The Lusin-Menchoff Theorem in Metric Space

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Introduction. A number of authors have investigated the d -topology in Euclidean n -space E_n .

This topology is defined by Goffman and Waterman [1] as that topology in which the open sets are those Lebesgue measurable sets having each of its points as a point of density.

A point x is said to be a point of density of a measurable set A if

$$\lim_{d(I) \rightarrow 0} \frac{|A \cap I|}{|I|} = 1$$

where I is an oriented interval containing x , $d(I)$ is the diameter of I , and $|E|$ denotes the Lebesgue measure of E .

Goffman, Nishiura and Neugebauer [2] showed that if the I 's in the above definition are allowed to be arbitrary oriented intervals, then the d -topology is not completely regular. They showed further that if only those intervals having the ratio of the lengths of their sides bounded, above and below, are considered, then the d -topology is completely regular.

Using oriented squares, Troyer and Ziemer [3] proved complete regularity of the corresponding topology for outer Caratheodory measures defined on E_n .

Since then, the complete regularity of the d -topology has been used in various ways by Zink [4] and Ionescu-Tulcea [5, pp. 54–65]. It seems, therefore, that a generalization to more abstract spaces would be useful as well as intrinsically interesting.

In this paper, we prove the crucial lemma, the Lusin-Menchoff theorem, in metric space. A Morse halo condition [6] on a class of distinguished sets takes the place of regularity conditions on the intervals.

Our proof is simpler than Troyer and Ziemer's, yet it includes their result. Using the Lusin-Menchoff theorem and a lemma of Zahorski [7], the complete regularity of the d -topology in metric space follows as in the previous cases.

The Lusin-Menchoff theorem. Let Φ denote a finite regular outer Caratheodory measure on a metric space M with metric ρ . If $A \subset M$, then $d(A)$ will

denote the diameter of A . A ball of radius r will be denoted $S(r)$; the ball of radius r centered at x will be denoted $S(x, r)$.

In what follows M is assumed to contain $\mathcal{g}$, a distinguished family of sets. To each $I \in \mathcal{g}$ corresponds $I' = \{a \mid \rho(a, I) \leq 2d(I)\}$. The family $\mathcal{g}$ has the following properties:

1. If I is in $\mathcal{g}$, then I is Φ -measurable and $\Phi(I) > 0$.
2. There is a positive constant λ so that $\Phi(I') \leq \lambda\Phi(I)$.
3. If $x \in M$ and $\epsilon > 0$, then there is an $I \in \mathcal{g}$ so that $x \in I$ and $d(I) < \epsilon$.

If A is a measurable set, then A is said to be Φ -dense at x if

$$\lim_{\substack{d(I) \rightarrow 0 \\ x \in I}} \frac{\Phi(A \cap I)}{\Phi(I)} = 1.$$

Assume that $\Phi(S(x, r))$ is positive, that Φ is finite and, finally that

$$\alpha(r) = \inf_{x \in M} \Phi(S(x, r)) > 0.$$

Note that this last assumption holds if M is a compact metric space.

Remark 1. $d(I) \geq d(I')/5$ and I' contains a sphere $S(d(I))$.

Remark 2. If the elements of the family $\mathcal{g}$ are closed sets, then the family is a Vitali cover, and as Morse [6] has shown, the above conditions are sufficient in order that we have a Vitali covering theorem, which enables one to establish the existence of the d -topology.

Lemma. If $\gamma(\delta) = \inf_{I \in \mathcal{g}} \{\Phi(I) \mid d(I) \geq \delta\}$, $\delta > 0$, then $\gamma(\delta) > 0$.

Proof. Assume that there is a $\delta' > 0$ so that $\gamma(\delta') = 0$. For any $\epsilon > 0$ there is an I with $d(I) \geq \delta'$ such that $\Phi(I) < \epsilon$. Then I' contains an $S(\delta')$ and

$$0 < \alpha(\delta') \leq \Phi(S(\delta')) \leq \Phi(I') < \lambda\Phi(I) < \lambda\epsilon.$$

Since the choice of $\epsilon > 0$ was arbitrary, we have a contradiction.

The Lusin-Menchoff Theorem. If X is a closed set contained in E , a Φ -measurable set, and if every point of X is a point of Φ -density of E , then there is a closed set F , $X \subset F \subset E$, such that every point of X is a point of density of F .

Proof. Define for each positive integer n

$$R_n = \left\{ a \mid \frac{1}{n+1} < \rho(a, X) \leq \frac{1}{n} \right\} \cap E.$$

Without loss of generality, we may assume that

$$E = \left(\bigcup_1^\infty R_n \right) \cup X.$$

The R_n 's are Φ -measurable. By the regularity of Φ , for each $n = 1, 2, \dots$, there is a closed $F_n \subset R_n$ such that

$$\Phi(R_n \setminus F_n) < \frac{\gamma(1/5n)}{2^n}.$$

Put $F = (\bigcup_1^\infty F_n) \cup X$; then F is closed.

We need only consider the case when x is a boundary point of X . Let $\{I_i\}$ be a sequence of sets of $\mathcal{G}$ so that $d(I_i) \rightarrow 0$ and $x \in \bigcap_1^\infty I_i$.

Since x is a boundary point, the I'_j , $j = 1, 2, \dots$, are neighborhoods of x , and $d(I'_j) \rightarrow 0$, there is a j such that $I'_j \cap R_{n_i} \neq \emptyset$ and $I'_j \cap R_n = \emptyset$ if $n < n_i$. Therefore I'_j has diameter no smaller than $1/(n_i + 1)$, and so $d(I_i) \geq 1/5(n_i + 1)$.

The lemma implies that

$$\Phi(I_i) \geq \gamma(1/5(n_i + 1)) > 0.$$

Since the F_n 's, I_i 's, and X are measurable sets and γ is nondecreasing, we have

$$\begin{aligned} \Phi(I_i \cap F) &= \Phi\left(I_i \cap \left[\bigcup_1^\infty F_n \cup X\right]\right) \\ &= \Phi\left(I_i \cap \left[\bigcup_{n_i}^\infty F_n \cup X\right]\right) \\ &= \Phi\left(I_i \cap \bigcup_{n_i}^\infty F_n\right) + \Phi(I_i \cap X) \\ &\geq \sum_{n_i}^\infty \Phi(I_i \cap R_n) - \sum_{n_i}^\infty \frac{\gamma(1/5n)}{2^n} + \Phi(I_i \cap X) \\ &\geq \sum_{n_i}^\infty \Phi(I_i \cap R_n) + \Phi(I_i \cap X) - \frac{\gamma(1/5n_i)}{2^{n_i-1}} \\ &= \Phi(I_i \cap E) - \frac{\gamma(1/5n_i)}{2^{n_i-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\Phi(I_i \cap F)}{\Phi(I_i)} &\geq \frac{\Phi(I_i \cap E)}{\Phi(I_i)} - \frac{\gamma(1/5n_i)}{\Phi(I_i) \cdot 2^{n_i-1}} \\ &\geq \frac{\Phi(I_i \cap E)}{\Phi(I_i)} - \frac{\gamma(1/5n_i)}{2^{n_i-1} \cdot \gamma(1/5n_i)}. \end{aligned}$$

As $j \rightarrow \infty$, the right hand side in the above inequality goes to one, and the theorem is proved.

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