A Converse of Taylor's Theorem

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Nelson [5] has given a simple proof of a converse of Taylor's theorem which had been proved earlier [1], [3], [4] by different means. It will be shown here that a modification of Nelson's approach yields a more general result and eliminates the need for a certain theorem of multilinear algebra.

We recall first some definitions and standard notation (see *e.g.* [2]). Let E, F be Banach spaces. Then $L(E, F)$ denotes the Banach space of continuous linear mappings from E into F with the usual operator norm, $L^k(E, F)$ denotes the Banach space of continuous k -multilinear mappings $v: E \times \cdots \times E \rightarrow F$ which may be identified with the spaces defined inductively as follows:

$$L^0(E, F) = F, L^k(E, F) = L(E, L^{k-1}(E, F)) = L^{k-1}(E, L(E, F)).$$

If $v \in L^k(E, F)$ and $(z_1, \dots, z_k) \in E^k$, we shall sometimes write $v.z_1 \cdots z_k$ instead of $v(z_1, \dots, z_k)$ and $v.z^k$ instead of $v(z, \dots, z)$. The symmetrization Sv of v is defined by

$$Sv.z_1 \cdots z_k = \frac{1}{k!} \sum_{\sigma} v.z_{\sigma(1)} \cdots z_{\sigma(k)},$$

where the sum is taken over all permutations σ of the set $\{1, \dots, k\}$.

Now let U be an open set in E . A mapping $f: U \rightarrow F$ is said to be differentiable at $x \in U$ if there exists a mapping $Df(x) \in L(E, F)$ such that

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - Df(x) \cdot h\|}{\|h\|} = 0.$$

In this case f is continuous at x and $Df(x)$, which is unique, is called the derivative of f at x . One defines in the usual manner higher order derivatives $D^k f: U \rightarrow L^k(E, F)$ and the notions of a C^k -mapping and a C^∞ -mapping.

The following statement is an immediate consequence of Taylor's theorem with an integral remainder (see [2]): if f is a C^r -mapping, $r \geq 0$, then for every $x_0 \in U$ there exist neighborhoods V of x_0 in U and W of 0 in E such that for every $(x, h) \in V \times W$

$$f(x+h) = \sum_{k=0}^r \frac{D^k f(x) \cdot h^k}{k!} + \rho(x, h),$$

where $\rho: V \times W \rightarrow F$ is continuous, $\rho(x, 0) = 0$ and

$$\lim \frac{\|\rho(x, h)\|}{\|h\|^r} = 0 \quad \text{as } (x, h) \rightarrow (x_0, 0).$$

Our aim is to prove the following

Theorem. Let $\Omega = \{(x, h) \in U \times E \mid x + h \in U\}$, where E is a Banach space and U is open in E . Let F be a Banach space, and $f: U \rightarrow F$, $\varphi_k: U \rightarrow L^k(E, F)$, $0 \leq k \leq r$, be given mappings ($r \geq 0$). Assume that the mapping $\rho_r: \Omega \rightarrow F$ defined by

$$(1) \quad \rho_r(x, h) = f(x + h) - \sum_{k=0}^r \frac{\varphi_k(x) \cdot h^k}{k!}, \quad (x, h) \in \Omega,$$

satisfies the following conditions:

- (i) For every $x \in U$, $\rho_r(x, 0) = 0$.
- (ii) For every $x_0 \in U$, $\lim \|\rho_r(x, h)\|/\|h\|^r = 0$ as $(x, h) \rightarrow (x_0, 0)$.

Then f is a C^r -mapping, $D^k f = S\varphi_k$ for all k , $0 \leq k \leq r$, and ρ_r is continuous on Ω .

Remarks. 1. The present theorem differs from the others quoted in that it does not assume the continuity of any of the mappings f , φ_k , ρ_r .

2. Condition (ii) cannot be weakened to $\|h\|^{-r} \|\rho_r(x_0, h)\| \rightarrow 0$ for $h \rightarrow 0$. Example: the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, $f(0) = 0$ is not of class C^1 .

3. Taylor's Theorem and the above converse imply immediately the known fact that if f is a C^r -mapping, then $D^k f(x)$ is symmetric for all $x \in U$, $2 \leq k \leq r$, cf. [5].

Proof. We say that a function f which satisfies the hypotheses of the theorem has a Taylor expression of order r , and we prove the statement by induction on r . The case $r = 0$ is immediate: for $(x, h) \in \Omega$,

$$f(x + h) = \varphi_0(x) + \rho_0(x, h)$$

and (i), (ii) imply that $\varphi_0 = f$ and f is continuous on U . Since ρ_0 is the difference of continuous functions, it is itself one.

Assume now that the theorem is valid for all Taylor expressions of order $\leq r - 1$ ($r \geq 1$), and let f have a Taylor expression of order r . Let $x_0 \in U$ and choose a neighborhood V of x_0 and an open ball B of radius $2a$ centered at 0 such that $V + B \subset U$. Let $x \in V$ and $h, \ell \in E$ with $\|\ell\| < \|h\| < a$. We expand $f(x + h + \ell)$ once about the "center" x and once about the "center" $x + h$:

$$f(x + h + \ell) = \sum_{k=0}^r \varphi_k(x) \cdot (h + \ell)^k / k! + \rho_r(x, h + \ell)$$

$$f(x + h + \ell) = \sum_{k=0}^r \varphi_k(x + h) \cdot \ell^k / k! + \rho_r(x + h, \ell).$$

Let

$$\sigma_r(x, h, \ell) = \rho_r(x, h + \ell) - \rho_r(x + h, \ell).$$

By subtracting the expressions for $f(x + h + \ell)$ and collecting the coefficients of ℓ^k , we get

$$(2) \quad \sigma_r(x, h, \ell) = g_0(x, h) + g_1(x, h) \cdot \ell + \cdots + g_r(x, h) \cdot \ell^r,$$

where $g_p(x, h) \in L^p(E, F)$, $0 \leq p \leq r$. Since $\|\ell\| < \|h\|$ we have

$$\frac{\|\sigma_r(x, h, \ell)\|}{\|h\|^r} < \frac{\|\rho_r(x, h + \ell)\|}{\|h + \ell\|^r} + \frac{\|\rho_r(x + h, \ell)\|}{\|\ell\|^r}$$

and hence, as $(x, h) \rightarrow (x_0, 0)$, with $\|\ell\| < \|h\|$, we have $\|h\|^{-r} \|\sigma_r(x, h, \ell)\| \rightarrow 0$.

Now let ℓ, h be fixed, $0 < \|\ell\| < \|h\|$, and choose $r + 1$ distinct real numbers $\lambda_0, \dots, \lambda_r$, such that $|\lambda_k| \leq 1$, $0 \leq k \leq r$. Consider the system of $r + 1$ linear equations in $g_k(x, h) \cdot \ell^k$, $k = 0, \dots, r$, obtained from (2) by replacing ℓ by $\lambda_k \ell$. The determinant of the coefficient matrix is a Vandermonde determinant and hence is non-zero. Therefore the quantities $g_k(x, h) \cdot \ell^k$ can be represented as linear combinations of $\sigma_r(x, h, \lambda_0 \ell), \dots, \sigma_r(x, h, \lambda_r \ell)$. Consequently, for all k , $0 \leq k \leq r$,

$$\|h\|^{-r} \|g_k(x, h) \cdot \ell^k\| \rightarrow 0$$

as $(x, h) \rightarrow (x_0, 0)$ with $\|\ell\| < \|h\|$.

We shall apply this result to g_1 . From its definition we get

$$\begin{aligned} g_1(x, h) \cdot \ell &= \varphi_1(x + h) \cdot \ell - \varphi_1(x) \cdot \ell - \varphi_2(x)(\ell, h)/2! - \varphi_2(x)(h, \ell)/2! \\ &\quad - \cdots - \varphi_r(x)(\ell, h, \dots, h)/r! - \varphi_r(x)(h, \ell, \dots, h)/r! \\ &\quad - \cdots - \varphi_r(x)(h, h, \dots, h, \ell)/r! \end{aligned}$$

or

$$(3) \quad g_1(x, h) \cdot \ell = \left[\varphi_1(x + h) - \sum_{k=1}^r \frac{S\varphi_k(x) \cdot h^{k-1}}{(k-1)!} \right] \cdot \ell,$$

where, for $1 \leq k \leq r$, $S\varphi_k(x) \in L^{k-1}(E, L(E, F))$.

We estimate the norm of $g_1(x, h) \in L(E, F)$ for $h \neq 0$. We have

$$\|g_1(x, h)\| = \sup_{\|\ell\| < \|h\|} \frac{\|g_1(x, h) \cdot \ell\|}{\|\ell\|}.$$

It follows that for each $x_0 \in U$ we have

$$(4) \quad \lim_{(x, h) \rightarrow (x_0, 0)} \frac{\|g_1(x, h)\|}{\|h\|^{r-1}} = 0.$$

Now, by (3) and (4), φ_1 has a Taylor expression of order $r - 1$ whose remainder term, g_1 , satisfies (i) and (ii). It follows from the induction hypothesis that φ_1 is a C^{r-1} -mapping and $D^i \varphi_1 = S\varphi_{i+1}$ for $0 \leq i \leq r - 1$.

It remains to identify $S\varphi_k$ as $D^k f$. To this end we write

$$(5) \quad \begin{aligned} f(x+h) - \varphi_0(x) - \varphi_1(x)h &= \sum_{k=2}^r \frac{\varphi_k(x)h^k}{k!} + \rho_r(x, h) \\ &= \rho_1(x, h). \end{aligned}$$

We note that $S\varphi_k(x) \cdot h^k = \varphi_k(x) \cdot h^k$ for $1 \leq k \leq r$. It was shown just above that the $S\varphi_k$ are continuous and hence bounded in norm in a neighborhood of any $x_0 \in U$. Also ρ_r satisfies (ii). Therefore ρ_1 satisfies (i) and (ii). We read off from (5) that $f = \varphi_0$, $Df = \varphi_1$, and φ_0 is continuous on U . Finally, it follows from (1) that ρ_r is continuous on Ω .

Remark. It seems necessary to apply the following fact from multilinear algebra to complete Nelson's proof that $D^k f(x) = a_k(x)$ [5; page 8]. For any symmetric multilinear mapping $\nu: E^k \rightarrow F$ one can show that

$$\begin{aligned} \sum_{i_1=0}^1 \cdots \sum_{i_k=0}^1 (-1)^{i_1+\cdots+i_k} \nu(x_1 + (-1)^{i_1}x_2 + \cdots + (-1)^{i_k}x_k)^k \\ = 2^{k-1}k! \nu(x_1, x_2, \cdots, x_k). \end{aligned}$$

It follows that the norm of a symmetric multilinear mapping can be estimated by considering its restriction to the set $\Delta = \{(h, h, \cdots, h) \in E^k\}$. Our approach enables us to avoid this argument.

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