Two approximation properties and a Radon-Nikodym derivative for lattices of sets

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Abstract. We show that the classical approximation property: an algebra $\mathfrak{A}_1$ of sets is dense in the sigma algebra $\mathfrak{A}_2$ which it generates when the topology is determined by a probability measure on $\mathfrak{A}_2$, is also valid for lattices of sets: a lattice $\mathfrak{M}_1$ of sets is dense in the sigma lattice $\mathfrak{M}_2$ which it generates. A second approximation property asserts that appropriate $\mathfrak{M}_2$ -measurable functions can be approximated by simple $\mathfrak{M}_1$ -measurable functions with respect to an Orlicz space norm satisfying the Δ_2 -condition. The two approximation properties and results of S. Johansen are then applied to obtain a Radon-Nikodym derivative for additive functions defined on lattices of sets.

I. Introduction. Throughout this paper, $\mathfrak{M}$ will denote a lattice of subsets of a non-empty set Ω (*i.e.*, both the empty set, $\emptyset$, and Ω belong to $\mathfrak{M}$, and both $A \cup B$ and $A \cap B$ are in $\mathfrak{M}$ whenever A and B are elements of $\mathfrak{M}$).

A real valued function μ on $\mathfrak{M}$ is said to be additive if

- (i) $\mu(\emptyset) = 0$ and
- (ii) $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$, $A, B \in \mathfrak{M}$.

Suppose each of λ and μ is a bounded, additive function on $\mathfrak{M}$ and μ is non-negative (*i.e.*, $\mu(A) \leq \mu(B)$ if $A \subset B$).

We shall construct the Radon-Nikodym derivative of λ with respect to μ . Denote by $\mathfrak{A}$ the algebra of subsets of Ω that $\mathfrak{M}$ generates.

Results of B. J. Pettis [6, Theorem 1.2, p. 186] assert that each of λ and μ has a unique additive extension to $\mathfrak{A}$. These extensions will also be denoted by λ and μ . It is easy to check that $\mathfrak{A}$ is the set of finite disjoint unions of elements of $\mathfrak{F}$, where

$$\mathfrak{F} = \{A \cap B^c; A, B \in \mathfrak{M}, B^c = \Omega - B\}.$$

Referring the reader to pages 97, 164–167, 240, 274–276, and 311–313 of Dunford and Schwartz [2] for details, we recall that there is a compact Hausdorff space Ω_1 and an isomorphism τ of $\mathfrak{A}$ onto the algebra $\mathfrak{A}_1$ of closed and open

subsets of Ω_1 such that the images λ_1 and μ_1 of λ and μ (i.e., $\mu_1(\tau A) = \mu(A)$) are countably additive on $\mathfrak{A}_1$. Consequently λ_1 and μ_1 have unique countably additive extensions λ_2 and μ_2 to the sigma algebra $\mathfrak{A}_2$ generated by $\mathfrak{A}_1$. Moreover, $\mathfrak{A}_1$ is dense in $\mathfrak{A}_2$ with respect to the semi-metric ρ_1 defined by $\rho_1(\tau A, \tau B) = \mu_1((\tau A) \triangle (\tau B)) = \mu(A \triangle B) = \mu(A \cup B) - \mu(A \cap B)$, where $A \triangle B$ denotes the symmetric difference $A \cup B - A \cap B$. The lattice $\mathfrak{N}$ is mapped by τ onto a lattice $\mathfrak{N}_1$ of subsets of Ω_1 , and $\mathfrak{N}_1$ generates a sigma lattice $\mathfrak{N}_2$ contained in $\mathfrak{A}_2$. The first step in our program is to establish the lattice analog of the classical approximation theorem [3, p. 56, Theorem D]: we show that $\mathfrak{N}_1$ is dense in $\mathfrak{N}_2$ with respect to ρ_1 .

II. A lattice of sets is dense in the generated sigma lattice.

Theorem 1. *Suppose $F \in \mathfrak{N}_2$ and $\epsilon > 0$. Then there exists $H \in \mathfrak{N}_1$ such that $\mu_2(F \triangle H) < \epsilon$.*

Proof. Let

$$\mathfrak{N}_1 = \left\{ F : F = \bigcup_{i=1}^{\infty} E_i, E_i \subset E_{i+1}, E_i \in \mathfrak{N}_1 \right\}.$$

If $F_i = \bigcup_{n=1}^{\infty} E_{i,n}$ is in $\mathfrak{N}_1$ for $i = 1, 2, \dots$, then $\bigcup_{i=1}^{\infty} F_i = \bigcup_{k=1}^{\infty} H_k$, where

$$H_k = E_{1,k} \cup E_{2,k-1} \cup \dots \cup E_{k,1},$$

and

$$F_i \cap F_j = \bigcup_{k=1}^{\infty} (E_{i,k} \cap E_{j,k}).$$

Since $H_k \subset H_{k+1}$ and $(E_{i,k} \cap E_{j,k}) \subset (E_{i,k+1} \cap E_{j,k+1})$, we have $\bigcup_{i=1}^{\infty} F_i \in \mathfrak{N}_1$ and $F_i \cap F_j \in \mathfrak{N}_1$. So $\mathfrak{N}_1$ is a lattice which contains $\mathfrak{N}_1$ and is closed under countable unions. Similarly, let

$$\mathfrak{N}_2 = \left\{ F : F = \bigcap_{i=1}^{\infty} E_i, E_i \supset E_{i+1}, E_i \in \mathfrak{N}_1 \right\}.$$

If $F_i = \bigcap_{n=1}^{\infty} E_{i,n}$ is in $\mathfrak{N}_2$ for $i = 1, 2, \dots$, then $\bigcap_{i=1}^{\infty} F_i = \bigcap_{k=1}^{\infty} P_k$, where

$$P_k = E_{1,k} \cap E_{2,k-1} \cap \dots \cap E_{k,1},$$

and

$$F_i \cup F_j = \bigcap_{k=1}^{\infty} (E_{i,k} \cup E_{j,k}).$$

So $\mathfrak{N}_2$ is a lattice which contains $\mathfrak{N}_1$ and it is closed under countable intersections. Define

$$m^*(F) = \inf_{F \subset E \in \mathfrak{N}_1} \mu_2(E),$$

$$m_*(F) = \sup_{F \supset E \in \mathfrak{N}_2} \mu_2(E),$$

and let

$$\mathcal{L} = \{F: m_*(F) = m^*(F)\}.$$

Notice that $m^*(F) \geq \mu_2^*(F) \geq \mu_{2*}(F) \geq m_*(F)$, where μ_2^* and μ_{2*} denote the standard outer and inner measures induced on the subsets of Ω_1 by μ_2 . Hence $\mathcal{L}$ is contained in the sigma algebra of μ_2 -measurable sets. If $F \in \mathfrak{N}_1$, then $m^*(F) = \mu_2(F) = m_*(F)$; hence $\mathfrak{N}_1 \subset \mathcal{L}$.

Suppose $F \in \mathcal{L}$ and $\epsilon > 0$. Then there exist $E \in \mathfrak{N}_2$ and $H = \bigcup_{n=1}^{\infty} H_n \in \mathfrak{N}_1$, where $H_n \in \mathfrak{N}_1$ for $n = 1, 2, \dots$, such that $E \subset F \subset H$ and $\mu_2(H - E) < \epsilon$. Choose n large enough so that $\mu_2(H - H_n) < \epsilon$; $F - H_n \subset H - H_n$ and $H_n - F \subset H - E$, so $\mu_2(F \triangle H_n) \leq \mu_2(H - H_n) + \mu_2(H - E) < 2\epsilon$. Consequently, $\mathfrak{N}_1$ is dense in $\mathcal{L}$. If we can show that $\mathcal{L}$ is a sigma lattice, then $\mathfrak{N}_2 \subset \mathcal{L}$ and we have $\mathfrak{N}_1$ dense in $\mathfrak{N}_2$. To this end, suppose that F_i is a sequence of elements of $\mathcal{L}$. It suffices to establish

$$(a) \bigcup_{i=1}^{\infty} F_i \in \mathcal{L} \quad \text{and} \quad (b) \bigcap_{i=1}^{\infty} F_i \in \mathcal{L}.$$

Let $\epsilon > 0$ and take $E_i \in \mathfrak{N}_2$, $H_i \in \mathfrak{N}_1$ such that $E_i \subset F_i \subset H_i$ and $\mu_2(H_i - E_i) < \epsilon/2^i$.

Proof of (a).

$$\begin{aligned} \mu_2\left(\bigcup_{i=1}^{\infty} E_i\right) &= \lim_{n \rightarrow \infty} \mu_2\left(\bigcup_{i=1}^n E_i\right) \leq m_*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq m_*\left(\bigcup_{i=1}^{\infty} F_i\right) \\ &\leq m^*\left(\bigcup_{i=1}^{\infty} F_i\right) \leq m^*\left(\bigcup_{i=1}^{\infty} H_i\right) = \mu_2\left(\bigcup_{i=1}^{\infty} H_i\right), \end{aligned}$$

and

$$\begin{aligned} \mu_2\left(\bigcup_{i=1}^{\infty} H_i\right) - \mu_2\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu_2\left(\bigcup_{i=1}^{\infty} H_i - \bigcup_{i=1}^{\infty} E_i\right) \\ &= \mu_2\left(\bigcup_{i=1}^{\infty} (H_i - \bigcup_{i=1}^{\infty} E_i)\right) \leq \sum_{i=1}^{\infty} \mu_2(H_i - \bigcup_{i=1}^{\infty} E_i) \\ &\leq \sum_{i=1}^{\infty} \mu_2(H_i - E_i) < \epsilon. \end{aligned}$$

So $m_*(\bigcup_{i=1}^{\infty} F_i) = m^*(\bigcup_{i=1}^{\infty} F_i)$.

Proof of (b).

$$\begin{aligned} \mu_2\left(\bigcap_{i=1}^{\infty} E_i\right) &\leq m_*\left(\bigcap_{i=1}^{\infty} F_i\right) \leq m^*\left(\bigcap_{i=1}^{\infty} F_i\right) \leq m^*\left(\bigcap_{i=1}^{\infty} H_i\right) \\ &\leq \lim_{n \rightarrow \infty} m^*\left(\bigcap_{i=1}^n H_i\right) = \lim_{n \rightarrow \infty} \mu_2\left(\bigcap_{i=1}^n H_i\right) = \mu_2\left(\bigcap_{i=1}^{\infty} H_i\right) \end{aligned}$$

and

$$\begin{aligned}
\mu_2\left(\bigcap_{i=1}^{\infty} H_i\right) - \mu_2\left(\bigcap_{i=1}^{\infty} E_i\right) &= \mu_2\left(\bigcap_{i=1}^{\infty} H_i - \bigcap_{i=1}^{\infty} E_i\right) \\
&= \mu_2\left(\bigcup_{i=1}^{\infty} \left(\bigcap_{j=1}^{\infty} H_j - E_i\right)\right) \leq \sum_{i=1}^{\infty} \mu_2\left(\bigcap_{j=1}^{\infty} H_j - E_i\right) \\
&\leq \sum_{i=1}^{\infty} \mu_2(H_i - E_i) < \epsilon.
\end{aligned}$$

So $m_*(\bigcap_{i=1}^{\infty} F_i) = m^*(\bigcap_{i=1}^{\infty} F_i)$ and $\mathfrak{L}$ is a σ -lattice, thus completing a proof of Theorem 1.

III. A second approximation property. In order to state this approximation property succinctly, we recall some definitions.

A real valued function f defined on Ω_1 is said to be $\mathfrak{M}_2$ -measurable if $[f > a] \in \mathfrak{M}_2$, $a \in R$; if the range of f is a finite set, then f is called a simple function.

The Orlicz space $L^\Phi(R) = L^\Phi(\Omega_1, \mathfrak{A}_2, \mu_2, R)$ and the norm $N_\Phi(f)$ for $f \in L^\Phi(R)$ are defined on page 32 of [7]; R denotes the real numbers.

Theorem 2. *Suppose f is a $\mathfrak{M}_2$ -measurable function and $f \in L^\Phi(R)$, where Φ is a continuous Young's function satisfying the Δ_2 -condition. Then there exists a sequence $\{g_n\}$ of simple $\mathfrak{M}_1$ -measurable functions in $L^\Phi(R)$ satisfying $\lim_{n \rightarrow \infty} N_\Phi(f - g_n) = 0$.*

Proof. It suffices to establish Theorem 2 for non-negative functions. We shall first explain how the general case follows from the non-negative case and then establish the non-negative case. To begin, notice that $f = f \vee 0 + f \wedge 0$, where both $f \vee 0$ and $f \wedge 0$ are $\mathfrak{M}_2$ -measurable and $f \vee 0$ is non-negative. Denote $-(f \wedge 0)$ by g . Then g is a non-negative $\mathfrak{M}_2^c$ -measurable function, where $\mathfrak{M}_2^c$ denotes the sigma lattice comprised of the complements A^c of the elements A and $\mathfrak{M}_2$. Since $\mathfrak{M}_2^c$ is the sigma lattice generated by the lattice $\mathfrak{M}_1^c$, the non-negative case, applied to $\mathfrak{M}_1^c$ and $\mathfrak{M}_2^c$, implies the existence of a sequence $\{-h_n\}$ of simple $\mathfrak{M}_1^c$ -measurable functions approximating g . The sequence $\{h_n\}$ of simple $\mathfrak{M}_1$ -measurable functions approximates $-g = f \wedge 0$. Denote the simple $\mathfrak{M}_1$ -measurable functions approximating $f \vee 0$ by g_n and let $k_n = g_n + h_n$. Then, using the triangle inequality, we obtain

$$\begin{aligned}
N_\Phi(f - k_n) &= N_\Phi((f \vee 0 + f \wedge 0) - (g_n + h_n)) \\
&\leq N_\Phi(f \vee 0 - g_n) + N_\Phi(f \wedge 0 - h_n).
\end{aligned}$$

Hence, $\lim_{n \rightarrow \infty} N_\Phi(f - k_n) = 0$.

Turning now to the non-negative case, suppose $f \geq 0$. Since Φ satisfies the Δ_2 -condition (i.e., there exists a positive real number K such that $\Phi(2x) \leq K\Phi(x)$ for $x \geq 0$), by [5, Theorem 9.4, p. 76] it suffices to show that $\lim_{n \rightarrow \infty} \int_{\Omega_1} \Phi(f - g_n) d\mu_2 = 0$.

For each positive integer n , split $[0, n]$ into $n2^n$ equal parts of length $1/2^n$.

Since f is $\mathfrak{M}_2$ -measurable, by Theorem 1, we can find sets $E_{n,k} \in \mathfrak{M}_1$, where $k = 1, \dots, n2^n + 1$, such that

$$\mu_2\left(\left[f > n - \frac{j}{2^n}\right] \Delta E_{n,n2^n+1-j}\right) < \epsilon_n, \quad \text{where } j = 0, 1, \dots, n2^n - 1,$$

and $\mu_2([f \geq 0] \Delta E_{n,1}) < \epsilon_n$, where ϵ_n remains to be specified. (Note: $[f \geq 0] = \Omega_1$, so we can assume $E_{n,1} = \Omega_1$).

To simplify the notation, we shall temporarily fix the positive integer n and suppress it in the notation; thus we simply denote $E_{n,n2^n+1-j}$ by E_j and define the simple $\mathfrak{M}_1$ -measurable function

$$g_n = g = \begin{cases} n & \text{on } E_0 \\ n - \frac{j}{2^n} & \text{on } E_j \cap \left(\bigcup_{i=0}^{j-1} E_i\right)^c, \quad j = 1, \dots, n2^n. \end{cases}$$

To further simplify the notation, let $N = [f > n]$;

$$N_j = \left[n - \frac{j+1}{2^n} < f \leq n - \frac{j}{2^n}\right], \quad j = 0, \dots, n2^n - 2;$$

$$N_j = \left[0 \leq f \leq \frac{1}{2^n}\right], \quad j = n2^n - 1; \quad \text{and}$$

$$M_j = E_{j+1} \cap \left(\bigcup_{i=0}^j E_i\right)^c, \quad j = 0, \dots, n2^n - 1.$$

Then $(M_j)^c = (E_{j+1})^c \cup \left(\bigcup_{i=0}^j E_i\right)$. Hence

$$\int_{\Omega_1} \Phi(f - g) d\mu_2 = \int_N \Phi(f - g) d\mu_2 + \sum_{j=0}^{n2^n-1} \int_{N_j} \Phi(f - g) d\mu_2.$$

To handle the first term in the latter expression, notice that $\int_N \Phi(f - g) d\mu_2 \leq \int_N \Phi(f) d\mu_2 \rightarrow 0$ as $n \rightarrow \infty$ because Φ is increasing, $f \in L^\Phi(R)$, and $0 \leq f - g \leq f$ on N since $0 \leq g \leq n$ and $f > n$ on N . To treat the second part, split each term:

$$\int_{N_j} \Phi(f - g) d\mu_2 = \int_{N_j \cap M_j} \Phi(f - g) d\mu_2 + \int_{N_j \cap M_j^c} \Phi(f - g) d\mu_2,$$

and examine each resulting piece. Firstly,

$$\int_{N_j \cap M_j} \Phi(f - g) d\mu_2 \leq \int_{N_j \cap M_j} \Phi\left(\frac{1}{2^n}\right) d\mu_2 \leq \int_{N_j} \Phi\left(\frac{1}{2^n}\right) d\mu_2 = \Phi\left(\frac{1}{2^n}\right) \mu_2(N_j).$$

Moreover, for $j = 0, \dots, n2^n - 2$,

$$\{N_j \cap (E_{j+1})^c\} \subset \left\{\left[f > n - \frac{j+1}{2^n}\right] \Delta (E_{j+1})\right\}$$

and

$$\{N_i \cap E_i\} \subset \left\{ \left[f > n - \frac{i}{2^n} \right] \Delta(E_i) \right\} \quad \text{for } i = 0, \dots, j.$$

Finally, for $j = n2^n - 1$,

$$\{N_i \cap (E_{i+1})^c\} \subset \{[f \geq 0] \Delta(E_{i+1})\} \quad \text{and}$$

$$\{N_i \cap E_i\} \subset \left\{ \left[f > n - \frac{i}{2^n} \right] \Delta(E_i) \right\}, \quad \text{for } i = 0, \dots, j.$$

Hence $\mu_2(N_i \cap (M_i)^c) \leq (j+2)\epsilon_n \leq (n2^n + 1)\epsilon_n$ and

$$\int_{N_i \cap M_i^c} \Phi(f - g) d\mu_2 \leq \int_{N_i \cap M_i^c} \Phi(n) d\mu_2 \leq \Phi(n)(n2^n + 1)\epsilon_n$$

for $j = 0, 1, \dots, n2^n - 1$. It follows that

$$\begin{aligned} \sum_{i=0}^{n2^n-1} \int_{N_i} \Phi(f - g) d\mu_2 &\leq \Phi\left(\frac{1}{2^n}\right) \sum_{i=0}^{n2^n-1} \mu_2(N_i) + (n2^n)\Phi(n)(n2^n + 1)\epsilon_n \\ &< \Phi\left(\frac{1}{2^n}\right) \mu_2(\Omega_1) + \Phi(n)(n2^n)(n2^n + 1)\epsilon_n \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ if the ϵ_n are chosen so that $\Phi(n)(n2^n)(n2^n + 1)\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Thus, $\lim_{n \rightarrow \infty} \int_{\Omega_1} \Phi(f - g_n) d\mu_2 = 0$.

In [4] S. Johansen defined, constructed, and established the uniqueness of a Radon-Nikodym derivative for sigma additive set functions defined on sigma lattices of sets. We shall obtain information about λ and μ by applying Johansen's results to λ_2 and μ_2 . In order to explain our application we transcribe an appropriate version of Johansen's result.

Definition 1. We call f a derivative of λ_2 with respect to μ_2 on $\mathfrak{N}_2$ if f is an extended real valued function defined on Ω_1 , and if

- (1) f is $\mathfrak{N}_2$ -measurable,
- (2) $\lambda_2(A \cap [f < b]) \leq b\mu_2(A \cap [f < b])$, $A \in \mathfrak{N}_2$, $b \in \mathbb{R}$,
- (3) $\lambda_2(B^c \cap [f > a]) \geq a\mu_2(B^c \cap [f > a])$, $B \in \mathfrak{N}_2$, $a \in \mathbb{R}$.

Theorem 3. (Johansen). *There exists a derivative f of λ_2 with respect to μ_2 ; f is finite μ_2 almost surely. If g is also a derivative of λ_2 with respect to μ_2 , then $\mu_2([f \neq g]) = 0$.*

In [1] R. B. Darst established the analogous definition, construction, and uniqueness of a Radon-Nikodym derivative for non-negative, bounded, finitely additive set functions defined on lattices of sets.

Definition 2. We call a sequence $\{f_n\}$ of non-negative simple $\mathfrak{M}$ -measurable functions defined on Ω a derivative of λ with respect to μ if

$$\{f_n\} \text{ is a } L_1\text{-Cauchy sequence: } \int_{\Omega} |f_m - f_n| d\mu \rightarrow 0,$$

and for every $\epsilon > 0$, $M > 0$ there is a positive integer N such that if $n > N$, $a, b \in R$, $b < M$, and $A, B \in \mathfrak{M}$, then

$$(2-i) \lambda(A \cap [f_n \leq b]) < b\mu(A \cap [f_n \leq b]) + \epsilon,$$

$$(2-ii) \lambda(B^c \cap [f_n > a]) > a\mu(B^c \cap [f_n > a]) - \epsilon.$$

Suppose we let $\Phi(x) = |x|$, $x \in R$. Then $L^{\Phi} = L_1$, and applying Theorem 2 to the function f obtained via Theorem 3 yields a L_1 -Cauchy sequence $\{g_n\}$ of simple $\mathfrak{M}_1$ -measurable functions. The inverse image, under τ , of the sequence $\{g_n\}$ is a sequence $\{f_n\}$ of simple $\mathfrak{M}$ -measurable functions which is L_1 -Cauchy with respect to μ . When λ is non-negative, the sequence $\{f_n\}$ can be shown to be a Radon-Nikodym derivative of λ with respect to μ (i.e., $\{f_n\}$ satisfies the requirements of Definition 2). Whether or not λ is non-negative, it seems reasonable to call $\{f_n\}$ a Radon-Nikodym derivative of λ with respect to μ . Unfortunately, even when $\lambda \geq 0$, the characterization is ugly: the appearance of M in Definition 2 is a technical inconvenience made necessary by the fact that $[f_n \leq b] = \Omega$ for fixed n and large positive b . When λ is not assumed to be non-negative, yet another constraint must be introduced to account for the fact that $[f_n > a] = \Omega$ for fixed n and large negative a . The requisite modification should be clear to the reader.

The isomorphisms and approximations associated with maps $\lambda \rightarrow \lambda_1 \rightarrow \lambda_2$ and $\mu \rightarrow \mu_1 \rightarrow \mu_2$ permit us to obtain a variety of results. For example, considering additive functions defined on a lattice of sets with values in a Banach lattice leads to a study analogous to that which J. J. Uhl began in [7] for Banach space valued additive functions defined on an algebra of sets. Several of these results will appear in a subsequent paper. Unfortunately, even when the functions are real valued, statements of some of these results are complicated. Nevertheless, all of the results for algebras which one can reasonably expect to establish for lattices seem to be true. For instance, the Radon-Nikodym derivative of an L^{Φ} -function is an L^{Φ} -function, the appropriate L^{Φ} -martingale convergence theorem is valid, and, when $\Phi(x) = |x|^2$, the Radon-Nikodym derivative (conditional expectation) of a L^{Φ} -function, f , is the best admissible L^{Φ} -approximation to f . Furthermore, appropriate V^{Φ} versions of these theorems obtain, where V^{Φ} denotes the completion of L^{Φ} .

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