Perturbations of Isometries on Hilbert Space

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1. Introduction. This paper is concerned with perturbations of isometries on infinite-dimensional Hilbert space. Our choice of studying isometries has been motivated by the fact that they (in particular, the shifts) provide many useful examples and counterexample to all parts of Hilbert space theory (see Halmos [1]). A brief description of our main result is as follows. Let H, H' be complex separable infinite-dimensional Hilbert spaces. Let S be an isometry on H and P be a bounded operator (throughout this paper, all operators are linear) on H which can be factored formally as the product B^*A of two bounded operators A, B on H to H' . Under certain assumptions on the behavior of A, B relative to S , we are able to construct a nonsingular operator $W(\mu)$ such that $(S + \mu P) = W(\mu)SW(\mu)^{-1}$ for all sufficiently small complex parameters μ . It follows that S and $S + \mu P$ are similar operators on H . This result has an interesting application to perturbation problem of the unilateral shift

$$S: (x_0, x_1, x_2, \dots) \rightarrow (0, x_0, x_1, \dots)$$

on $\ell^2(0, \infty)$. It is well known that the invariant subspaces of S have been characterized by Beurling, using function algebra representation [2]. In a paper of Duren [3], a study was made on certain "tridiagonal operators" on $\ell^2(0, \infty)$ whose lattices of invariant subspaces are isomorphic to that of S . Later Freeman [4] shows that for a large class of strictly lower-triangular matrices $P = (p_{mn})$, satisfying

$$|P| = \sum_{m,n=0}^{\infty} |p_{mn}| < \infty \quad \text{and} \quad p_{n+1,n} \neq -1,$$

the operators S and $S + P$ are similar to each other. In this paper we are able to establish the similarity between S and $S + P$ with only size restriction on P . More precisely, we show that $S + \mu P$ and S are similar operators on $\ell^p(0, \infty)$ ($1 \leq p \leq \infty$) for all infinite matrices $P = (p_{mn})$ with $|P| = \sum_{m,n=0}^{\infty} |p_{mn}| < \infty$ and for all complex parameters μ with $|\mu| < 1/2 |P|$.

Our approach is motivated by our previous success in treating perturbation of generators of continuous semi-groups [5] based on the theory of wave operators and smooth perturbation [6]. The idea is to construct a nonsingular bounded

operator $W(\mu)$ explicitly (in terms of the perturbation μP) which actually implements the similarity.

2. The main results. Throughout the paper $\mathfrak{B}(H, H')$ will denote the Banach space of all bounded operators on H to H' , and $\mathfrak{B}(H) = \mathfrak{B}(H, H)$. We use I to denote the identity operator on $H(H')$. If u is a vector and A is a bounded operator, then we write $\|u\|$, $\|A\|$, and A^* for the norm of u , the norm of A and the adjoint of A , respectively. We use the symbol $\langle \cdot, \cdot \rangle$ both for the inner products in H and H' . The symbol $Y = \ell^2(0, \infty; H')$ denotes the linear space of all sequences $x = \{x_n\}$, $n = 0, 1, 2, \dots$, $x_n \in H'$ for which the norm

$$\|x\| = \left(\sum_{n=0}^{\infty} \|x_n\|^2 \right)^{1/2}$$

is finite.

The main tool in our proof of the similarity between S and $S + \mu P$ is the nonsingular operator $W(\mu)$ defined by

$$\langle W(\mu)u, v \rangle = \langle u, v \rangle + \mu \sum_{n=0}^{\infty} \langle AS^{*n-1}u, B(S + \mu P)^{*n}v \rangle$$

for all $u, v \in H$ (note that $P = B^*A$). In order to show the existence of $W(\mu)$ and that it does intertwine between S and $S + \mu P$, we must first obtain an estimate of the form

$$\sum_{n=0}^{\infty} \|B(S + \mu P)^{*n}u\|^2 \leq K^2 \|u\|^2,$$

for all $u \in H$ and some constant $K > 0$. This step turns out to be the most difficult part of the entire proof. For there seems to be no easy way to get a reasonable estimate for $\|B(S + \mu P)^{*n}u\|$ ($n = 0, 1, 2, \dots$), due to the fact that S and P do not necessarily commute and the lack of a simple expansion formula for $(S + \mu P)^n$ in terms of S and P . To go around these difficulties, we first construct a family $\{U(n, \mu)\}$ ($n = 0, 1, 2, \dots$) of operators in $\mathfrak{B}(H)$ satisfying the inequality

$$\sum_{n=0}^{\infty} \|BU(n, \mu)^*u\|^2 \leq K^2 \|u\|^2$$

for some $K > 0$ and $u \in H$ and then show that $U(n, \mu) = (S + \mu P)^n$. Our method of constructing $U(n, \mu)$ is motivated by the following observation. Let G be the infinitesimal generator of a semi-group $T(t; G)$ on H . If $P \in \mathfrak{B}(H)$, then the resolvent $R(\lambda; G + P)$ of $G + P$ can be represented as the sum $\sum_{n=0}^{\infty} R(\lambda; G) \cdot [PR(\lambda; G)]^n$, where $R(\lambda; G)$ (the resolvent of G) is the Laplace transform of $T(t; G)$. We recall that the product of the Laplace transforms is the Laplace transform of their convolution. Thus, if $G + P$ also generates a semi-group $T(t; G + P)$, one would expect to have

$$T(t; G + P) = \sum_{n=0}^{\infty} S_n(t),$$

where $S_0(t) = T(t; G)$ and $S_n(t)u = \int_0^t T(t-r; G)PS_{n-1}(r)u \, dr$ (cf. [7, Chapter XIII]). This leads to our notion of the discrete convolution operator. Let S be an isometry on H and $A, B \in \mathfrak{B}(H, H')$. We set

$$D(A, B, S) = \left\{ f \in Y : \sum_{n=1}^{\infty} \left\| \sum_{r=1}^n AS^{n-r}B^*f(r-1) \right\|^2 < \infty \right\}.$$

We define

$$T(A, B, S)f(n) = \begin{cases} 0, & n = 0; \\ \sum_{r=1}^n AS^{n-r}B^*f(r-1), & n \geq 1, \end{cases}$$

for each $f \in D(A, B, S)$. The operator $T(A, B, S)$ is easily seen to be a linear operator from $D(A, B, S)$ to Y . It is a discrete version of the convolution operator C defined in [5].

It should be noted that in general the set $D(A, B, S)$ may consist of the zero element alone.

Definition 2.1. Let S be an isometry on H and $A \in \mathfrak{B}(H, H')$. The operator A is said to be S -smooth, if there exists a constant $M < \infty$ such that

$$(1) \quad \sum_{n=0}^{\infty} \|AS^n u\|^2 \leq M^2 \|u\|^2 \quad \text{and} \quad \sum_{n=0}^{\infty} \|AS^{*n} u\|^2 \leq M^2 \|u\|^2,$$

for all $u \in H$. In this case, we set $|A|_S$ to be the infimum of all constants M satisfying (1).

Definition 2.2. A class $\mathfrak{R}(S) \subset \mathfrak{B}(H)$ of operators is called S -admissible, if each $P \in \mathfrak{R}(S)$ can be factored as the product $P = B^*A$ of two operators $A, B \in \mathfrak{B}(H, H')$, having the following properties:

- (i) both A and B are S -smooth, and
- (ii) the operator $T(A, B, S)$ is a bounded operator of Y into Y .

To each $P \in \mathfrak{R}(S)$, we set

$$(2) \quad |P| = \max \{ \|T(A, B, S)\|, |A|_S |B|_S \}.$$

We can now state the main results of this paper.

Theorem 2.3. Let S be an isometry on H . Then for each $P = B^*A$ that belongs to the S -admissible class $\mathfrak{R}(S)$ and for any complex number μ with $|\mu| < 1/2 |P|$, the operator $S + \mu P$ is similar to S . The operator $W(\mu)$ given by the following formula

$$\langle W(\mu)u, v \rangle = \langle u, v \rangle + \mu \sum_{n=0}^{\infty} \langle AS^{*n-1}u, B(S + \mu P)^{*n}v \rangle,$$

for all $u, v \in H$, is a nonsingular bounded operator on H which implements the similarity, i.e., we have $(S + \mu P) = W(\mu)SW(\mu)^{-1}$.

Remark 2.4. In Theorem 2.3, we can establish the existence of the operator $W(\mu)$ by only assuming that $|\mu| < 1/|P|$. It is for nonsingularity of $W(\mu)$ that we require $|\mu| < 1/2 |P|$; this requirement can be dropped, however, if S is assumed to be unitary. For in this case the following operator

$$\langle Z(\mu)u, v \rangle = \langle u, v \rangle + \mu \sum_{n=0}^{\infty} \langle A(S + \mu P)^n u, B S^{n+1} v \rangle$$

exists for $|\mu| < 1/|P|$, and one can show that the relation $W(\mu)Z(\mu) = Z(\mu)W(\mu) = I$ holds.

Remark 2.5. Theorem 2.3 actually holds for any operator S (not necessarily isometric) which admits a left inverse $Q(QS = I)$. In this case, a natural candidate for $W(\mu)$ would be:

$$\langle W(\mu)u, v \rangle = \langle u, v \rangle + \mu \sum_{n=0}^{\infty} \langle A Q^{n+1} u, B(S + \mu P)^n v \rangle,$$

of course, with the assumption that $\sum_{n=0}^{\infty} \|A Q^n u\|^2 \leq K^2 \|u\|^2$ for some $K > 0$ (a slight modification of the S -admissible class will do). This operator $W(\mu)$ will intertwine S and $S + \mu P$.

Theorem 2.6. Let S be the unilateral shift on $\ell^2(0, \infty)$. Then all infinite matrices $P = (p_{mn})$ with $|P| = \sum_{m,n=0}^{\infty} |p_{mn}| < \infty$ belong to the S -admissible class $\mathfrak{R}(S)$.

Corollary 2.7. Let S be the unilateral shift on $\ell^2(0, \infty)$. If $P = (p_{mn})$ is an infinite matrix with $|P| = \sum_{m,n=0}^{\infty} |p_{mn}| < \infty$, then for any complex number μ with $|\mu| < 1/2 |P|$, the operator $S + \mu P$ is similar to S .

It is easy to see that Corollary 2.7 follows immediately from Theorems 2.6 and 2.3.

Remark 2.8. The result of Corollary 2.7 is valid on all $\ell^p(0, \infty)$, $1 \leq p \leq \infty$. This is a consequence of Remark 2.5 and the fact that the unilateral shift on $\ell^p(0, \infty)$ admits a left inverse

$$Q: (x_0, x_1, x_2, \dots) \rightarrow (x_1, x_2, x_3, \dots)$$

(cf. Remark 4.3).

3. The proof of theorem 2.3. Our first task is to construct the perturbed discrete semi-group $\{U(n, \mu)\} (n = 0, 1, 2, \dots)$ of operators in $\mathfrak{B}(H)$ for each given $P \in \mathfrak{R}(S)$. To each fixed $P = B^*A \in \mathfrak{R}(S)$ and fixed $u \in H$, set

$$V_0 u(k) = \begin{cases} 0, & k = 0; \\ A S^{k-1} u, & k \geq 1. \end{cases}$$

Since A is S -smooth, we have $V_0 u \in Y$ and $\|V_0 u\| \leq |A|_S \|u\|$. We write T for $T(A, B, S)$ and define

$$V_n u = T(V_{n-1} u) \quad \text{for all } n \geq 1.$$

It is clear that $V_n u \in Y$. Moreover, an easy computation shows that

$$\|V_n u\| \leq |P|^n |A|_S \|u\|, \quad \text{for all } n \geq 0.$$

Now we define $U_0(k) = S^k$ and

$$(3) \quad \langle U_n(k)u, v \rangle = \sum_{r=0}^k \langle V_{n-1} u(r), BS^{*(k-r)} v \rangle,$$

for all $u, v \in H$ and for $n \geq 1$. Since $\|V_{n-1} u\| \leq |P|^{n-1} |A|_S \|u\|$, and the finite sum on the right hand side of (3) has its absolute value dominated by $\|V_{n-1} u\| \cdot |B|_S \|v\|$, it follows that $U_n(k) \in \mathfrak{B}(H)$ for all $n, k = 0, 1, 2, \dots$ and we have $\|U_n(k)\| \leq |P|^n$ uniformly in k for each n . Note that $U_0(0) = I$ and $U_n(0) = 0$ for $n \geq 1$.

With this family $\{U_n(k)\}$ of operators at our disposal, we set, for $k = 0, 1, 2, \dots$,

$$(4) \quad U(k, \mu)u = \sum_{n=0}^{\infty} \mu^n U_n(k)u, \quad \text{for } u \in H \quad \text{and} \quad |\mu| < 1/|P|.$$

It can be seen easily that the infinite series on the right converges absolutely and uniformly in k and $\|U(k, \mu)u\| \leq (1 - |\mu| |P|)^{-1} \|u\|$. Thus $U(k, \mu) \in \mathfrak{B}(H)$ for all $k \geq 0$ and all $|\mu| < 1/|P|$.

Lemma 3.1. *Let S be an isometry on H . For each given $P = B^*A \in \mathfrak{K}(S)$ and each $|\mu| < 1/|P|$, the family of operators $\{U(n, \mu)\}$ ($n = 0, 1, 2, \dots$) constructed above satisfies*

- (i) $\{\sum_{n=0}^{\infty} \|BU(n, \mu)^* u\|^2\}^{1/2} \leq |B|_S (1 - |\mu| |P|)^{-1} \|u\|$, and
- (ii) $U(k, \mu)u = (S + \mu P)^k u$, for all $k \geq 0$ and $u \in H$.

Proof of (i). Let $P = B^*A$ be S -admissible. Recall that the operator $T = T(A, B, S)$ associated with A and B is given by:

$$Tf(n) = \begin{cases} 0, & n = 0; \\ \sum_{r=1}^n AS^{n-r}B^*f(r-1), & n \geq 1, \end{cases}$$

for $f \in Y = \ell^2(0, \infty; H')$. A direct computation shows that the adjoint of T is the operator

$$T^*g(r) = \sum_{k=r+1}^{\infty} BS^{*k-r-1}A^*g(k), \quad g \in Y \quad \text{and} \quad r \geq 0.$$

For a fixed $u \in H$ and $k \geq 0$, let us define

$$g_k(r) = \begin{cases} BS^{*k-r}u, & 0 \leq r \leq k; \\ 0, & k < r, \end{cases}$$

and

$$U_{n,k}^*(r)u = \begin{cases} U_n(k-r)^*u, & 0 \leq r \leq k; \\ 0, & k < r, \end{cases}$$

where $U_n(k)^*$ is just the adjoint of $U_n(k)$ defined in (3). We claim that

$$(5) \quad BU_{n,k}^*(r)u = T^{*n}g_k(r), \quad \text{for all } n \geq 0.$$

Note that for $n = 0$, $BU_{0,k}^*(r)u = BU_0(k-r)^*u = BS^{*k-r}u = g_k(r)$; thus (5) holds for $n = 0$. The general statement follows by considering the inner product $\langle h, T^{*n+1}g_k(r) \rangle$ for $h \in H'$ and by mathematical induction. Thus, as elements of Y , $BU_{n,k}^*(\cdot)u = T^{*n}g_k$, and $\|BU_{n,k}^*(\cdot)u\| = \|T^{*n}g_k\| \leq \|T^*\|^n \|g_k\| \leq |P|^n \|g_k\|$. By the definition of g_k , we have $\|g_k\| \leq (\sum_{n=0}^{\infty} \|BS^{*n}u\|)^{1/2} \leq |B|_s \|u\|$ uniformly in $k \geq 0$. Hence $\|BU_{n,k}^*(\cdot)u\|_Y \leq |P|^n |B|_s \|u\|$.

Observe that

$$\begin{aligned} \sum_{r=0}^k \|BU_n(r)^*u\|^2 &= \sum_{r=0}^k \|BU_n(k-r)^*u\|^2 = \sum_{r=0}^k \|BU_{n,k}^*(r)u\|^2 \\ &= \|BU_{n,k}^*(\cdot)u\|_Y^2 \leq (|P|^n |B|_s \|u\|)^2 \end{aligned}$$

uniformly in $k \geq 0$. We therefore have

$$(6) \quad \|BU_n(\cdot)^*u\|_Y \leq |P|^n |B|_s \|u\|.$$

Since $U(k, \mu)^*u = \sum_{n=0}^{\infty} \bar{\mu}^n U_n(k)^*u$ (the series converges strongly and uniformly in k , cf. (4)) and $B \in \mathfrak{B}(H, H')$, we can write $BU(k, \mu)^*u = \sum_{n=0}^{\infty} \bar{\mu}^n BU_n(k)^*u$, i.e., $BU(\cdot, \mu)^*u = \sum_{n=0}^{\infty} \bar{\mu}^n BU_n(\cdot)^*u$ as a member of Y .

Consequently

$$\begin{aligned} \left(\sum_{k=0}^{\infty} \|BU(k, \mu)^*u\|^2 \right)^{1/2} &= \|BU(\cdot, \mu)^*u\|_Y \leq \sum_{n=0}^{\infty} |\mu|^n \|BU_n(\cdot)^*u\|_Y \\ &\leq |B|_s \left\{ \sum_{n=0}^{\infty} (|\mu| |P|)^n \right\} \|u\| = |B|_s (1 - |\mu| |P|)^{-1} \|u\|, \quad \text{by (6)}. \end{aligned}$$

Proof of (ii). Since $U(0, \mu) = I$, the statement $U(k, \mu) = (S + \mu P)^k$ for all integers $k \geq 0$ will follow if we can show that $(S + \mu P)U(k, \mu) = U(k+1, \mu)$ for all $k \geq 0$. To this end, let $u, v \in H$, $n \geq 1$, and $k \geq 0$. We have (see (3))

$$\begin{aligned} \langle U_n(k+1)u, v \rangle &= \sum_{r=0}^{k+1} \langle V_{n-1}u(r), BS^{*k+1-r}v \rangle \\ &= \sum_{r=0}^k \langle V_{n-1}u(r), BS^{*k+1-r}v \rangle + \langle V_{n-1}u(k+1), Bv \rangle \\ &= \beta_{n,k} + \langle V_{n-1}u(k+1), Bv \rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned}
 \langle (S + \mu P)U_n(k)u, v \rangle &= \langle U_n(k)u, (S + \mu P)^*v \rangle \\
 &= \sum_{r=0}^k \langle V_{n-1}u(r), BS^{*k-r}(S + \mu P)^*v \rangle \\
 &= \sum_{r=0}^k \langle V_{n-1}u(r), BS^{*k+1-r}v \rangle + \sum_{r=0}^k \langle V_{n-1}u(r), BS^{*k-r}(\mu P)^*v \rangle \\
 &= \beta_{n,k} + \mu \sum_{r=0}^k \langle AS^{*r}B^*V_{n-1}u(r), Bv \rangle \\
 &= \beta_{n,k} + \mu \sum_{r=1}^{k+1} \langle AS^{*r-1}B^*V_{n-1}u(r-1), Bv \rangle \\
 &= \beta_{n,k} + \mu \langle V_nu(k+1), Bv \rangle,
 \end{aligned}$$

by noting that $P = B^*A$ and $V_nu = T(V_{n-1}u)$.

It follows that

$$\begin{aligned}
 \langle U(k+1)u, v \rangle &= \sum_{n=0}^{\infty} \mu^n \langle U_n(k+1)u, v \rangle = \langle U_0(k+1)u, v \rangle \\
 &\quad + \sum_{n=1}^{\infty} \mu^n \beta_{n,k} + \sum_{n=1}^{\infty} \mu^n \langle V_{n-1}u(k+1), Bv \rangle, \quad \text{and}
 \end{aligned}$$

$$\begin{aligned}
 \langle (S + \mu P)U(k, \mu)u, v \rangle &= \sum_{n=0}^{\infty} \mu^n \langle (S + \mu P)U_n(k)u, v \rangle \\
 &= \langle (S + \mu P)U_0(k)u, v \rangle + \sum_{n=1}^{\infty} \mu^n \{ \beta_{n,k} + \mu \langle V_nu(k+1), Bv \rangle \} \\
 &= \langle (S + \mu P)U_0(k)u, v \rangle + \sum_{n=1}^{\infty} \mu^n \beta_{n,k} + \sum_{n=1}^{\infty} \mu^{n+1} \langle V_nu(k+1), Bv \rangle.
 \end{aligned}$$

This implies that

$$\begin{aligned}
 \langle U(k+1)u, v \rangle - \langle (S + \mu P)U(k, \mu)u, v \rangle &= \langle U_0(k+1)u, v \rangle \\
 &\quad + \sum_{n=0}^{\infty} \mu^{n+1} \langle V_nu(k+1), Bv \rangle - \langle (S + \mu P)U_0(k)u, v \rangle \\
 &\quad - \sum_{n=1}^{\infty} \mu^{n+1} \langle V_nu(k+1), Bv \rangle = \langle S^{k+1}u, v \rangle - \langle SS^ku, v \rangle \\
 &\quad - \mu \langle PS^ku, v \rangle + \mu \langle V_0u(k+1), Bv \rangle = 0, \quad \text{since } \langle V_0u(k+1), Bv \rangle \\
 &= \langle AS^ku, Bv \rangle = \langle B^*AS^ku, v \rangle = \langle PS^ku, v \rangle.
 \end{aligned}$$

Consequently we have

$$U(k+1, \mu) = (S + \mu P)U(k, \mu) \quad \text{for all } k \geq 0.$$

Proof of Theorem 2.3 Let us write $W(\mu) = I + \mu Q(\mu)$, where $\langle Q(\mu)u, v \rangle = \sum_{n=0}^{\infty} \langle AS^{*n+1}u, B(S + \mu P)^{*n}v \rangle$, for all $u, v \in H$. Since

$$\left\{ \sum_{n=0}^{\infty} \|B(S + \mu P)^{*n}v\|^2 \right\}^{1/2} = \left\{ \sum_{n=0}^{\infty} \|BU(n, \mu)^{*n}v\|^2 \right\}^{1/2} \\ \leq |B|_S (1 - |\mu| |P|)^{-1} \|v\|$$

by Lemma 3.1 and A is S -smooth by assumption, a direct estimation shows that

$$|\langle Q(\mu)u, v \rangle| \leq |P| (1 - |\mu| |P|)^{-1} \|u\| \|v\|.$$

It follows that $Q(\mu) \in \mathfrak{B}(H)$ with $\|Q(\mu)\| \leq |P| (1 - |\mu| |P|)^{-1}$, hence $W(\mu) = I + \mu Q(\mu)$ also belongs to $\mathfrak{B}(H)$. The condition that $|\mu| < 1/2 |P|$ implies that $\|\mu Q(\mu)\| < 1$. Consequently the operator $W(\mu) = I + \mu Q(\mu)$ is nonsingular.

It remains to show that $(S + \mu P)W(\mu) = W(\mu)S$. To this end, let $u, v \in H$. Since $S^*S = I$, we have

$$\begin{aligned} \langle W(\mu)Su, v \rangle &= \langle Su, v \rangle + \mu \sum_{n=0}^{\infty} \langle AS^{*n+1}Su, B(S + \mu P)^{*n}v \rangle \\ &= \langle Su, v \rangle + \mu \sum_{n=0}^{\infty} \langle AS^{*n}u, B(S + \mu P)^{*n}v \rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle (S + \mu P)W(\mu)u, v \rangle &= \langle W(\mu)u, (S + \mu P)^*v \rangle \\ &= \langle u, (S + \mu P)^*v \rangle + \mu \sum_{n=0}^{\infty} \langle AS^{*n+1}u, B(S + \mu P)^{*n+1}v \rangle \\ &= \langle (S + \mu P)u, v \rangle + \mu \sum_{n=0}^{\infty} \langle AS^{*n}u, B(S + \mu P)^{*n}v \rangle - \mu \langle Au, Bv \rangle \\ &= \langle Su, v \rangle + \mu \sum_{n=0}^{\infty} \langle AS^{*n}u, B(S + \mu P)^{*n}v \rangle, \end{aligned}$$

by noting that $\mu \langle Au, Bv \rangle = \mu \langle B^*Au, v \rangle = \mu \langle Pu, v \rangle$. We see clearly that $\langle W(\mu)Su, v \rangle = \langle (S + \mu P)W(\mu)u, v \rangle$ for all $u, v \in H$. This completes the proof of Theorem 2.3.

4. The proof of theorem 2.6. Throughout this section, let $H = \ell^2(0, \infty)$ and $H' = \ell^2((0, \infty) \times (0, \infty))$. Given $P = (p_{mn})$ with $\sum_{m,n=0}^{\infty} |p_{mn}| < \infty$, we define A, B from H to H' as follows,

$$(Au)_{mn} = \alpha_{mn} |p_{mn}|^{1/2} u_n, \quad \text{and} \\ (Bu)_{mn} = |p_{mn}|^{1/2} u_m, \quad \text{for } u = (u_m) \in H,$$

where $\alpha_{mn} = p_{mn}/|p_{mn}|$ for $p_{mn} \neq 0$ and $\alpha_{mn} = 0$ when $p_{mn} = 0$. Clearly $\|Au\|^2 = \sum_m \sum_n |(Au)_{mn}|^2 = \sum_m \sum_n |p_{mn}| |u_n|^2 \leq |P| \|u\|^2$. This implies that $\|A\| \leq |P|^{1/2}$ and hence $A \in \mathfrak{B}(H, H')$. It is easily seen that B also belongs to $\mathfrak{B}(H, H')$ with the same bound. A direct computation shows that

$$(B^*v)_m = \sum_n |p_{mn}|^{1/2} v_{mn}, \quad \text{for } v \in H';$$

and $B^*A = P$ on H .

Lemma 4.1. *The operators A , B defined above are S -smooth with $|A|_S$ and $|B|_S$ both bounded by $|P|^{1/2}$.*

Proof. It is easily seen that the behavior of A relative to S is quite the same as that of B (they differ essentially by constants of absolute value one). Therefore we need only to prove the statement for, say, B . Since

$$(BS^k u)_{mn} = \begin{cases} |p_{mn}|^{1/2} u_{m-k} & \text{for } m \geq k \text{ and} \\ 0 & \text{for } m < k, \end{cases}$$

we have $\|BS^k u\|^2 = \sum_m \sum_n |(BS^k u)_{mn}|^2 = \sum_{m \geq k} \sum_n |p_{mn}| |u_{m-k}|^2$. It follows that $\sum_{k=0}^{\infty} \|BS^k u\|^2 = \sum_{k=0}^{\infty} \sum_{m \geq k} \sum_n |p_{mn}| |u_{m-k}|^2 = \sum_m \sum_n p_{mn} (\sum_{k=0}^m |u_{m-k}|^2) \leq |P| \|u\|^2$. On the other hand,

$$(BS^{*k} u)_{mn} = |p_{mn}|^{1/2} u_{m+k}, \quad \text{for } m, k \geq 0.$$

A similar computation shows that $\sum_{k=0}^{\infty} \|BS^{*k} u\|^2 \leq |P| \|u\|^2$. Hence B is S -smooth with $|B|_S \leq |P|^{1/2}$.

Lemma 4.2. *The operator $T = T(A, B, S)$ associated with the operators A , B defined above is a bounded linear operator of Y into Y .*

Proof. Let $f \in Y$ and $v \in H'$. Then, by definition,

$$\langle Tf(n), v \rangle = \begin{cases} \sum_{r=1}^n \langle AS^{n-r} B^* f(r-1), v \rangle, & \text{for } n \geq 1, \\ 0, & n = 0. \end{cases}$$

Hence

$$\begin{aligned} |\langle Tf(n), v \rangle| &\leq \sum_{r=1}^n |\langle f(r-1), BS^{*n-r} A^* v \rangle| \\ &\leq \sum_{r=1}^n \sum_i \sum_m |f(r-1)_{im}| |(BS^{*n-r} A^* v)_{im}|. \end{aligned}$$

Since

$$\begin{aligned} (BS^{*n-r} A^* v)_{im} &= |p_{im}|^{1/2} (S^{*n-r} A^* v)_i \quad \text{and} \quad |(S^{*n-r} A^* v)_i| \\ &= |(A^* v)_{i+n-r}| \leq \sum_{k=0}^{\infty} |p_{k, i+n-r}|^{1/2} |v_{k, i+n-r}|, \end{aligned}$$

it follows that

$$\begin{aligned} |\langle Tf(n), v \rangle| &\leq \sum_r \sum_i \sum_m \sum_k |f(r-1)_{im}| |p_{im}|^{1/2} |p_{k, i+n-r}| |v_{k, i+n-r}| \\ &\leq \left\{ \sum_r \sum_i \sum_m \sum_k |f(r-1)_{im}|^2 |p_{k, i+n-r}| \right\}^{1/2} \\ &\quad \cdot \left\{ \sum_r \sum_i \sum_m \sum_k |p_{im}| |v_{k, i+n-r}|^2 \right\}^{1/2} \leq |P|^{1/2} \|v\| \beta(n), \end{aligned}$$

where $\beta(n) = (\sum \sum \sum \sum |f(r-1)_{im}|^2 |p_{k,i+n-r}|)^{1/2}$, $n \geq 1$. Consequently $\|Tf(n)\| \leq |P|^{1/2} \beta(n)$. Since

$$\begin{aligned} \sum_{n=0}^{\infty} \beta(n)^2 &= \sum_n \sum_r \sum_i \sum_m \sum_k |f(r-1)_{im}|^2 |p_{k,i+n-r}| \\ &\leq \sum_r \sum_i \sum_m |f(r-1)_{im}|^2 \left(\sum_n \sum_k |p_{k,i+n-r}| \right) \\ &\leq |P| \|f\|^2, \end{aligned}$$

we have $\sum_{n=0}^{\infty} \|Tf(n)\|^2 \leq |P|^2 \|f\|^2$. This means that $T \in \mathfrak{B}(Y)$ with $\|T\| \leq |P|$.

Lemmas 4.1 and 4.2 yield Theorem 2.6.

Remark 4.3. In the case of the Banach space $\ell^p(0, \infty)$, $1 \leq p \leq \infty$, the results above will remain valid under a slight modification. We define

$A: \ell^p(0, \infty) \rightarrow \ell^p((0, \infty) \times (0, \infty))$ by

$$(Au)_{mn} = \alpha_{mn} |p_{mn}|^{1/p} u_n \text{ and}$$

$B: \ell^q(0, \infty) \rightarrow \ell^q((0, \infty) \times (0, \infty))$ by

$$(Bu)_{mn} = |p_{mn}|^{1/q} u_m,$$

respectively, where $1/p + 1/q = 1$. The operator $T(A, B, S)$ is then a bounded operator from $\ell^p(0, \infty; H')$ to itself, where $H' = \ell^p((0, \infty) \times (0, \infty))$. All the arguments above follow through verbatim.

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