indices of length a , it follows from the above estimates that it suffices to establish the finiteness of the sum

$$(5.8) \quad \sum_{A,K} b_{|A|,|K|} |d_A^K| r^{|A|+|K|} |A|^{\mu|A|} |K|^{|K|}$$

for all $r > 0$, where $b_{a,k}$ is given by (4.6). But $T \in \mathfrak{I}_\lambda(\mathfrak{g})$, so that

$$\sum_{A,K} |d_A^K| r^{|A|+|K|} |A|^{\lambda|A|} |K|^{\lambda|K|} < \infty$$

for all $r > 0$. Hence the sum (5.8) will be finite if we can establish an estimate

$$(5.9) \quad b_{a,k} a^{(\mu-\lambda)a} k^{(1-\lambda)k} \leq C^{a+k}$$

for some constant C and all integers a, k .

Recall from estimate (4.9) of [6] that

$$(5.10) \quad b_{a,k} \leq \begin{cases} C^{a+k}, & \text{when } k \leq an \\ C^{a+k} (k - an)^{(\mu-1)(k-an)}, & \text{when } k > an, \end{cases}$$

with n determined by $ad X$ as before. Let $\epsilon > 0$. Then by (5.10) we have

$$(5.11) \quad \sum_{a,k} \epsilon^{a+k} b_{a,k} a^{(\mu-\lambda)a} k^{(1-\lambda)k} \leq \sum_{k \leq an} (C_1 \epsilon)^{a+k} a^{\sigma a} + \sum_{a,k \geq 0} (C_1 \epsilon)^{a+k} a^{\sigma a} k^{(\mu-\lambda)k},$$

where $\sigma = (1 - \lambda)n + (\mu - \lambda)$ and C_1 is a constant (independent of ϵ). But $\lambda = (n + \mu)/(n + 1)$, so $\sigma = 0$ and $\mu - \lambda \leq 0$. Hence series (5.11) converges for ϵ sufficiently small, which implies that estimate (5.9) holds for C sufficiently large, finishing the proof of Theorem 5.1.

The shift $\lambda \rightarrow \mu$ which occurs in Theorem 5.1 precludes the use of (5.2) in studying the algebra $\mathfrak{Q}_\lambda(\mathfrak{g})$, $\lambda < 1$. To eliminate this difficulty, we first observe that the proof of Theorem 5.1 also proves the following:

Corollary 5.1. *If $T \in \bigcap_{\lambda < 1} \mathfrak{I}_\lambda(\mathfrak{g})$, then there exists $M \in \bigcap_{\lambda < 1} \mathfrak{M}_\lambda$ such that $T - M \in \bigcap_{\lambda < 1} \mathfrak{Q}_\lambda(\mathfrak{g})$.*

Next we establish a generalization of the other half of (5.1) ($\mathfrak{M}_\lambda$ as in Theorem 5.1):

Theorem 5.2. *Let L be the left regular representation of G_0 on $L_2(G_0)$, and*

let $\lambda < 1$ be as in Theorem 4.2. Denote by φ_λ the associated representation of $\mathfrak{J}_\lambda(\mathfrak{g})$ on $S_\lambda(L)$. Then φ_λ is faithful on $\mathfrak{N}_\lambda$. In particular,

$$(5.12) \quad \mathfrak{N}_\lambda \cap \mathfrak{O}_\lambda(\mathfrak{g}) = 0.$$

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\beta = (\beta_1, \dots, \beta_d)$ and the space $S_\alpha^\beta(\mathbf{R}^d)$ be as in the proof of Theorem 4.2. Let $g(x) = \exp x_1 X_1 \cdots \exp x_d X_d$, $x \in \mathbf{R}^d$. If $F \in S_\alpha^\beta(\mathbf{R}^d)$ and f is the function on G_0 given by $f(g(x)) = F(x)$, then we claim that f lies in the space $S_{\lambda_0}(L)$ (λ_0 determined as in the proof of Theorem 4.2). Indeed, the proof of Theorem 4.2 shows that

$$z \rightarrow L(\exp zX_i)f$$

is an entire function from $\mathbf{C}$ to $L_2(G)$ of order $\leq (1 - \mu_i)^{-1}$, $j = 1, 2, \dots, d$, so we can apply Corollary 4.1.

Suppose now that $T = \sum a_K X^K \in \mathfrak{N}_{\lambda_0}$ and $\varphi_{\lambda_0}(T) = 0$. Applying $\varphi_{\lambda_0}(T)$ to functions $f \in S_{\lambda_0}(L)$ as above and using equation (3.5), we obtain the relation

$$(5.13) \quad \sum (-1)^{|K|} a_K (\partial/\partial x)^K F(0) = 0$$

for all $F \in S_\alpha^\beta(\mathbf{R}^d)$. (The number λ_0 determined in the proof of Theorem 4.2 satisfies $\lambda_0 \geq \beta_i$; since $T \in \mathfrak{N}_{\lambda_0}$ it follows that the series (5.13) is absolutely convergent for any $F \in S_\alpha^\beta(\mathbf{R}^d)$.) The space $S_\alpha^\beta(\mathbf{R}^d)$ is translation-invariant (with respect to the action of the additive group $\mathbf{R}^d$). Since the coefficients a_K in (5.13) are scalars, we conclude that

$$(5.14) \quad \sum (-1)^{|K|} a_K (\partial/\partial x)^K F = 0$$

for all $F \in S_\alpha^\beta(\mathbf{R}^d)$. Choosing $F \neq 0$ and taking the (Euclidean) Fourier transform of (5.14), we see that the entire function $G(\zeta) = \sum a_K \zeta^K$, $\zeta \in \mathbf{C}^d$, must vanish identically. Hence $a_K = 0$ for all K . Q.E.D.

As a consequence of Corollary 5.1 and Theorem 5.2, we obtain the following complete generalization of (5.1):

Corollary 5.2. *If $\mathfrak{g}$ is a nilpotent Lie algebra, then*

$$(5.15) \quad \bigcap_{\lambda < 1} \mathfrak{J}_\lambda(\mathfrak{g}) = \bigcap_{\lambda < 1} \mathfrak{N}_\lambda + \bigcap_{\lambda < 1} \mathfrak{O}_\lambda(\mathfrak{g}) \quad (\text{direct sum}).$$

Motivated by Corollary 5.2, we introduce the algebra

$$(5.16) \quad \mathcal{E}(\mathfrak{g}) \equiv \bigcap_{\lambda < 1} \mathfrak{J}_\lambda(\mathfrak{g}) / \bigcap_{\lambda < 1} \mathfrak{O}_\lambda(\mathfrak{g}).$$

By the proof of Proposition 1.1 we have $\mathfrak{U}(\mathfrak{g})$ naturally embedded in $\mathcal{E}(\mathfrak{g})$. The collection $\{\|\cdot\|_{\lambda, r}\}_{0 < \lambda < 1, 0 < r < \infty}$ of norms on $\bigcap_{\lambda < 1} \mathfrak{J}_\lambda(\mathfrak{g})$ make this space into a Fréchet algebra (the projective limit of the algebras $\mathfrak{J}_\lambda(\mathfrak{g})$, $\lambda < 1$), and $\bigcap_{\lambda < 1} \mathfrak{O}_\lambda(\mathfrak{g})$ is a closed ideal. We give $\mathcal{E}(\mathfrak{g})$ the quotient topology. Then by (5.15) and the closed graph theorem, $\mathcal{E}(\mathfrak{g})$ is isomorphic as a topological vector space to the closed subspace $\bigcap_{\lambda < 1} \mathfrak{N}_\lambda$ of $\bigcap_{\lambda < 1} \mathfrak{J}_\lambda(\mathfrak{g})$. Equivalently, $\mathcal{E}(\mathfrak{g})$ may be identified as the completion of $\mathfrak{U}(\mathfrak{g})$ relative to the family of norms

$$(5.17) \quad \|\sum C_K X^K\|_\lambda = \sum |C_K| |K|^{\lambda|K|},$$

$0 < \lambda < 1$ (allowing λ to vary makes the parameter r irrelevant; we set $r = 1$ for convenience). In (5.17) the monomial X^K is defined in $\mathfrak{U}(\mathfrak{g})$, relative to the ordered Jordan–Hölder basis $\{X_i\}$.

Remark. Since $\bigcap_{\lambda < 1} \mathfrak{N}_\lambda$ is not stable under $\text{Aut } \mathfrak{g}$, we cannot use the direct sum decomposition (5.15) to obtain the structure of $\mathcal{E}(\mathfrak{g})$ as an $\text{Aut } (\mathfrak{g})$ module, and we have no useful analogue of Corollary 2.2 for $\mathcal{E}(\mathfrak{g})$.

§6. Representations of $\mathcal{E}(\mathfrak{g})$, $\mathfrak{g}$ nilpotent. In this section we continue the study of the algebra $\mathcal{E}(\mathfrak{g})$ introduced in §5, with $\mathfrak{g}$ a complex nilpotent Lie algebra. Let G_0 and G be as in §4. If π is a strongly continuous Banach space representation of G_0 on $\mathcal{H}(\pi)$, let $\mathcal{H}_\infty^\omega(\pi)$ be the subspace of entire vectors for π , and let π_ω be the holomorphic representation of G on $\mathcal{H}_\infty^\omega(\pi)$ obtained from π by analytic continuation (cf. [5] and [7]).

Lemma 6.1. *If $\lambda < 1$, then $\pi_\omega(G)$ leaves the spaces $S_{\lambda,t}(\pi)$ invariant, and the function $g \rightarrow \pi_\omega(g)v$, $v \in S_\lambda(\pi)$, is holomorphic from G to $S_\lambda(\pi)$.*

Proof. Let $X \in \mathfrak{g}$, and let $E(X) = \sum (n!)^{-1} X^n$, considered as an element of $\mathfrak{F}_\lambda(\mathfrak{g})$. Then by standard inequalities from the theory of entire functions, we have

$$(6.1) \quad \|E(X)\|_{\lambda,r} \leq A \exp \{B(r|X|)^q\},$$

where $q = (1 - \lambda)^{-1}$ and A, B are constants depending only on λ and the choice of basis for $\mathfrak{g}$. It follows from estimate (3.3) that if $g = \exp X$, then

$$\pi_\omega(g) : S_{\lambda,t}(\pi) \rightarrow S_{\lambda,t}(\pi),$$

and that for $0 < s < r < t$, one has

$$(6.2) \quad \|\pi_\omega(\exp X)v\|_{\lambda,r} \leq A \exp \{B(\delta s|X|)^q\} \|v\|_{\lambda,s},$$

where $\delta = r^\lambda(r - s)^{-\lambda}$. Furthermore, the series $\sum (n!)^{-1} \partial \pi(X)^n v$ is absolutely convergent in $S_{\lambda,t}(\pi)$ for any $v \in S_{\lambda,t}(\pi)$. Q.E.D.

Let $\mathcal{Q}(G)$ be the convolution algebra of analytic functionals on G , studied in [12] and [13]. The representation π_ω of G on $\mathcal{H}_\infty^\omega(\pi)$ extends uniquely to a representation of $\mathcal{Q}(G)$ on $\mathcal{H}_\infty^\omega(\pi)$, which we shall also denote by π_ω . By Lemma 6.1 and the results of [13], $\mathcal{Q}(G)$ leaves $S_\lambda(\pi)$ invariant, $\lambda < 1$. The algebra $\mathcal{E}(\mathfrak{g})$ also acts continuously on $S_\lambda(\pi)$, $\lambda < 1$, via the natural homomorphism of $\mathcal{E}(\mathfrak{g})$ into $\mathcal{Q}_\lambda(\mathfrak{g})$ composed with the representation Φ_λ of Proposition 3.1.

Proposition 6.1. *There is a unique injective homomorphism $\gamma : \mathcal{Q}(G) \rightarrow \mathcal{E}(\mathfrak{g})$ with the following property:*

If π is a Banach-space representation of G_0 and $T \in \mathcal{Q}(G)$, then the action of $\pi_\omega(T)$ on $S_\lambda(\pi)$, $\lambda < 1$, coincides with the action of $\gamma(T)$ on $S_\lambda(\pi)$.

Remarks. 1. In particular, the complex group G is embedded in both $\mathcal{Q}(G)$

and $\mathfrak{E}(\mathfrak{g})$, where $g = \exp X$ corresponds to the point mass $\delta_x \in \mathfrak{A}(G)$ and the element $\sum (n!)^{-1} X^n$ of $\mathfrak{E}(\mathfrak{g})$.

2. The proof of Proposition 6.1 shows that $\gamma(\mathfrak{A}(G))$ is a proper subalgebra of $\mathfrak{E}(\mathfrak{g})$.

Proof of Proposition 6.1. Let $\{X_i\}$ be the Jordan-Hölder basis for $\mathfrak{g}_0$ employed in §4, and set $g(t) = \exp t_1 X_1 \cdots \exp t_d X_d$, $t \in \mathbb{C}^d$. This gives a global holomorphic parametrization of G (cf. [16], Part II, Ch. II, §2). If v is an entire vector for the representation π , let $\vartheta(g) = \pi_\omega(g)v$. Then by definition $\pi_\omega(T)v = T(\vartheta)$, for $T \in \mathfrak{A}(G)$. Now

$$(\partial/\partial t)^K \vartheta(g(t))|_{t=0} = \partial\pi(X^K)v.$$

Hence by Taylor's formula we have

$$(6.2) \quad \pi_\omega(T)v = \sum c_K \partial\pi(X^K)v,$$

where $c_K = T(f_K)$, f_K being the entire function on G given by $f_K(g(t)) = (K!)^{-1} t^K$.

Since T is a continuous linear function on the space of entire functions on G , there exists an $r > 0$ and a constant M such that

$$|T(f_K)| \leq M(K!)^{-1} \max_{|t|=r} |t^K| = Mr^{1(K)}(K!)^{-1}.$$

Hence

$$(6.3) \quad \sum c_K s^{1(K)} K! < \infty$$

for some $s > 0$. Define

$$\tilde{T} = \sum c_K X^K$$

(X^K now being defined in $\mathfrak{J}(\mathfrak{g})$). By (6.3) we have $\tilde{T} \in \bigcap_{\lambda < 1} \mathfrak{J}_\lambda(\mathfrak{g})$, and we let $\gamma(T)$ be the image of $\tilde{T}$ in $\mathfrak{E}(\mathfrak{g})$ under the quotient map.

It follows from (6.2) that the action of $\gamma(T)$ on $S_\lambda(\pi)$, $\lambda < 1$, agrees with that of $\pi_\omega(T)$. The map γ is obviously linear, and by Corollary 5.2 it is injective. Let L be the left regular representation of G_0 on $L_2(G_0)$, and let λ be as in Theorem 4.2. Then the representation of $\mathfrak{E}(\mathfrak{g})$ on $S_\lambda(L)$ is faithful (Theorem 5.2 and Corollary 5.2). It follows by the above remarks that γ is multiplicative. Q.E.D.

While the results of [13] indicate that $\mathfrak{A}(G)$ is a natural group algebra to employ in connection with arbitrary holomorphic representations of G , the larger algebra $\mathfrak{E}(\mathfrak{g})$ seems more closely connected with the holomorphic representations on $S_\lambda(\pi)$, $\lambda < 1$, obtained from a unitary representation π of the real form G_0 . In particular, $\mathfrak{E}(\mathfrak{g})$ is semi-simple relative to such representations:

Theorem 6.1. *Let $T \in \mathfrak{E}(\mathfrak{g})$, $T \neq 0$. Then there exists $\lambda < 1$ and an irreducible unitary representation π of G_0 such that T acts non-trivially on $S_\lambda(\pi)$.*

Proof. By Theorem 5.2 and Corollary 5.2, the left regular representation

L of G_0 on $L_2(G_0)$ furnishes a faithful representation of $\mathfrak{E}(\mathfrak{g})$ on $S_\lambda(L)$ for some $\lambda < 1$. Apply the proof of Corollary 3.1 to this representation.

§7. Some applications. In this section we return to the case of an arbitrary finite-dimensional complex Lie algebra $\mathfrak{g}$, with real form $\mathfrak{g}_0$ and corresponding simply-connected group G_0 . We will illustrate some uses of the algebras $\mathfrak{A}_\lambda(\mathfrak{g})$ in the study of operators of the form $\partial\pi(T)$, where $T \in \mathfrak{U}(\mathfrak{g})$ and π is a strongly continuous representation of G_0 on Banach space.

One of the basic goals of operator theory is the construction of an "operational calculus," i.e., given an operator A , one seeks a homomorphism $f \rightarrow f(A)$ from some algebra of functions $f(z)$ of one variable to operators, such that for the function $f(z) = z$, one has $f(A) = A$. In connection with the operators $A = \partial\pi(T)$, we will consider the following algebras of functions:

Let $E_p(\mathbf{C})$, $0 < p < \infty$, be the space of entire functions $f(z)$ of one complex variable, for which

$$\|f\|_{p,r} \equiv \sup_z |f(z)| \exp(-r|z|^p) < \infty$$

for all $r > 0$. Give $E_p(\mathbf{C})$ the locally convex topology defined by the family of norms $\{\|f\|_{p,r}\}_{r>0}$. In this topology $E_p(\mathbf{C})$ is a Fréchet algebra under pointwise multiplication.

Proposition 7.1. *Let $T \in \mathfrak{U}(\mathfrak{g})$ be of degree N , and let $0 < p < \infty$. There exists a unique continuous homomorphism*

$$\omega_T : E_p(\mathbf{C}) \rightarrow \mathfrak{A}_\lambda(\mathfrak{g}),$$

where $\lambda = (Np)^{-1}$, such that $\omega_T(1) = 1$ and $\omega_T(z) = T$.

Proof. If $f \in E_p(\mathbf{C})$, $f(z) = \sum a_n z^n$, then by the Cauchy inequalities

$$(7.1) \quad |a_n| \leq (rep)^{n/p} n^{-n/p} \|f\|_{p,r}$$

(cf. [11]), and the power series for f converges in $E_p(\mathbf{C})$. Hence if ω_T exists, we must have

$$(7.2) \quad \omega_T(f) = \sum_n a_n T^n.$$

On the other hand, since T is of degree N , elementary estimates show that

$$(7.3) \quad \|T^n\|_{\lambda,r} \leq C^n (1+r)^{nN} n^{\lambda n N}$$

(where we consider $\mathfrak{U}(\mathfrak{g})$ embedded in $\mathfrak{A}_\lambda(\mathfrak{g})$, and the norm in (7.3) is obtained from (1.1) by passing to the quotient). If $\lambda N = 1/p$, then (7.1) and (7.3) show that the series (7.2) converges absolutely in $\mathfrak{A}_\lambda(\mathfrak{g})$, and $f \rightarrow \omega_T(f)$ is continuous. Since it is a homomorphism on the dense subspace of polynomials, it is a homomorphism on $E_p(\mathbf{C})$. Q.E.D.

Let A be an (unbounded) operator on a Banach space $\mathcal{H}$ with domain $\mathcal{D}(A)$,

and let $f(z) = \sum a_n z^n$ be a formal power series. Define the operator $f(A)$ on $\mathcal{H}$ by

$$f(A)v = \sum a_n A^n v,$$

where $\mathcal{D}(f(A))$ consists of all vectors $v \in \bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$ such that $\sum |a_n| \|A^n v\| < \infty$. Suppose now that $A = \partial\pi(T)$, where $T \in \mathcal{U}(\mathfrak{g})$ is of degree N , and π is a strongly continuous representation of the group G_0 on $\mathcal{H}$. The “abstract” operational calculus of Proposition 7.1 together with the results of §3 concerning representations of the algebras $\mathcal{A}_\lambda(\mathfrak{g})$ yield the following information about the operator $f(A)$ when $f \in E_p(\mathbb{C})$:

Proposition 7.2. *If $f \in E_p(\mathbb{C})$ and $\lambda = (Np)^{-1}$, then $\mathcal{D}(f(A)) \supseteq S_\lambda(\pi)$, and*

$$f(A) : S_{\lambda,t}(\pi) \rightarrow S_{\lambda,t}(\pi)$$

continuously for every $t > 0$. The map $f \rightarrow f(A)|_{S_{\lambda,t}(\pi)}$ is a continuous homomorphism from $E_p(\mathbb{C})$ to $\mathcal{L}_b(S_{\lambda,t}(\pi))$.

In particular, since the space $S_1(\pi)$ of analytic vectors for π is always dense in $\mathcal{H}(\pi)$ [14], we obtain

Corollary 7.1. *If $f \in E_p(\mathbb{C})$ and $Np \leq 1$, then $f(A)$ is densely defined.*

Remark. If $f(z)$ is an entire function of order p and finite (non-zero) type, i.e., $|f(z)| \leq C \exp(B|z|^p)$ for some B , then we still have $\mathcal{D}(f(A)) \supseteq S_{\lambda,t}(\pi)$, with $\lambda = (Np)^{-1}$ and t depending on B . Thus, e.g., if π is unitary and $Np = 1$, then $f(A)$ is still densely defined provided B is sufficiently small (since $S_{1,t}(\pi)$ is dense in $\mathcal{H}(\pi)$ for t sufficiently large; cf. [5]).

Consider now the “abstract Cauchy problem”

$$(7.4) \quad \begin{cases} (d/dt)^m u(t) = Au(t) \\ (d/dt)^k u(0) = u_k, \quad 0 \leq k \leq m-1, \end{cases}$$

where $A = \partial\pi(T)$, $T \in \mathcal{U}(\mathfrak{g})$ of degree N as above.

Theorem 7.1. *Assume that the initial data $\{u_k\}$ in (7.4) lie in the space $S_{\lambda,r}(\pi)$.*

(i) *If $\lambda N = m$, then there exists a unique solution $t \rightarrow u(t)$ of (7.4) in the space $S_\lambda(\pi)$, defined for*

$$|t| < C(1+r)^{-N},$$

where C is a constant independent of $\{u_k\}$ and r .

(ii) *If $\lambda N < m$, then there exists a unique solution $t \rightarrow u(t)$ of (7.4) in the space $S_{\lambda,r}(\pi)$, defined for all t .*

In both cases the solution is analytic in t and depends continuously on $\{u_k\}$ in the topology of $S_\lambda(\pi)$.

Proof. If $t \rightarrow u(t)$ is a map of class C^m from a neighborhood of 0 in $\mathbb{R}$ to $S_\lambda(\pi)$, satisfying (7.4), then it is automatically C^∞ , and the higher derivatives satisfy the equations

$$(7.5) \quad u^{(mn+k)}(t) = A^n u^{(k)}(t)$$

$(u^{(n)}(t) \equiv (d/dt)^n u(t))$, since $A : S_{\lambda,r}(\pi) \rightarrow S_{\lambda,r}(\pi)$ continuously ([6], Theorem 2.1). Thus the Taylor series of $u(t)$ at 0 is formally given by

$$(7.6) \quad \sum_{k=0}^{m-1} t^k \sum_{n=0}^{\infty} \frac{t^{mn}}{(mn+k)!} A^n u_k.$$

Conversely, if $\lambda N \leq m$, then we can establish the convergence of the series (7.6) as follows:

Consider first the case $\lambda N < m$. The function

$$f_{m,k}(z) = \sum_{n=0}^{\infty} \frac{z^n}{(mn+k)!}$$

is entire of exponential order $1/m$. Hence by Proposition 7.2 the operator $f_{m,k}(A)$ leaves the space $S_{\lambda,r}(\pi)$ invariant. Set

$$u(t) = \sum_{k=0}^{m-1} t^k f_{m,k}(t^m A) u_k.$$

Then $u(t) \in S_{\lambda,r}(\pi)$ for all t , and the series defining $f_{m,k}(t^m A) u_k$ converges absolutely in $S_{\lambda,r}(\pi)$. Hence $u(t)$ is also given by the series (7.6), and satisfies (7.4).

In the case $\lambda N = m$, we obtain from (3.3) and (7.3) the estimate

$$(7.7) \quad \|A^n v\|_{\lambda,2s} \leq C^n (1+s)^{nN} n^{mn} \|v\|_{\lambda,s},$$

where C is a constant independent of v and s . This shows that the series (7.6) converges absolutely in $S_{\lambda}(\pi)$ provided $|t| < C_1(1+r)^{-N}$, C_1 a constant independent of r and $\{u_k\}$. The sum $u(t)$ is analytic in t and satisfies (7.4).

To establish uniqueness, it suffices by the above to show that any C^∞ solution $u(t)$ of (7.4) in the space $S_{\lambda}(\pi)$, $\lambda \leq m/N$, is necessarily analytic in t . Let the domain of definition of $u(t)$ include the open interval $(t_0 - \epsilon, t_0 + \epsilon)$. The set $\{u^{(k)}(t) : |t - t_0| \leq \epsilon/2, 0 \leq k \leq m-1\}$ is compact in $S_{\lambda}(\pi)$, hence bounded. By Lemma 1.2 of [6] we conclude that there exists an $s > 0$ and a constant C such that

$$(7.8) \quad \sup_{\substack{|t-t_0| \leq \epsilon/2 \\ 0 \leq k \leq m-1}} \|u^{(k)}(t)\|_{\lambda,s} \leq C.$$

Now using the equation $u^{(mn+k)}(t) = A^n u^{(k)}(t)$ and Taylor's formula, we can write

$$u^{(mn+k)}(t) = \sum_{j=0}^{m-k-1} \frac{t^j}{j!} A^n u^{(k+j)}(t_0) + \frac{1}{(m-k-1)!} \int_{t_0}^t (t-\tau)^{m-k-1} A^{n+1} u(\tau) d\tau.$$

Hence by estimates (7.7) and (7.8), we conclude that

$$\sup_{\substack{|t-t_0| \leq \epsilon/2 \\ 0 \leq k \leq m-1}} \|u^{(mn+k)}(t)\|_{\lambda,2s} \leq C^n n^{\lambda n N}$$

for some constant C . Since $\lambda N \leq m$, this last estimate shows that $u(t)$ is analytic on a neighborhood of t_0 . Q.E.D.

Remarks. 1. Theorem 7.1 can be used to obtain existence and uniqueness theorems for solutions to the adjoint Cauchy problem in spaces in duality with $S_\lambda(\pi)$, following the general procedure of [4], Chapter II, §2.

2. When $\mathfrak{g}$ is nilpotent and π is unitary and irreducible, then π may be realized on $L^2(\mathbb{R}^n)$, such that $\partial\pi(\mathfrak{U}(\mathfrak{g}))$ consists of the differential operators with polynomial coefficients. The spaces $S_\lambda(\pi)$ are then closely related to the spaces S_α^β of Gel'fand–Shilov, as was shown in §5 of [6]. In this context T. Yamanaka [18] has obtained some explicit uniqueness theorems for the Cauchy problem (7.4), using the general method of Gel'fand–Shilov. His techniques could probably be adapted to treat the abstract problem (7.4) with A depending on t .

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This research was supported in part by NSF Grant GP 22822.

For a treatment of the Sobolev inequality similar to that in reference [15], cf. L. Auslander & J. Brezin, *Uniform distribution in solmanifolds*, Advances in Math. **7** (1971), 111–144.

Massachusetts Institute of Technology

Date communicated: FEBRUARY 2, 1971