Slices and Potentials

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1. The operation of slicing an m dimensional normal current T in U by a locally Lipschitzian map f of U into $\mathbf{R}^n$, with $m \geq n$, was studied in [2, §3] and [3, §4.3] through relative differentiation of measures. It was shown that the slice $\langle T, f, y \rangle$ exists for $\mathcal{L}^n$ almost all points y in $\mathbf{R}^n$, as an $m - n$ dimensional flat chain in U , and is $\mathcal{L}^n$ summable with respect to y . Here it will be proved that the Hausdorff dimension of the exceptional set, consisting of those points y in $\mathbf{R}^n$ for which $\langle T, f, y \rangle$ fails to exist, never exceeds $n - 1$. The new method combines techniques from potential theory and geometric measure theory, and yields also a residue formula for slices.

We use the terminology of [3], which contains glossaries of basic notation on pages 669 to 671.

2. In this section we discuss a few facts which we need from potential theory. Suppose M is the set of all (nonnegative) Radon measures over $\mathbf{R}^n$ with compact support. Whenever $\mu \in M$ and $0 < s < \infty$ the potential P_μ^s is given by the formula

$$P_\mu^s(y) = \int |z - y|^{-s} d\mu z \quad \text{for } y \in \mathbf{R}^n,$$

where $0^{-s} = \infty$. Letting

$$M^s = M \cap \{\mu : P_\mu^s(y) \leq 1 \quad \text{for } y \in \text{spt } \mu\},$$

we define the s dimensional capacity first for open subsets V of $\mathbf{R}^n$ as

$$\text{Cap}^s(V) = \sup \{\mu(V) : \mu \in M^s \quad \text{and} \quad \text{spt } \mu \subset V\},$$

then for arbitrary subsets A of $\mathbf{R}^n$ as

$$\text{Cap}^s(A) = \inf \{\text{Cap}^s(V) : A \subset V \subset \mathbf{R}^n \quad \text{and} \quad V \text{ is open}\}.$$

The following nine propositions hold:

(1) *If V is an open subset of $\mathbf{R}^n$, then*

$$\text{Cap}^s(V) = \sup \{\mu(V) : \mu \in M^s\}.$$

- (2) If $A \subset \bigcup_{i=1}^{\infty} B_i \subset \mathbf{R}^n$, then $\text{Cap}^s(A) \leq \sum_{i=1}^{\infty} \text{Cap}^s(B_i)$.
- (3) If $\mu \in M$ and $\rho > 0$, then
- $$\sup \left\{ \int_{\mathbf{U}(y, \rho)} |z - y|^{-s} d\mu z : y \in \mathbf{R}^n \right\} \leq 2^s \sup \left\{ \int_{\mathbf{U}(y, 2\rho)} |z - y|^{-s} d\mu z : y \in \text{spt } \mu \right\}.$$
- (4) If $\mu \in M$ and $c > 0$, then $c \text{Cap}^s \{y : P_\mu^s(y) > c\} \leq 2^s \mu(\mathbf{R}^n)$.
- (5) If $\mu \in M$ and $H = \{y : P_\mu^s(y) = \infty\}$, then $\text{Cap}^s(H) = 0$ and for every $\epsilon > 0$ there exists a neighborhood V of H in $\mathbf{R}^n$ such that $\text{Cap}^s(V) < \epsilon$ and

$$\int_{\mathbf{U}(y, \rho)} |z - y|^{-s} d\mu z \downarrow 0, \quad \text{uniformly for } y \in \mathbf{R}^n \sim V, \quad \text{as } \rho \downarrow 0.$$

- (6) If V is an open subset of $\mathbf{R}^n$, then $\text{Cap}^s(V) \leq (\text{diam } V)^s$.
- (7) If $A \subset \mathbf{R}^n$ and $\mathcal{H}^s(A) < \infty$, then $\text{Cap}^s(A) = 0$.
- (8) If $A \subset \mathbf{R}^n$, $\text{Cap}^s(A) = 0$ and $s < t < \infty$, then $\mathcal{H}^t(A) = 0$.
- (9) If $A \subset \mathbf{R}^n$, then $\inf \{s : \text{Cap}^s(A) = 0\} = \inf \{s : \mathcal{H}^s(A) = 0\}$.

Certainly the above facts, with the possible exception of (5), can be learned from standard works on potential theory such as [4], [5], [1]. However, for the reader's convenience, we will include their simple proofs.

To verify (1) we consider any $\mu \in M^s$ and $k < \mu(V)$; choosing a compact subset C of V with $\mu(C) > k$ we see that

$$\mu \llcorner C \in M^s, \text{ spt } (\mu \llcorner C) \subset V, \text{ and } \text{Cap}^s(V) \geq (\mu \llcorner C)(V) = \mu(C) > k.$$

Using (1) one easily checks (2).

To prove (3) we suppose $y \in \mathbf{R}^n$ and $\rho > \text{dist}(y, \text{spt } \mu) = |y - w|$ with $w \in \text{spt } \mu$. Then $\mathbf{U}(y, \rho) \subset \mathbf{U}(w, 2\rho)$ and

$$|z - w| \leq |z - y| + |w - y| \leq 2|z - y| \quad \text{for all } z \in \text{spt } \mu,$$

$$\int_{\mathbf{U}(y, \rho)} |z - y|^{-s} d\mu z \leq \int_{\mathbf{U}(w, 2\rho)} 2^s |z - w|^{-s} d\mu z.$$

The set $W = \{y : P_\mu^s(y) > c\}$ occurring in (4) is open, because Fatou's lemma shows that P_μ^s is lower semicontinuous. If $\gamma \in M^s$ with $\text{spt } \gamma \subset W$, then (3) implies $P_\gamma^s(y) \leq 2^s$ for all $y \in \mathbf{R}^n$, and it follows from the Fubini theorem that

$$c\gamma(W) \leq \int P_\mu^s d\gamma = \int P_\gamma^s d\mu \leq 2^s \mu(\mathbf{R}^n).$$

To prove (5) we first choose an integer $i > 2^{s+1} \mu(\mathbf{R}^n) \epsilon^{-1}$ and infer from (4) that $W = \{y : P_\mu^s(y) > i\}$ is a neighborhood of H with $\text{Cap}^s(W) < \epsilon/2$. For each positive integer j we select a compact set C_j so that

$$C_j \subset H \quad \text{and} \quad \mu(H \sim C_j) < 2^{-2j-3} \epsilon.$$

Observing that

$$\int_{\mathbf{U}(\mathbf{y}, \rho)} |z - y|^{-s} d\mu z \downarrow 0 \quad \text{as } \rho \downarrow 0 \quad \text{whenever } y \in \mathbf{R}^n \sim H,$$

we also select a compact set K_i and a number ρ_i so that

$$\begin{aligned} K_i \subset \mathbf{R}^n \sim H, \quad \mu[(\mathbf{R}^n \sim H) \sim K_i] &< 2^{-2i-3}\epsilon, \\ 0 < \rho_i < \text{dist}(C_i, \mathbf{R}^n \sim W), \end{aligned}$$

$$\int_{\mathbf{U}(\mathbf{y}, 2\rho_i)} |z - y|^{-s} d\mu z \leq 2^{-i-1} \quad \text{whenever } y \in K_i.$$

Since $\text{spt}(\mu \llcorner K_i) \subset K_i$, it follows from (3) that

$$\int_{\mathbf{U}(\mathbf{y}, \rho_i)} |z - y|^{-s} d(\mu \llcorner K_i) z \leq 2^{s-i-1} \quad \text{whenever } y \in \mathbf{R}^n.$$

Defining $\gamma_i = \mu \llcorner [\mathbf{R}^n \sim (C_i \cup K_i)]$ and $Z_i = \{y : P_{\gamma_i}^s(y) > 2^{s-i-1}\}$ we infer from (4) that

$$\text{Cap}^s(Z_i) \leq 2^{i+1}\gamma_i(\mathbf{R}^n) < 2^{i+1}2 \cdot 2^{-2i-3}\epsilon = 2^{-i-1}\epsilon.$$

Now, if $y \in \mathbf{R}^n \sim (W \cup Z_i)$, then

$$\mathbf{U}(y, \rho_i) \cap C_i = \emptyset, \quad \mu \llcorner \mathbf{U}(y, \rho_i) \leq (\mu \llcorner K_i) + \gamma_i,$$

$$\int_{\mathbf{U}(\mathbf{y}, \rho_i)} |z - y|^{-s} d\mu z \leq 2^{s-i-1} + P_{\gamma_i}^s(y) \leq 2^{s-i}.$$

Accordingly the conclusion of (5) holds with

$$V = W \cup \bigcup_{i=1}^{\infty} Z_i, \quad \text{Cap}^s(V) \leq \text{Cap}^s(W) + \sum_{i=1}^{\infty} \text{Cap}^s(Z_i) < \epsilon.$$

To verify (6) we assume V bounded. Whenever $0 \neq \mu \in M^s$ and $\text{spt } \mu \subset V$, we choose $y \in \text{spt } \mu$ and infer

$$(\text{diam } V)^{-s}\mu(V) \leq P_{\mu}^s(y) \leq 1.$$

To prove (7) in case A is bounded we apply (5) with $\mu = \mathcal{H}^s \llcorner A$. We find that $\text{Cap}^s(H) = 0$, and

$$\limsup_{\rho \downarrow 0} \rho^{-s} \mathcal{H}^s[A \cap \mathbf{B}(y, \rho)] \leq \lim_{\rho \downarrow 0} \int_{\mathbf{B}(\mathbf{y}, \rho)} |z - y|^{-s} d\mu z = 0$$

whenever $y \in \mathbf{R}^n \sim H$, hence $\mathcal{H}^s(A \sim H) = 0$ by [3, §2.10.19(2)]. Using (6) and (2) we infer $\text{Cap}^s(A \sim H) = 0$.

To prove (8) we first choose a Borel set B with $A \subset B$ and $\text{Cap}^s(B) = 0$. Assuming $\mathcal{H}^t(B) > 0$ we could apply [3, §2.10.48] to secure a compact set $C \subset B$ with $0 < \mathcal{H}^t(C) < \infty$, infer from [3, §2.10.19(5)] that

$$\limsup_{\rho \downarrow 0} \rho^{-t} \mathcal{H}^t[C \cap \mathbf{B}(y, \rho)] \leq \alpha(t)$$

for $\mathcal{H}^t$ almost all y in C , and hence construct a compact set $K \subset C$ with $\mathcal{H}^t(K) > 0$, $\text{diam } K \leq 1$, and

$$\lambda = \sup \{ \rho^{-t} \mathcal{H}^t[C \cap B(y, \rho)] : y \in K, 0 < \rho \leq 1 \} < \infty.$$

Taking $\mu = \mathcal{H}^t \llcorner K$ we would find that

$$\begin{aligned} P_\mu^s(y) &\leq \sum_{i=1}^{\infty} \int_{\{z : 2^{-i} < |z-y| \leq 2^{1-i}\}} |z-y|^{-s} d\mu z \\ &\leq \sum_{i=1}^{\infty} 2^{is} \lambda 2^{(1-i)s} = \lambda / (2^{-s} - 2^{-t}) \end{aligned}$$

whenever $y \in K$, hence $\lambda^{-1}(2^{-s} - 2^{-t})\mu \in M^s$ and

$$\text{Cap}^s(B) \geq \text{Cap}^s(K) \geq \lambda^{-1}(2^{-s} - 2^{-t})\mu(K) > 0.$$

Finally, we deduce (9) from (7) and (8).

3. Suppose $Y_1, \dots, Y_n$ are the standard coordinate functions on $\mathbf{R}^n$. We define the differential forms $\Omega, \Omega_{y,\rho}$ of degree n and $\Psi_y, \Gamma_y, \Gamma_{y,\rho}$ of degree $n-1$ on $\mathbf{R}^n$ corresponding to $y \in \mathbf{R}^n$ and $0 < \rho < \infty$ as follows: If $z \in \mathbf{R}^n$, then

$$\begin{aligned} \Omega(z) &= Y_1 \wedge \dots \wedge Y_n, \\ \Omega_{y,\rho}(z) &= \alpha(n)^{-1} \rho^{-n} \Omega(z) \quad \text{in case } |z-y| \leq \rho, \\ \Omega_{y,\rho}(z) &= 0 \quad \text{in case } |z-y| > \rho, \\ \Psi_y(z) &= (z-y) \lrcorner (Y_1 \wedge \dots \wedge Y_n) \\ &= \sum_{i=1}^n (-1)^{i-1} (z_i - y_i) Y_1 \wedge \dots \wedge Y_{i-1} \wedge Y_{i+1} \wedge \dots \wedge Y_n, \\ \Gamma_y(z) &= n^{-1} \alpha(n)^{-1} |z-y|^{-n} \Psi_y(z) \quad \text{in case } z \neq y, \\ \Gamma_y(y) &= 0, \\ \Gamma_{y,\rho}(z) &= n^{-1} \alpha(n)^{-1} \rho^{-n} \Psi_y(z) \quad \text{in case } |z-y| \leq \rho, \\ \Gamma_{y,\rho}(z) &= \Gamma_y(z) \quad \text{in case } |z-y| > \rho. \end{aligned}$$

We note that

$$\begin{aligned} \|\Psi_y(z)\| &= |z-y| \quad \text{and} \quad d\Psi_y(z) = n\Omega(z), \\ \|\Gamma_y(z)\| &= n^{-1} \alpha(n)^{-1} |z-y|^{1-n} \quad \text{and} \quad d\Gamma_y(z) = 0 \quad \text{in case } z \neq y. \end{aligned}$$

Also, if $n = 1$, then $\Psi_y(z) = z - y$ and $\Gamma_y(z) = (1/2) \text{sign}(z - y)$.

Lemma. If $S \in \mathbf{N}_n(\mathbf{R}^n)$, $y \in \mathbf{R}^n$ and $0 < \rho < \infty$, then

$$S(\Omega_{y,\rho}) = (\partial S)(\Gamma_{y,\rho}).$$

Proof. We observe that $\Gamma_{\nu,\rho}$ is Lipschitzian, $\Omega_{\nu,\rho}$ is bounded, and

$$d\Gamma_{\nu,\rho}(z) = \Omega_{\nu,\rho}(z) \quad \text{for } z \in W = \mathbf{R}^n \cap \{w: |w - y| \neq \rho\}.$$

Smoothing $\Gamma_{\nu,\rho}$ and $\Omega_{\nu,\rho}$ as in [3, §4.1.2], we find that

$$d(\Phi_\epsilon * \Gamma_{\nu,\rho}) = \Phi_\epsilon * \Omega_{\nu,\rho} \quad \text{for } 0 < \epsilon < \infty,$$

and that, as ϵ approaches 0,

$$\Phi_\epsilon * \Gamma_{\nu,\rho} \text{ converges uniformly on } \mathbf{R}^n \text{ to } \Gamma_{\nu,\rho},$$

$$\Phi_\epsilon * \Omega_{\nu,\rho} \text{ converges boundedly on } W \text{ to } \Omega_{\nu,\rho},$$

$$\begin{aligned} (\partial S)(\Gamma_{\nu,\rho}) &= \lim_{\epsilon \downarrow 0} (\partial S)(\Phi_\epsilon * \Gamma_{\nu,\rho}) \\ &= \lim_{\epsilon \downarrow 0} S(\Phi_\epsilon * \Omega_{\nu,\rho}) = S(\Omega_{\nu,\rho}) \end{aligned}$$

because $\mathcal{L}^n(\mathbf{R}^n \setminus W) = 0$ and $\|S\|$ is absolutely continuous with respect to $\mathcal{L}^n$, according to [3, §4.1.21].

Corollary. If $S \in \mathbf{N}_n(\mathbf{R}^n)$, $y \in \mathbf{R}^n$ and $\int |z - y|^{1-n} d\|\partial S\| < \infty$, then

$$\lim_{\rho \downarrow 0} S(\Omega_{\nu,\rho}) = (\partial S)(\Gamma_\nu).$$

Proof. Since $\int \|\Gamma_\nu\| d\|\partial S\| < \infty$, $\|\Gamma_{\nu,\rho}\| \leq \|\Gamma_\nu\|$ for $0 < \rho < \infty$, and $\Gamma_{\nu,\rho}(z) \rightarrow \Gamma_\nu(z)$ as $\rho \downarrow 0$ for each $z \in \mathbf{R}^n$, we infer that

$$(\partial S)\Gamma_{\nu,\rho} = \int \langle \vec{\partial S}, \Gamma_{\nu,\rho} \rangle d\|\partial S\| \rightarrow \int \langle \vec{\partial S}, \Gamma_\nu \rangle d\|\partial S\| = (\partial S)\Gamma_\nu \quad \text{as } \rho \downarrow 0.$$

Theorem. Whenever U is an open subset of $\mathbf{R}^n$, K is a compact subset of U , $T \in \mathbf{N}_{m,K}(U)$, $\nu \geq m \geq n$,

$f: U \rightarrow \mathbf{R}^n$ is a locally Lipschitzian map,

$$\mu = f_\#(|T| + \|\partial T\|), \quad D = \mathbf{R}^n \cap \{y: \mu\{y\} > 0\},$$

$$E = \mathbf{R}^n \cap \left\{ y: \lim_{\rho \downarrow 0} \int_{\{z: 0 < |z - y| < \rho\}} |z - y|^{1-n} d\mu z > 0 \right\},$$

the following five statements hold:

(1) If $y \in \mathbf{R}^n$, $0 < \rho < \infty$ and either f is of class 1 or $n = 1$, then

$$T \llcorner f^* \Omega_{\nu,\rho} = (\partial T) \llcorner f^* \Gamma_{\nu,\rho} + (-1)^n \partial(T \llcorner f^* \Gamma_{\nu,\rho}).$$

(2) If $y \in \mathbf{R}^n \setminus E$ and either f is of class 1 or $n = 1$, then $(\partial T) \llcorner f^* \Gamma_\nu$ and $T \llcorner f^* \Gamma_\nu$ are flat chains with finite mass and

$$\langle T, f, y \rangle = (\partial T) \llcorner f^* \Gamma_\nu + (-1)^n \partial(T \llcorner f^* \Gamma_\nu),$$

$$\mathbf{F}_K(\langle T, f, y \rangle - T \llcorner f^* \Omega_{\nu,\rho}) \leq \lambda \int_{\{z: 0 < |z - y| < \rho\}} |z - y|^{1-n} d\mu z$$

for $0 < \rho < \infty$, where $\lambda = n^{-1}\alpha(n)^{-1} [\text{Lip } (f \mid \text{spt } T)]^{n-1}$.

(3) If $y \in \mathbf{R}^n \sim E$, then $\langle T, f, y \rangle \in \mathbf{F}_{m-n, K}(U)$.

(4) If $n > 1$, then $D \cup E = \mathbf{R}^n \cup \{y: \int |x - y|^{1-n} d\mu = \infty\}$,

$$\text{Cap}^{n-1}(D \cup E) = 0,$$

and for each $\epsilon > 0$ there exists a neighborhood V of $D \cup E$ in $\mathbf{R}^n$ such that $\text{Cap}^{n-1}(V) < \epsilon$ and the function $\langle T, f, \cdot \rangle$ is $\mathbf{F}_K$ continuous relative to $\mathbf{R}^n \sim V$.

(5) If $n = 1$, then $E = \emptyset$ and the function $\langle T, f, \cdot \rangle$ is $\mathbf{F}_K$ continuous at each point of $\mathbf{R} \sim D$.

Proof. We first treat the case when either f is of class 1 or $n = 1$.

For each $\psi \in \mathfrak{D}^{m-n}(U)$ we apply the preceding lemma with $S = f_*(T \llcorner \psi)$, and also recall [3, §4.3.1, §4.1.14, §4.1.7], to compute

$$\begin{aligned} (-1)^{n(m-n)}(T \llcorner f^* \Omega_{v, \rho})(\psi) &= [f_*(T \llcorner \psi)](\Omega_{v, \rho}) = [\partial f_*(T \llcorner \psi)](\Gamma_{v, \rho}) \\ &= [f_* \partial(T \llcorner \psi)](\Gamma_{v, \rho}) = (-1)^{m-n}[(\partial T) \llcorner \psi - T \llcorner d\psi](f^* \Gamma_{v, \rho}) \\ &= (-1)^{m-n}[(-1)^{(m-n)(n-1)}((\partial T) \llcorner f^* \Gamma_{v, \rho})(\psi) - (-1)^{(m-n+1)(n-1)}(T \llcorner f^* \Gamma_{v, \rho})(d\psi)]. \end{aligned}$$

From this we infer the conclusion (1).

We observe that, for $z \in \mathbf{R}^n$,

$$\begin{aligned} \Gamma_v(z) - \Gamma_{v, \rho}(z) &= 0 \quad \text{if } z = y \quad \text{or} \quad |z - y| \geq \rho, \\ ||\Gamma_v(z) - \Gamma_{v, \rho}(z)|| &= n^{-1}\alpha(n)^{-1}(|z - y|^{-n} - \rho^{-n})|z - y| \\ &\leq n^{-1}\alpha(n)^{-1}|z - y|^{1-n} \quad \text{if } 0 < |z - y| < \rho. \end{aligned}$$

Consequently

$$\begin{aligned} \int ||f^*(\Gamma_v - \Gamma_{v, \rho})|| d(||\partial T|| + ||T||) \\ \leq \int [\text{Lip } (f \mid \text{spt } T)]^{n-1} (||\Gamma_v - \Gamma_{v, \rho}|| \circ f) d(||\partial T|| + ||T||) \\ = [\text{Lip } (f \mid \text{spt } T)]^{n-1} \int ||\Gamma_v - \Gamma_{v, \rho}|| d\mu \\ \leq \lambda \int_{\{z: 0 < |z - y| < \rho\}} |z - y|^{1-n} d\mu z. \end{aligned}$$

In case $y \notin E$, the last integral is finite and tends to 0 as ρ approaches 0, hence $(\partial T) \llcorner f^* \Gamma_v$ and $T \llcorner f^* \Gamma_v$ are flat chains with finite mass and

$$\begin{aligned} \mathbf{F}_K[(\partial T) \llcorner f^* \Gamma_v + (-1)^n \partial(T \llcorner f^* \Gamma_v) - T \llcorner f^* \Omega_{v, \rho}] \\ \leq \mathbf{M}[(\partial T) \llcorner f^*(\Gamma_v - \Gamma_{v, \rho})] + \mathbf{M}[T \llcorner f^*(\Gamma_v - \Gamma_{v, \rho})] \\ \leq \int ||f^*(\Gamma_v - \Gamma_{v, \rho})|| d(||\partial T|| + ||T||) \rightarrow 0 \quad \text{as } \rho \downarrow 0, \end{aligned}$$

whence we deduce the conclusion (2).

Assuming $n > 1$ we apply the proposition (5) of Section 2 with $s = n - 1$. Clearly $D \cup E = H$. For each resulting set V we see from our conclusion (2) that

$$\mathbf{F}_K(\langle T, f, y \rangle - T \sqsubset f^* \Omega_{y,\rho}) \rightarrow 0, \text{ uniformly for } y \in \mathbf{R}^n \sim V, \text{ as } \rho \downarrow 0.$$

Since $T \sqsubset f^* \Omega_{y,\rho}$ is $\mathbf{F}_K$ continuous with respect to y , for each ρ , we infer the conclusion (4).

Assuming $n = 1$ we obtain from (2) the estimate

$$\mathbf{F}_K(\langle T, f, y \rangle - T \sqsubset f^* \Omega_{y,\rho}) \leq 2^{-1} \mu\{z: 0 < |z - y| < \rho\},$$

and hence deduce the conclusion (5). We also note that (5) was proved previously in [3, §4.3.4].

Finally, the construction described in [3, §4.3.7] shows that the conclusions (3) and (4) remain valid for the more general case where f is locally Lipschitzian.

4. Combining the preceding theorem with the proposition (9) of Section 2 we conclude that

$$\mathcal{H}^s(\mathbf{R}^n \cap \{y: \langle T, f, y \rangle \text{ does not exist}\}) = 0$$

for all $s > n - 1$, while we see from Example 1 below that this equation need not hold for any $s < n - 1$. It is not known whether the equation always holds for $s = n - 1$. This problem cannot be settled by the potential theoretic method of the present paper, because Example 2 shows that the set E need not even be countably $\mathcal{H}^{n-1}$ measurable. In the special case where f is the identity map of $\mathbf{R}^n$ we know from [3, §4.5.9 (24)] that the equation holds for $s = n - 1$.

The problem of representability of slices by integration is quite different from the problem of their existence. It was proved in [3, §4.3.2] that

$$\mathcal{H}^s(\mathbf{R}^n \cap \{y: \mathbf{M}\langle T, f, y \rangle = \infty\}) = 0$$

for $s = n$. However this equation need not hold for any $s < n$, as illustrated by Example 3.

Example 1. Let V be a nonempty bounded open subset of $\mathbf{R}^n$. Using [3, §2.10.28, §2.10.48] one can construct a subset A of V so that $\mathcal{H}^{n-1}(A) = 0$, but $\mathcal{H}^s(A) > 0$ whenever $0 \leq s < n - 1$. For each positive integer i the set A can be covered by a countable family G_i of open balls contained in V with

$$\sum_{W \in G_i} (\text{diam } W)^{n-1} < 2^{-i}.$$

It follows that

$$T = \sum_{i=1}^{\infty} \sum_{W \in G_i} (\mathbf{E}^n \sqsubset W) \in \mathbf{I}_n(\mathbf{R}^n)$$

and, if $y \in A$, then $T(\Omega_{y,\rho}) \rightarrow \infty$ as $\rho \downarrow 0$, hence $\langle T, \mathbf{1}_A, y \rangle$ does not exist.

Example 2. For each nonnegative integer i let Q_i be the family of 2^i disjoint compact intervals with length $\sigma_i = \exp(1 - 2^i)$, constructed by Cantor's method so that $Q_0 = \{J\}$, $J = \{u: 0 \leq u \leq 1\}$, $\cup Q_i \subset \cup Q_{i-1}$ and $\cup Q_{i-1} \sim \cup Q_i$ is open in $\mathbf{R}$ when $i > 0$. Since

$$\sum_{i=0}^{\infty} 2^{-i} \log^+ (\sigma_i^{-1}) = \sum_{i=1}^{\infty} 2^{-i} (2^i - 1) = \infty,$$

it follows from [1, §4, Theorem 3, page 31] that the compact set

$$C = \bigcap_{i=0}^{\infty} Q_i$$

has logarithmic capacity 0; therefore, if γ is any Radon measure over $\mathbf{R}$ with $\text{spt } \gamma \subset C$, and if

$$N = C \cap \left\{ u: \int \log^+ (|v - u|^{-1}) d\gamma v = \infty \right\},$$

then $\gamma(C \sim N) = 0$. Choosing γ in particular so that

$$\gamma(K) = 2^{-i} \text{ whenever } K \in Q_i,$$

we conclude $\gamma(\mathbf{R}) = \gamma(C) = \gamma(J) = 1$, $\gamma(N) = 1$, N is uncountable. Moreover γ is the variation measure of the nondecreasing continuous function g defined by the formula

$$g(u) = \gamma\{v: v < u\} \quad \text{for } u \in \mathbf{R}.$$

One readily verifies that

$$[0, 1] \llcorner g \in \mathbf{N}_1(\mathbf{R}), \quad \partial([0, 1] \llcorner g) = \delta_1 - \gamma,$$

$$T = ([0, 1] \llcorner g) \times [0, 1] \in \mathbf{N}_2(\mathbf{R}^2),$$

$$\partial T = (\delta_1 - \gamma) \times [0, 1] - ([0, 1] \llcorner g) \times (\delta_1 - \delta_0),$$

$$||\partial T|| = (\delta_1 + \gamma) \times (\mathcal{L}^1 \llcorner J) + [(\mathcal{L}^1 \llcorner J) \llcorner g] \times (\delta_1 + \delta_0).$$

Furthermore, if $y \in N \times J$, then

$$\begin{aligned} \int |z - y|^{-1} d ||\partial T|| z &\geq \int |z - y|^{-1} d[\gamma \times (\mathcal{L}^1 \llcorner J)]z \\ &\geq \iint_J (|z_1 - y_1| + |z_2 - y_2|)^{-1} d\mathcal{L}^1 z_2 d\gamma z_1 \\ &\geq \iint_0^{1/2} (|z_1 - y_1| + \tau)^{-1} d\mathcal{L}^1 \tau d\gamma z_1 \\ &= \int [\log(|z_1 - y_1| + \tfrac{1}{2}) + \log^+ (|z_1 - y_1|^{-1})] d\gamma z_1 = \infty. \end{aligned}$$

Thus $N \times J$ is contained in the set E of our theorem, with $f = \mathbf{1}_{\mathbf{R}^*}$.

Example 3. Whenever $A \subset \{y: 0 < y < 1\}$, A is compact, $\mathcal{L}^1(A) = 0$ and $\epsilon > 0$, one can construct a map $f: \mathbf{R} \rightarrow \mathbf{R}$ with $\text{Lip}(f) < 1 + \epsilon$ and with the following property:

For all $y \in A$ and $0 < k \in \mathbf{Z}$ there exist $0 < u_1 < v_1 < u_2 < v_2 < \dots < u_k < v_k < 1$ such that $f(u_i) < y < f(v_i)$ for $i = 1, \dots, k$.

Then one can select $\delta > 0$ so that $0 < u_i - \delta, v_i + \delta < 1$,

$$f(x) < y \text{ for } u_i - \delta < x < u_i + \delta, f(x) > y \text{ for } v_i - \delta < x < v_i + \delta,$$

secure $\phi \in \mathcal{D}^0(\mathbf{R})$ with $\text{spt } \phi \subset \{x: 0 < x < 1\}$, $0 \leq \phi(x) \leq 1$ for $x \in \mathbf{R}$,

$$\phi(u_i - \delta) = 0, \quad \phi(u_i + \delta) = 1 = \phi(v_i - \delta), \quad \phi(v_i + \delta) = 0,$$

$$\text{spt}(d\phi) \subset \bigcup_{i=1}^k (\{x: u_i - \delta < x < u_i + \delta\} \cup \{x: v_i - \delta < x < v_i + \delta\}),$$

and infer that

$$\begin{aligned} \langle [0, 1], f, y \rangle(\phi) &= [(\delta_1 - \delta_0) \lfloor f^* \Gamma_v \rfloor](\phi) - [[0, 1] \lfloor f^* \Gamma_v \rfloor](d\phi) \\ &= -[[0, 1] \lfloor \tfrac{1}{2} \text{sign}(f - y) \rfloor](d\phi) \\ &= \frac{1}{2} \sum_{i=1}^k ([u_i - \delta, u_i + \delta] - [v_i - \delta, v_i + \delta])(d\phi) = k, \end{aligned}$$

hence $\mathbf{M}\langle [0, 1], f, y \rangle \geq k$.

Thus $A \subset \{y: \mathbf{M}\langle [0, 1], f, y \rangle = \infty\}$. Moreover A can be chosen so that $\mathcal{H}^s(A) > 0$ for $0 \leq s < 1$.

5. Applying our theorem to the subtraction

$$f: \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n, f(u, v) = u - v \quad \text{for } (u, v) \in \mathbf{R}^n \times \mathbf{R}^n,$$

we will now find a representation of an arbitrary normal current

$$S \in \mathbf{N}_k(\mathbf{R}^n)$$

by a pair of differential forms. For each $\psi \in \mathcal{D}^k(\mathbf{R}^n)$ we know from [3, §4.3.20] that

$$\begin{aligned} S(\psi) &= (S \lfloor \psi)(1) = [S \cap (\mathbf{E}^n \lfloor \psi)](1) \\ &= (-1)^{n-k} \langle S \times (\mathbf{E}^n \lfloor \psi), f, 0 \rangle(1). \end{aligned}$$

Observing that

$$\iint_K |u - v|^{1-n} d\gamma u d\mathcal{L}^n v \leq n\alpha(n) \text{diam}(K) \gamma(\mathbf{R}^n)$$

for every compact subset K of $\mathbf{R}^n$ and every Radon measure γ over $\mathbf{R}^n$, we see that the origin of $\mathbf{R}^n$ does not belong to the exceptional set E of our theorem, hence

$$\begin{aligned}
S(\psi) &= (-1)^{n-k}[(\partial S) \times (\mathbf{E}^n \llcorner \psi) - S \times (\mathbf{E}^n \llcorner d\psi)](f^* \Gamma_0) \\
&= (-1)^{n-k}[(\mathbf{E}^n \llcorner \psi)(f^* \Gamma_0)_{\partial S} - (\mathbf{E}^n \llcorner d\psi)(f^* \Gamma_0)_S] \\
&= (-1)^{n(k-1)}[\mathbf{E}^n \llcorner (f^* \Gamma_0)_{\partial S}](\psi) + (-1)^{nk}[\mathbf{E}^n \llcorner (f^* \Gamma_0)_S](d\psi).
\end{aligned}$$

In case $k = 0$ the first term should be omitted, in case $k = n$ the second. Thus we obtain the representation

$$S = (-1)^{n(k-1)} \mathbf{E}^n \llcorner (f^* \Gamma_0)_{\partial S} + (-1)^{nk} \partial[\mathbf{E}^n \llcorner (f^* \Gamma_0)_S],$$

where the differential forms $(f^* \Gamma_0)_{\partial S}$ and $(f^* \Gamma_0)_S$ of degrees $n - k$ and $n - k - 1$ are defined $\mathcal{L}^n$ almost everywhere on $\mathbf{R}^n$ by the method of [3, §4.1.8], are analytic in $\mathbf{R}^n \sim \text{spt } \partial S$ and $\mathbf{R}^n \sim \text{spt } S$, respectively, and are $\mathcal{L}^n \llcorner K$ summable for every compact subset K of $\mathbf{R}^n$.

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