A_n -Actions on Fibre Spaces

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Communicated by A. WALLACE

1. Introduction. A familiar problem in the study of fibre spaces is deciding when a given fibre space $F \rightarrow E \rightarrow B$ is fibre homotopy equivalent to one induced from a path fibration $\Omega B \rightarrow \mathcal{L}B \rightarrow B$. The problem of characterizing such fibre spaces was first raised by Massey in [9], where he asked when an inclusion $F \subset E$, is, up to homotopy equivalence, the inclusion of a fibre into the total space of a fibre space.

Meyer in [10], shows that $F \rightarrow E \rightarrow B$ is an induced fibre space if F has at most $m - 1$ non-vanishing homotopy groups in consecutive dimension, provided that F has the homotopy type of a loop space and B is $(m - 1)$ -connected. In [7], Ganea gives results which allows F to have $2m - 1$ non-vanishing homotopy groups in consecutive dimensions, provided B is $(m - 1)$ -connected and the homotopy equivalence between F and the loop space fulfills a certain condition which involves the operation of ΩB on F .

Dold and Lashof in [3], give the following definition of a principal fibre space $p: E \rightarrow B$.

Definition 1.1. Let $p: E \rightarrow B$ be a fibre space, and let (F, m) be an H -space with unit e . An operation (or action) of F in p is a continuous map $m': E \times F \rightarrow E$ satisfying

- 1) $m'(x, e) = x, \quad x \in E,$
- 2) $pm'(x, f) = p(x), \quad x \in E, f \in F,$
- and
- 3) the map $m'_x: F \rightarrow p^{-1}(p(x))$ given by $m'_x(f) = m'(x, f) = xf$ induces an isomorphism

$$m'_x{}^*: \pi_i(F) \cong \pi_i(p^{-1}(p(x)))$$

for all $x \in E, i \geq 0$.

If, in addition (F, m) is an associative H -space and F operates in p such that $m' \circ (1 \times m) = m' \circ (m' \times 1)$, then the fibre space is called principal.

If p is induced from $\Omega A \rightarrow \mathcal{L}A \rightarrow A$, then p is a principal fibre space. The action of ΩA on E is given by

$$m'((b, \lambda), \mu) = (b, \lambda + \mu)$$

where

$$\lambda : [0, r] \rightarrow A$$

$$\mu : [0, s] \rightarrow A$$

with

$$\lambda(0) = \mu(0) = \mu(s) = * \text{ and } \lambda(r) = b.$$

$\lambda + \mu$ runs along μ from 0 to s and then along λ from s to $s + r$. This notation is non-standard but useful to us. The map m'_x (where $x = (b, \lambda)$) is a homotopy equivalence. An inverse is given by

$$(b, \lambda') \rightarrow -\lambda + \lambda'$$

where $-\lambda + \lambda'$ means “ λ' , then back along λ ”.

Let $\Sigma = \{E, p, B, F\}$ be a principal fibre space. In [3], Dold and Lashof describe a construction of quasifibrations which when applied to a principal quasifibration can be iterated, giving a sequence: $\Sigma, DL(\Sigma), DL^2(\Sigma), \dots$ of principal quasifibrations. (Here $DL^j(\Sigma)$ refers to the j^{th} iterative in the Dold and Lashof construction.) By taking the limit in the proper way, they obtain a universal principal quasifibration

$$DL^\infty(\Sigma) = \{DL^\infty(E), DL^\infty(p), DL^\infty(B), F\}.$$

We will now consider weakening conditions. Meyer in [10] gives a generalization of principal fibre space, when he introduces the notion of an H -fibration. This is a concept intermediate between that of an arbitrary fibre space and of a principal fibre space. It stands in the same relation to principal fibre space (loop fibration) as the notion of H -space does to that of associative H -space (loop space).

Definition 1.2. A fibre space $F \xrightarrow{i} E \xrightarrow{p} B$ is an H -fibration if there are maps m and m' such that the diagram

$$\begin{array}{ccccc} F \times F & \xrightarrow{i \times 1} & E \times F & \xrightarrow{p \times 0} & B \times * \\ \downarrow m & & \downarrow m' & & \parallel \\ F & \xrightarrow{i} & E & \xrightarrow{p} & B \end{array}$$

strictly commutes and m is an H -structure on F .

Demanding that E and B have the homotopy type of CW -complexes and with the conditions mentioned earlier, Ganea [7], proves that an H -fibration $F \rightarrow E \rightarrow B$ is induced by some map $B \rightarrow A$ if there is an H -map $r : F \rightarrow \Omega A$, where H -map is defined as follows.

Definition 1.3. A map $r : F \rightarrow G$ where F and G are H -spaces, is an H -map if homotopy commutativity holds in the following diagram.

$$\begin{array}{ccc}
 F \times F & \xrightarrow{r \times r} & G \times G \\
 \downarrow m & & \downarrow m' \\
 F & \xrightarrow{r} & G
 \end{array}$$

Porter in [13] gives the following definition.

Definition 1.4. A fibre preserving map $g : \Sigma \rightarrow \Sigma'$ between principal fibrations is called a homomorphism if

$$\begin{array}{ccc}
 F \times E & \xrightarrow{\sigma \mid F \times g} & F' \times E' \\
 \downarrow m' & & \downarrow \mu' \\
 E & \xrightarrow{g} & E'
 \end{array}$$

commutes.

Homotopy commutativity of the preceding diagram is not sufficient to ensure the existence of a map between the associated universal quasifibrations. However, Porter gives higher homotopy conditions, which, if satisfied, allows the construction of such maps. Porter calls maps satisfying these higher homotopy conditions strong homotopy homomorphisms.

Using the concept of strong homotopy homomorphisms, he is able to show that there exists $f : B \rightarrow B'$ such that $F \rightarrow E \rightarrow B$ is equivalent to the fibre space induced by f from $F' \rightarrow E' \rightarrow B'$.

The present paper will characterize certain fibre spaces which are equivalent to induced fibre spaces. We restrict our attention to strictly associative H -spaces which operate in fibre spaces satisfying certain appropriate homotopy conditions. Our approach will essentially include Meyer's notion of H -fibrations.

We deal with fibre spaces $\Sigma = \{E, p, B, F\}$ in which an associative H -space (F, m) operates, but not necessarily associatively. For instance, instead of requiring that $m' \circ (1 \times m) = m' \circ (m' \times 1)$ in the definition of a principal fibre space, we might demand only that the maps be homotopic, i.e., $m' \circ (1 \times m) \simeq m' \circ (m' \times 1)$. That is, $m' : E \times F \rightarrow E$ is a homotopy associative operation. We will call such a fibre space A_2 -principal. A fibre space in which F operates and no further conditions are required is called an A_1 -principal fibre space. For instance, Meyer's H -fibration is an example of an A_1 -principal space if it satisfies the remaining conditions. In general, A_n -principal fibre spaces are defined to be fibre spaces in which the operation of F on E satisfies higher homotopy conditions. We will also have the notion of an A_∞ -principal fibre space.

In section 5, we introduce a construction of A_n -principal fibre spaces (actually for technical reasons, the construction is of A_n -principal quasifibrations) which we call an A_n -construction. This construction is iterative, giving a sequence of A_n -principal fibre spaces. Starting with the trivial fibre space $\pi : F \rightarrow *$, where F is an associative H -space, we apply the A_n -construction iteratively and obtain in the limit a fibre space which provides us with $u_n : UE_n \rightarrow B_{H(n)}$, a universal example of A_n -principal fibre spaces. Thus, we will have the following classification theorem.

Theorem 1.5. *If F is an associative H -space, which is a finite dimensional CW-complex, there is a space $B_{H(n)}$ such that $[\ , B_{H(n)}]$ and $A_n F(\)$ are naturally equivalent as functors from the category of finite-dimensional CW-complexes and homotopy classes of maps to the category of sets and functions.*

Here $[X, Y]$ denotes the set of homotopy classes of maps of X into Y , and $A_n F(X)$ denotes the set of A_n -equivalence classes of A_n -principal fibre spaces $p : E \rightarrow B$ (see section 6 for definitions).

The space $H(n)$ consists of families of maps $\Gamma = \{\gamma_1, \dots, \gamma_{n+1}\}$ where γ_1 is a homotopy equivalence between F and itself and that up to homotopy conditions γ_1 looks like a translation. This resemblance is examined in more detail in [12]. The space $H(0) = H(F)$, is the topological monoid of homotopy equivalences of F into itself. The space $B_{H(n)}$ refers to the classifying space of a monoid $H(n)$. For instance, the space $B_{H(1)}$ classifies Meyer's H -fibration. Essentially, $F \simeq H(\infty) \subset \dots \subset H(n) \subset H(n-1) \subset \dots \subset H(0) = H(F)$. Pulling back $f : X \rightarrow B_{H(n-1)}$ to $B_{H(n)}$ corresponds to the fibration admitting the additional structure.

In section 7, our A_∞ -construction is shown to be compatible with the Dold and Lashof construction. That is, there exists a map $\bar{\xi} : DL^\infty(*) \rightarrow B_{H(\infty)}$ which induces a map $\bar{\xi}^* : [X, DL^\infty(*)] \rightarrow [X, B_{H(\infty)}]$. The map $\bar{\xi}$ carries a classifying map for p as a principal fibre space into a classifying map for p regarded as an A_∞ -principal fibre space.

In section 8, it is shown that the map $\bar{\xi}$ is actually a homotopy equivalence induced by a homotopy equivalence between F and $H(\infty)$. This being the case, $\bar{\xi}^*$ is an isomorphism.

Thus, given a fibre space $p : E \rightarrow B$ which admits an A_∞ -action, we may give the interpretation that, up to homotopy, there is an action $m' : E \times F \rightarrow E$ which acts associatively. More precisely, we will have

Theorem 1.6. *$p : E \rightarrow B$ is fibre homotopy equivalent to an induced fibre space if and only if E admits an A_∞ -action of ΩB .*

We include an appendix which contains propositions relating quasi-fibrations and fibre spaces, which are appealed to at various times in the paper. In this paper, fibre space will mean Hurewicz fibre space.

All spaces, not otherwise specified are assumed to be k -spaces. If after performing a construction, the resulting space is not a k -space, it is to be assumed that we have retopologized. In [14], [20] and [21], it is shown that the constructions in question then have the desired properties (references will be quoted where appropriate).

The following definition is given by Steenrod in [20].

Definition 1.7. If X is a Hausdorff space, the associated compactly generated space $k(X)$ (i.e. k -space) is the set X with the topology defined as follows: a closed set of $k(X)$ is a set that meets each compact set of X in a closed set.

If $f : X \rightarrow Y$ is a mapping of Hausdorff spaces, $k(f)$ denotes the same function $k(X) \rightarrow k(Y)$.

We adopt Steenrod's conventions as to notation: for instance, if Y and Z are k -spaces, their product $Y \times Z$ is $k(Y \times_e Z)$, where $\times_e$ denotes the product with the usual cartesian topology. Similarly for function spaces, $Y^X = kC(X, Y)$, where $C(X, Y)$ denotes the space of continuous mappings $X \rightarrow Y$ with the compact-open topology.

This paper presents the main results of my Ph.D. thesis completed at the University of Notre Dame. I wish to express my great appreciation of the invaluable assistance and encouragement of James D. Stasheff.

2. The Dold and Lashof construction. In this section we review the results of Dold and Lashof. The definition and properties of quasifibrations are given in [4].

The Dold and Lashof construction embeds a quasifibration in which an H -space (F, m) operates, in a quasifibration $DL(p) : DL(E) \rightarrow DL(B)$ such that the inclusion map is null-homotopic. The construction is as follows. Consider the diagram

$$\begin{array}{ccccc} CE \times F & \longrightarrow & CE \times F \cup_m E & = & DL(E) \\ \downarrow & & \downarrow & & \downarrow DL(p) \\ CE & \longrightarrow & CE \cup_p B & = & DL(B) \end{array}$$

where CE is the cone over E with E as the base of the cone. The horizontal lines are quotient maps and the vertical line on the left is projection onto the first factor.

Attach $CE \times F$ to E by the map m' , identifying

$$(x, f) \in E \times F \subset CE \times F$$

with

$$m'(x, f) = xf$$

To be more precise, a point in $DL(E)$ is denoted by a triple (t, x, f) with t in $[0, 1]$, $x \in E$ and $f \in F$ subject to the conditions that:

- (1) $(0, x, f)$ is identified with $f \in F$
- and (2) $(1, x, f)$ is identified with $xf \in E$.

Similarly, attach CE to B by the map p , identifying

$$x \in E \subset CE$$

with

$$p(x) \in B.$$

A point in $DL(B)$ is denoted by a pair (t, x) with $t \in [0, 1]$, and $x \in E$ such that (t, x) and (t', x') represents the same point if:

- (1) $t = t' = 0$
 or (2) $t = t' = 1$ and $p(x) = p(x')$.

The projection is given by

$$DL(p)(t, x, f) = (t, x)$$

While the topology used by Dold and Lashof is actually stronger than the identification (quotient) topology, Stasheff points out in [17] that this does not affect the property of being a quasifibration, nor does using the k -space topology according to Steenrod [20]. The point is that being a quasifibration involves only map of cells, *i.e.*, compact spaces.

In proposition 2.3 of [3], it is shown that $DL(p)$ is a quasifibration such that the inclusion $E \subset DL(E)$ given by $x \rightarrow (1, x, e)$ is null homotopic. The contraction of E in $DL(E)$ is given by $x \rightarrow (t, x, e)$.

This construction can be applied to any quasifibration $p : E \rightarrow B$ in which a (not necessarily associative) H -space operates. It is desired that the construction be made iterative in the sense that there exist an operation

$$DL(m') : DL(E) \times F \rightarrow DL(E)$$

which extends m' . To this end, Dold and Lashof require a principal quasifibration (see Definition 1.1).

Proposition 2.1. *If $p : E \rightarrow B$ is a principal quasifibration over an associative H -space (F, m) , then $DL(p) : DL(E) \rightarrow DL(B)$ can be given the structure of a principal quasifibration over F . The operation $DL(m')$ of F in $DL(p)$ extending the given operation m' in p is given by*

$$DL(m')[(t, x, f), g] = (t, x, fg)$$

Proof. This is proposition 3.2 of [3], but because of the change in topology, the continuity of $DL(m')$ requires new justification. In [14], Rothenberg and Steenrod prove that $DL(p)$ in the k -topology has a continuous action of $DL(E) \times F \rightarrow DL(E)$. The point is that a product of quotients is a quotient of the product in the category of k -spaces.

We point out that the associativity condition in the definition of principal quasifibration is necessary to make $DL(m')$ well defined. It is then the associativity condition which allows the Dold and Lashof construction to be iterated, producing a sequence of principal quasifibrations over F as follows. Let $DL^0(\Sigma) = \Sigma = \{E, p, B, F\}$ be a principal quasifibration over F . Define

$$DL^{n+1}(\Sigma) = DL(DL^n(\Sigma))$$

that is, $DL^{n+1}(\Sigma)$ is the principal quasifibration obtained by the construction described above.

We then have the sequence of principal quasifibrations

$$\begin{array}{ccccccc} E & \longrightarrow & DL(E) & \longrightarrow & \cdots & \longrightarrow & DL^n(E) \longrightarrow \cdots \\ \downarrow & & \downarrow & & & & \downarrow \\ B & \longrightarrow & DL(B) & \longrightarrow & \cdots & \longrightarrow & DL^n(B) \longrightarrow \cdots \end{array}$$

where all the horizontal maps are inclusions.

The limit $DL^\infty(\Sigma) = \{DL^\infty(E), DL^\infty(p), DL^\infty(B), F\}$ is defined as follows. The sets $DL^\infty(E)$, $DL^\infty(B)$ are respectively the direct limits of the sets $DL^n(E)$ and $DL^n(B)$. Steenrod in [21] shows that infinite unions of k -spaces are k -spaces. The map $DL^\infty(p): DL^\infty(E) \rightarrow DL^\infty(B)$ is given by $DL^\infty(p) \mid DL^n(E) = DL^n(p)$.

It is proven in Theorem 3.5 of [3] that $DL^\infty(p)$ together with the operation $DL^\infty(m'): DL^\infty(E) \times F \rightarrow DL^\infty(E)$ given by $DL^\infty(m') \mid DL^n(E) \times F = DL^n(m')$ is a universal principal quasifibration.

By universal is meant that $DL^\infty(E)$ is aspherical. Of course, Dold and Lashof have a different topology on $DL^\infty(E)$ than we do, however, Rothenberg and Steenrod in [14] prove that $DL^\infty(p)$ in the k -topology has a continuous action $DL^n(E) \times F \rightarrow DL^n(E)$ for all n including ∞ .

Continuing to recall the results of Dold and Lashof, we note that there always exists a universal principal quasifibration with fibre F , namely $\Sigma_F = \{DL^\infty(F), DL^\infty(\pi), DL^\infty(*), F\}$ which is obtained by applying the construction to the trivial fibring $\pi: F \rightarrow *$ (a point), iterating and taking the limit in the manner described above. As an application, Dold and Lashof get a classification of fibre bundles with respect to fibre homotopy equivalence.

3. A_n -principal Quasifibrations. It is natural to wonder if any of the conditions required in the definition of principal quasifibration can be weakened, and still have an iterative construction of quasifibrations as in [3]. For instance, suppose instead of requiring strict associativity of the operation $m': E \times F \rightarrow E$, we require that it be fibrewise homotopy associative. That is, the following diagram is homotopy commutative

$$\begin{array}{ccc} E \times F \times F & \xrightarrow{1 \times m} & E \times F \\ \downarrow m' \times 1 & & \downarrow m' \\ E \times F & \xrightarrow{m} & E \end{array}$$

by a homotopy h_t such that $ph_t(x, f, g) = p(x)$.

Since the Dold and Lashof construction can be applied to any quasifibration in which an H -space operates, it in particular can be applied to the quasifibration $p: E \rightarrow B$ in which the operation of the associative H -space (F, m) in p is fibrewise homotopy associative. The problem is in attempting to define an operation of F in $DL(p)$. The operation given in Proposition 2.1 will not work, for in the present case, it is not well-defined.

Because of the homotopy associativity condition, there exists a map $H: I \times E \times F^2 \rightarrow E$ such that

$$(1) \ H(0, x, f, g) = x(fg),$$

$$(2) \ H(1, x, f, g) = (xf)g,$$

and

$$(3) \ pH(t, x, f, g) = p(x).$$

It is easily seen that the map

$$DL(m'): DL(E) \times F \rightarrow DL(E)$$

given by

$$DL(m')[[(t, x, f), g]] = \begin{cases} (2t, x, fg), & 0 \leq 2t \leq 1 \\ H(2t - 1, x, f, g), & 1 \leq 2t \leq 2, \end{cases}$$

is well-defined, extending m' .

Unfortunately, this map fails to be an operation in the sense of Dold and Lashof.

Suppose we were able to find a definition for $DL(m')$ which is an operation in the sense of Dold and Lashof and which extends m' . We would then be able to construct

$$DL^2(p) : DL^2(E) \rightarrow DL^2(B)$$

Even allowing this there still are great difficulties in attempting to define an operation of F in $DL^2(p)$ so as to extend $DL(m')$.

Considering the five ways of associating four points and the homotopies connecting them using H once, we have

$$(xf)(gh) \simeq ((xf)g)h \simeq (x(fg))h \simeq x((fg)h) = x(f(gh)) \simeq (xf)(gh).$$

This gives a map of $\delta I^2 \times E \times F^3 \rightarrow E$, and we would need to extend this map to $I^2 \times E \times F^3 \rightarrow E$ to define an operation of F in $DL^2(p)$ extending $DL(m')$. Thus we are led to require homotopy conditions more complicated than homotopy associative to ensure the existence of an operation meeting our needs in $DL^2(p)$. Since our goal is to have an iterative construction, we are led to make the following definitions.

Definition 3.1. Let (F, m) be an associative H -space. An A_n -action of F on a space X is a family of maps

$$\{\theta_i : I^{i-1} \times X \times F^i \rightarrow X, 1 \leq i \leq n\}$$

such that

$$\begin{aligned} & \theta_i(t_1, \dots, t_{i-1}, x, f_1, \dots, f_i) \\ &= \theta_{i-1}(t_1, \dots, \hat{t}_k, \dots, t_{i-1}, x, \dots, f_k f_{k+1}, \dots, f_i) \quad \text{if } t_k = 0, \\ &= \theta_{i-k}(t_{k+1}, \dots, t_{i-1}, \theta_k(t_1, \dots, t_{k-1}, x, \dots, f_k), \dots, f_i) \quad \text{if } t_k = 1. \end{aligned}$$

(Here $\hat{t}_k$ means that particular parameter is to be omitted.)

If F is a subspace of X we will require

$$\theta_i(t_1, \dots, t_{i-1}, f, f_1, \dots, f_i) = ff_1 \cdots f_i \quad \text{for all } i.$$

Actually since (F, m) is an associative H -space, it is clear that F acts on itself as an A_∞ -action (i.e., an A_n -action for all n).

Definition 3.2. Let (F, m) be an associative H -space. Let $p : E \rightarrow B$ be a quasifibration such that there exists an A_n -action of F on E . Then p is said to be an A_n -principal quasifibration if:

- (1) $\theta_1 | E \times \{e\} \simeq \pi_1 | E \times \{e\}$, where π_1 is projection on the first factor.
- (2) $p\theta_i(\tau, x, f_1, \dots, f_i) = kp(x)$, where $k : B \rightarrow B$ is homotopic to the identity and $\tau = (t_1, \dots, t_{i-1})$, $\forall i$.

and

- (3) The map $\theta_{1x} : F \rightarrow p^{-1}(p(x))$ given by $\theta_{1x}(f) = \theta_1(x, f) = xf$ induces an isomorphism

$$\theta_{1x}^* : \pi_1(F) \rightarrow \pi_1(p^{-1}(kp(x))) \quad \forall x \in E.$$

Note that any principal quasifibration as defined by Dold and Lashof admits an A_∞ -action, called the trivial A_∞ -action, and thus a principal quasifibration is an A_∞ -principal quasifibration.

4. The Dold and Lashof construction applied to A_n -principal quasifibrations. In section [3], we asserted that weakening the conditions of a principal quasifibration makes it difficult to find an operation in $DL(p)$, that respects all the structure of p . The fact is that in certain cases, no matter what operation is defined in an A_n -principal quasifibration $p : E \rightarrow B$, it is not possible to give $DL(p) : DL(E) \rightarrow DL(B)$ the structure of an A_n -principal quasifibration.

It is well known that S^7 can be given the structure of an H -space, thus having an A_1 -structure. In Theorem 1.4 of [8], James shows that S^7 cannot be given a homotopy associative multiplication; therefore, it cannot be given an A_2 -structure. We review these notions, offering an alternative proof that S^7 admits no homotopy associative multiplication.

What is new is the fact that starting with the trivial fibring $S^7 \rightarrow *$ applying the Dold and Lashof construction results in $S^{15} \rightarrow S^8$ and we show that S^7 does not operate in any non-trivial sense in any fibring $S^{15} \rightarrow S^8$.

To begin with, we recall that the form $X * Y$ is obtained from $X \cup X \times I \times Y \cup Y$ by identifying $(x, 0, y)$ with x and $(x, 1, y)$ with y for $x \in X, y \in Y$. In particular $S^r * S^t = S^{r+t+1}$. Assuming (X, m) is an arc-connected H -space, $X \rightarrow CX \times X \cup_m X \rightarrow SX$ is a quasifibration and $CX \times X \cup_m X$ has the homotopy type of $X * X$.

Supposing (X, m) to be a homotopy associative H -space (that is, it possesses an A_2 -structure), define

$$DL(m) : DL(X) \times X \rightarrow DL(X) \text{ by}$$

$$DL(m)[(t, x, y), z] = \begin{cases} (2t, x, yz), & 0 \leq 2t \leq 1 \\ H(2t - 1, x, y, z), & 1 \leq 2t \leq 2, \end{cases}$$

where H is a homotopy between $m \circ (m \times 1)$ and $m \circ (1 \times m)$.

Then we can define $DL^2(x)$ and $DL^2(p)$. $DL^2(x)$ has the homotopy type of $X * X * X$ if X is arc-connected. Altogether we have

$$\begin{array}{ccccc} X \text{ contractible in } & DL(X) & \text{contractible in} & & DL^2(X) \\ \downarrow p & & \downarrow DL(p) & & \downarrow DL^2(p) \\ * & \longrightarrow & S(X) & \longrightarrow & C[DL(X)] \cup_{DL(p)} SX \end{array}$$

In fact, if we have fibrings p_2 and p_3 such that

$$\begin{array}{ccccc}
X & \text{contractible in } E_1 & & \text{contractible in } E_2 & \\
\downarrow p_1 & & & \downarrow p_2 & \downarrow p_3 \\
* & \longrightarrow & B_2 & \longrightarrow & B_3
\end{array}$$

then X admits some homotopy associative multiplication by Theorem 1.4 of [22] if E_2 is a CW -complex containing E_1 as its subcomplex.

This is enough to prove our assertion that S^7 admits no homotopy associative multiplication and more important that S^7 does not act on any fibring $S^{15} \rightarrow S^8$, such as the Hopf fibring. For if it did, we would have a fibring

$$S^7 \rightarrow S^{23} \xrightarrow{\gamma} S^8 \cup_{\gamma} e^{16}$$

where γ is the Hopf map. By standard arguments, we could show that in $H^*(S^8 \cup_{\gamma} e^{16} \cup_p e^{24})$, the generator u of H^8 has non-trivial cube u^3 which generates H^{24} . But $u^3 \equiv 0 \pmod{3}$ since $u^3 \equiv \phi_3^4 u \pmod{3}$ while ϕ^4 decomposes through dimension 20 which is 0.

The previous argument that S^7 admits no homotopy associative multiplication is due to Adams [1].

Thus, in general, the Dold and Lashof construction applied to an A_n -principal quasifibration does not result in a quasifibration which can be given an A_n -principal structure.

There is one positive assertion that can be made; namely, given an A_n -principal quasifibration $p : E \rightarrow B$, the quasifibration $DL(p) : DL(E) \rightarrow DL(B)$ can be given an A_{n-1} principal structure. As we will not make use of this result in what is to follow, no proof will be given here.

5. The A_n -construction of quasifibrations. In this section, we describe a construction, to be called an A_n -construction, which can be iterated indefinitely, giving a sequence of A_n -principal quasifibration. Taking the limit of this sequence, one obtains a universal example and a classifying space for A_n -principal fibre spaces.

One construction is as follows. Let $p : E \rightarrow B$ be a quasifibration with fibres of the homotopy type of the associative H -space (F, m) and $F \subset E$. Let $\{\theta_i, 1 \leq i \leq n\}$ be an A_n -action of F on E . We associate with this quasifibration the map $A_n(p) : A_n(E) \rightarrow B$, where the space $A_n(E)$ is the set of all families of maps $\Phi = \{\varphi_1, \dots, \varphi_{n+1}\}$ subject to the conditions:

- (1) φ_1 is a homotopy equivalence between F and some fibre $p^{-1}(b)$, b in B .
- (2) $\varphi_i : F^{i-1} \times F^i \rightarrow p^{-1}(p(\varphi_1(F))) \subset E$, $1 \leq i \leq n+1$ such that

$$\varphi_i(t_1, \dots, t_{i-1}, f_1, \dots, f_i)$$

$$= \varphi_{i-1}(t_1, \dots, \hat{t}_k, \dots, t_{i-1}, f_1, \dots, f_k f_{k+1}, \dots, f_i) \quad \text{if } t_k = 0$$

$$= \theta_{i-k}(t_{k+1}, \dots, t_{i-1}, \varphi_k(t_1, \dots, t_{k-1}, f_1, \dots, f_k), \dots, f_i) \quad \text{if } t_k = 1.$$

$$(\text{Note: } p\theta_{i-k}(\dots \varphi_k \dots) = p\varphi_1.)$$

The map $A_n(p)$ is given by $A_n(p)(\Phi) = p(\varphi_1(F))$. The fibres of $A_n(p)$ have the homotopy type of the space $H(n)$, which consists of families of maps $\Gamma = \{\gamma_1, \dots, \gamma_{n+1}\}$ subject to the conditions:

(3) γ_1 is a homotopy equivalence between F and itself.

(4) $\gamma_i : I^{i-1} \times F^i \rightarrow F$ ($1 \leq i \leq n+1$) such that

$$\begin{aligned} \gamma_i(t_1, \dots, t_{i-1}, f_1, \dots, f_i) \\ &= \gamma_{i-1}(t_1, \dots, \hat{t}_k, \dots, t_{i-1}, f_1, \dots, f_k f_{k+1}, \dots, f_i) \quad \text{if } t_k = 0, \\ &= \gamma_k(t_1, \dots, t_{k-1}, f_1, \dots, f_k) \cdot (f_{k+1} f_{k+2} \cdots f_i) \quad \text{if } t_k = 1. \end{aligned}$$

Note that for $H(2)$, this says that γ_1 looks like a translation up to homotopy. In case γ_i exists for all i , we call γ_1 a homolation, that is a homotopy translation. Homolations are studied in more detail in [12].

The space $A_n(E)$ is a subspace of

$$M(F; E) \times M(I \times F^2; E) \times \cdots \times M(I^n \times F^{n+1}; E)$$

where $M(I^i \times F^{i+1}; E) = \{f : I^i \times F^{i+1} \rightarrow E\}$ (with the k -space topology derived from the compact open topology). Steenrod in [20], specifically shows that the k -topology of the compact open topology behaves well for all our needs. As an example, Stasheff's $\text{Prin}(E)$ in [16] can be thought of as $A_0(E)$. At the other extreme, the space $A_\infty(E)$ consists of all infinite families $\Phi = \{\varphi_1, \varphi_2, \dots\}$ satisfying the conditions given above. A family of maps belonging to $A_n(E)$ consists of maps which up to homotopy respect the A_n -structure of $p : E \rightarrow B$.

If we start with a principal quasifibration, which is an A_∞ -principal quasifibration, an example of such a family would be as follows. Let $\varphi_1 = M_x^1$ where

$$\begin{aligned} M_x^1 : F &\rightarrow p^{-1}(p(x)) \\ M_x^1(f) &= m'(x, f). \end{aligned}$$

If all the fibres have the homotopy type of CW -complexes, φ_1 is a homotopy equivalence. In general, we could define φ_n by

$$\varphi_n(t_1, \dots, t_{n+1}, f_1, \dots, f_n) = M_x^1(f_2 f_3 \cdots f_n)$$

for all values of t_i .

It is possible to define an operation

$$\mu : A_n(E) \times H(n) \rightarrow A_n(E)$$

by an argument similar to that given by Fuchs in [6]. Let Φ be in $A_n(E)$ and Γ be in $H(n)$. Define

$$\mu(\Phi, \Gamma) = \{\varphi_1 \gamma_1, \dots, \varphi_{n+1} \gamma_{n+1}\}$$

The continuity of the map μ will follow from the canonical nature (*i.e.*, independence of the maps on which it works) of the parameter splitting.

The map $\varphi_1 \gamma_1$ is defined to be $\varphi_1 \circ \gamma_1$ which is clearly a homotopy equivalence between F and some $p^{-1}(b)$, b in B , if φ_1 is a homotopy equivalence between F and $p^{-1}(b)$. The definition of $\varphi_k \gamma_k$ is given by a system of families which we now describe. The map $\varphi_k \gamma_k$ sends $I^{k-1} \times F^k \rightarrow E$, thus, consider I^{k-1} split into 2^{k-1} "cubes" of dimension $k-1$, $\varphi_k \gamma_k$ is then given by defining maps on each of these "cubes" $\times F^k \rightarrow E$ which fit together correctly on common boundaries. The maps we have in mind will be all meaningful maps of the form

$$\varphi_i \circ (\gamma_i \times d_1 \times \cdots \times d_{k-1})$$

where $1 \leq j \leq k$, $1 \leq \ell \leq k$ and the d 's are either the identity 1_r or a map indicating a particular way of associating a certain number of elements of F .

As an example, we may define

$$\begin{aligned} \varphi_2 \gamma_2(t, f, g) &= \begin{cases} \varphi_1 \circ \gamma_2(2t, f, g), & 0 \leq 2t \leq 1 \\ \varphi_2(2t - 1, \gamma_1(f), g), & 1 \leq 2t \leq 2, \end{cases} \\ \varphi_3 \gamma_3(t, s, f, g, h) &= \begin{cases} \varphi_1 \circ \gamma_3(2t, 2s, f, g, h), & 0 \leq 2t \leq 1, 0 \leq 2s \leq 1 \\ \varphi_2(2s - 1, \gamma_2(2t, f, g), h), & 0 \leq 2t \leq 1, 1 \leq 2s \leq 2 \\ \varphi_2(2t - 1, \gamma_1(f), gh), & 1 \leq 2t \leq 2, 0 \leq 2s \leq 1 \\ \varphi_3(2t - 1, 2s - 1, \gamma_1(f), g, h), & 1 \leq 2t \leq 2, 1 \leq 2s \leq 2, \end{cases} \end{aligned}$$

etc.

In the case of $\varphi_3 \gamma_3$, we have pictorially:

Returning to our A_n -construction, we note that $A_n(p)$ has the following property, the proof of which is similar to Lemma 7 of [16]. The only difference being that we must account for the fact that the members of $A_n(E)$ are families of maps rather than maps as in $\text{Prin}(E)$.

Lemma 5.1. $A_n(p)$ is a fibre space if p is an A_n -principal fibre space.

We now proceed to the main construction. Let $p : E \rightarrow B$ be a quasifibration with fibres of the homotopy type of an associative H -space (F, m) . Let $\{\theta_i\}$ be an A_n -action of F on E . Embed p in the space $P(H(n), p) : P(H(n)), E \rightarrow P(H(n), B)$. The total space is defined as

$$P(H(n), E) = C(A_n E) \times F \cup_q E$$

where $(1, \Phi, f)$ is identified with $\varphi_1(f) = q(\Phi, f)$. The base

$$P(H(n), B) = C(A_n E) \cup_v B$$

where $v = A_n(p)$, and $P(H(n), p)$ is defined as an extension of p by

$$\begin{array}{ccc} C(A_n E) \times F & \longrightarrow & P(H(n), E) \\ \downarrow & & \downarrow \\ C(A_n E) & \longrightarrow & P(H(n), B) \end{array}$$

where the vertical map on the left is projection onto the first factor, and the horizontal lines represent quotient maps.

Lemma 5.2. $P(H(n), p)$ is a quasifibration if p is.

Proof. A proof similar to that given by Dold and Lashof to prove proposition 2.3 of [3] works perfectly well here. To show the mapping of the fibre $P(H(n), p)^{-1}(t, \Phi)$ into $P(H(n), p)^{-1}(1, \Phi) = p^{-1}(P(H(n), p)(\Phi))$ given by $(t, \Phi, f) \rightarrow (1, \Phi, f) \sim \varphi_1(f)$ is a homotopy equivalence, we need only note that the map

$$q(\Phi, \cdot) = \varphi_1 : F \rightarrow E$$

is a homotopy equivalence of F into some fibre $p^{-1}(b)$, b in B .

Lemma 5.3. $A_n(P(H(n), p))$ is a quasifibration if $A_n(p)$ is.

Proof. Since we are dealing with k -spaces, we are permitted to identify $A_n(P(H(n), p))$ with $DL(A_n(p))$ by identifying $C(A_n E) \times H(n)$ with $A_n(C(A_n E) \times F)$. The identification is as follows:

if (t, Φ, Γ) is in $C(A_n E) \times H(n)$, then

$$(t, \Phi, \gamma_i) : I^{i-1} \times F^i \rightarrow C(A_n E) \times F$$

is given by

$$(t, \Phi, \gamma_i)(t_1, \dots, t_{i-1}, f_1, \dots, f_i) = (t, \Phi, \gamma_i(t_1, \dots, f_i)).$$

Thus, $A_n(P(H(n), p))$ will be a quasifibration if $DL(A_n(p))$ is. To prove the latter, it would be necessary to show that right translations by $H(n)$ is a homotopy equivalence. It would be sufficient to show that elements of $H(n)$ have "inverses" in $H(n)$.

The approach used in showing this is a modification of the technique used by Fuchs in [6] in showing that if $\{a_i\}$ is an H -homomorphism and α_* is a homotopy equivalence, then there exists $\{b_i\}$ which is an inverse H -homomorphism. We will not give further details here.

We are now in a position to prove the following theorem.

Theorem 5.4. *If $p : E \rightarrow B$ is an A_n -principal quasifibration with A_n -action of F on E given by $\{\theta_i\}$, it is possible to define an A_n -action of F on $P(H(n), E)$ making $P(H(n), p)$ an A_n -principal quasifibration.*

Proof. The proof of this theorem is not complicated. All one has to do is to define a family of maps

$$\{v_i : I^{i-1} \times P(H(n), E) \times F^i \rightarrow P(H(n), E), 1 \leq i \leq n\}$$

which can be shown to be an A_n -action of F on $P(H(n), E)$ satisfying the condition of Definition 3.2. Still we will have to motivate the definition of this family. The maps v_i will be defined by an inductive procedure, each map given in two parts to account for the fact that some points will lie in $P(H(n), E) - E$ while the remainder are in E . Therefore, assume that v_i has been defined by the inductive procedure described below for all $i < k$. At the start, ignore the identification in the space $P(H(n), E)$, thinking of it as the space $I \times A_n(E) \times F$. Thus, we will define an appropriate map of $I^k \times A_n(E) \times F^{k+1}$ into $P(H(n), E)$. Divide I^k into two pieces C_k and D_k where $C_k \cup D_k = I^k$ and $C_k \cap D_k = a$ ($k-1$)-dimensional cell in the interior of I^k together with its boundary. Only the boundary of the cell is dictated to us. We choose any division of I^k which is consistent with the division chosen for I^i in defining the maps v_i for $j < k$.

Define $v_k | C_k \times A_n(E) \times F^{k-1} \rightarrow I \times A_n(E) \times F \rightarrow P(H(n), E)$ where the first map is of the form $\mu_k \times 1 \times \pi$ and the second map is the quotient map. The map $\pi : F^{k+1} \rightarrow F$ is given by $\pi(f, f_1, \dots, f_i) = ff_1 \dots f_k$ and $\mu_k : C_k \rightarrow I$ is any map such that

$$\begin{aligned} \mu_k(0, t_1, \dots, t_{k-1}) &= 0 \\ \mu_k(t, t_1, \dots, t_{k-1}) &= \begin{cases} \mu_{k-1}(t, t_1, \dots, \hat{t}_i, \dots, t_{k-1}) & \text{if } t_i = 0 \\ \mu_{k-i}(t_{i+1}, \dots, t_{k-1}, \mu_i(t, t_1, \dots, t_{i-1})) & \text{if } t_i = i \end{cases} \end{aligned}$$

and $\mu_k(t, t_1, \dots, t_{k-1}) = 1$ if $(t, t_1, \dots, t_{k-1})$ is in $C_k \cap D_k$.

In defining $v_k | D_k \times A_n(E) \times F^{k+1}$, we make use of the following notation:

$$0_i : I^{k-1} \rightarrow I^k \text{ means}$$

$$0_i(t_1, \dots, t_{k-1}) = (t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{k-1})$$

and

$$1_i : I^{k-1} \rightarrow I^k \text{ means}$$

$$1_i(t_1, \dots, t_{k-1}) = (t_1, \dots, t_{i-1}, 1, t_i, \dots, t_{k-1})$$

Then define $v_k | D_k \times A_n(E) \times F^{k+1}$ to be the composite of the maps

$$D_k \times A_n(E) \times F^{k+1} \rightarrow I^k \times F^{k+1} \rightarrow E$$

where the first map is $\rho_k \times T$ and the second is φ_{k+1} . The map T is projection of $A_n(E) \times F^{k+1}$ onto F^{k+1} and ρ_k is any relative homomorphism of $D_k \rightarrow I^k$ which extends $\rho_k^* : \partial D_k \rightarrow \partial I^k$ where

$$\rho_k^*(1, t_1, \dots, t_{k-1}) = (1, t_1, \dots, t_{k-1}), \quad 0 \leq t_i \leq 1, j = 1, \dots, k-1$$

$$\rho_k^*(t, t_1, \dots, t_{k-1}) = \begin{cases} 0_k \circ \rho_{k-1}(t, t_1, \dots, \hat{t}_i, \dots, t_{k-1}) & \text{if } t_i = 0 \\ (0_1 \times \dots \times 0_i) \circ \rho_{k-i}(\mu_k(t, t_{i+1}, \dots, t_{k-1})) & \text{if } t_i = 1 \\ \text{and } (t, t_{i+1}, \dots, t_{k-1}) \in D_{k-i} \\ [1_i \circ p_i(t, t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{k-1})] & \text{if } t_i = 1 \\ \text{and } (t, t_1, \dots, t_{i-1}) \in D_i \end{cases}$$

and $\rho_k^*(t, t_1, \dots, t_{k-1}) = (0, \dots, 0)$ if $(t, t_1, \dots, t_{k-1}) \in C_k \cap D_k$.

In the k -topology, $C(A_n E) \times F^{k+1} \cup_q E \times F^k$ is homeomorphic to $P(H(n), E) \times F^k$, thus, we define v_k by

$$v_k(t_1, \dots, t_{k-1}, [t, \Phi, f], f_1, \dots, f_k) = \begin{cases} [\mu_k(\tau), \Phi, ff_1 \dots f_k] & \tau \in C_k \\ \varphi_{k+1}(\rho_k(\tau), f, f_1, \dots, f_k) & \tau \in D_k \end{cases} \quad \tau = (t, t_1, \dots, t_k)$$

This procedure then defines a family of maps $\{v_i\}$ which is an A_n -action of F on $P(H(n), E)$, and it is easily shown that $P(H(n), p)$ is an A_n -principal quasifibration.

The A_n -construction just described can be iterated as follows. Let $p : E \rightarrow B$ be an A_n -principal quasifibration. Relabel this quasifibration as $p_0 : E_0 \rightarrow B_0$. Inductively define $p_m : E_m \rightarrow B_m$ to be

$$P(H(n), p_{m-1}) : P(H(n), E_{m-1}) \rightarrow P(H(n), B_{m-1}),$$

and let $Ult_n(p) : Ult_n(E) \rightarrow Ult_n(B)$ be the limit of the sequence of quasifibrations $p_m : E_m \rightarrow B_m$. Of course, each of the quasifibrations $p_m : E_m \rightarrow B_m$ is given an A_n -principal structure by the method described in Theorem 5.4.

Lemma 5.5. *$Ult_n(p)$ is a quasifibration if p is. $A_n(Ult_n(p))$ is a quasifibration if $A_n(p)$ and p are quasifibrations.*

Proof. That $Ult_n(p)$ is a quasifibration follows as in the argument given by Dold and Lashof in Theorem 3.5 of [3]. Lemma 5.2 tells us that $P(H(n), p)$ is a quasifibration, and Lemma 5.3 tells us that $A_n(P(H(n), p))$ is a quasifibration. Thus, for each m , $A_n(p_m)$ is a quasifibration. Taking a limit of these quasifibrations, these same arguments give us that $A_n(Ult_n(p))$ is a quasifibration.

Lemma 5.6. *$A_n(Ult_n E)$ is aspherical.*

Proof. We must show that $\pi_i(A_n(Ult_n E)) = 0 \quad \forall i \geq 0$. To this end, let $f : S^n \rightarrow A_n(Ult_n E)$ be any continuous map. Then $A_n(Ult_n(p)) \circ f : S^n \rightarrow Ult_n(B)$. This maps S^n into some B_m , hence $f(S^n) \subset A_n(E_m)$. Thus it will be sufficient

to show that $A_n(E_m)$ is contractible in $A_n(E_{m+1})$, for in that case f will be null-homotopic. We may define a map

$$\begin{aligned} A_n(E_m) &\rightarrow C(A_n E_m) \times H(n) \text{ by} \\ \Phi &\rightarrow (1, \Phi, Id) \end{aligned}$$

where Id is the family of maps belonging to $H(n)$, corresponding to $1_F : F \rightarrow F$. As in Lemma 5.3, we identify $C(A_n E_m) \times H(n)$ with $A_n[C(A_n E_m) \times F]$. We thus have a map

$$A_n(E_m) \rightarrow C(A_n E_m) \times H(n) \rightarrow A_n(E_{m+1}),$$

the contraction in this latter case being given by

$$k_t(\Phi) = (t, \Phi, Id).$$

We now have an iterative construction which gives us a sequence of A_n -principal quasifibrations. In the next section, we make use of this result to obtain a universal example and a classifying space for certain A_n -principal fibre spaces. The justification for the change from quasifibration to fibre space will be given in the appendix.

6. The classification theorem. Before introducing the notion of A_n -equivalence, we will need the following definition.

Definition 6.1. Let $\Sigma_i = \{E_i, P_i, B_i, F\}$, $i = 0, 1$ be two A_n -principal fibre spaces. Let $\{\theta_k^i\}$ be their respective A_n -actions. A family of maps

$$\{w_k : I^{k-1} \times E_0 \times F^{k-1} \rightarrow E_1, 1 \leq k \leq n+1\}$$

is said to be admissible if:

- (1) $w_1 : E_0 \rightarrow E_1$ lies over $\bar{w}_1 : B_0 \rightarrow B_1$ (i.e., $p_1 w_1 = \bar{w}_1 p_0$).
- (2) The restriction $w_1^k : p_0^{-1}(b) \rightarrow p_1^{-1}(\bar{w}_1(b))$, $w_1^k(x) = w_1(x)$ for x in $p_0^{-1}(b)$ is a homotopy equivalence for all b in B .
- (3) The family "respects" A_n -actions.

An interpretation of condition (3) is as follows. In general $w_k : I^{k-1} \times E_0 \times F^{k-1} \rightarrow E_1$ satisfies the following conditions:

- (a) $p_1 w_k(\tau, x, f_1, \dots, f_{i-1}) = \bar{w}_1 p_0(x)$, $\tau = (t_1, \dots, t_{i-1})$.
- (b) $w_k(0, t_2, \dots, t_{k-1}, x, f_1, \dots, f_{k-1})$ is to be given by $k-1$ formulas representing

$$w_1 \circ \theta_{k-1}^0, w_2 \circ (\theta_{k-2}^0 \times 1) \cdots w_{k-1} \circ (\theta_1^0 \times 1 \times \cdots \times 1).$$

- (c) $w_k(t_1, \dots, t_{k-1}, x, f_1, \dots, f_{k-1})$
 $= w_{k-1}(t_1, \dots, \hat{t}_i, \dots, t_{k-1}, x, f_1, \dots, f_i f_{i+1}, \dots, f_{k-1})$
 if $t_i = 0$ and $j \neq 1$.

$$\begin{aligned}
& \text{(d) } w_k(t_1, \dots, t_{k-1}, x, f_1, \dots, f_{k-1}) \\
& \quad = \theta_{k-i}^1(t_{i+1}, \dots, t_{k-1}, w_i(t_1, \dots, t_{i-1}, x, f_1, \dots, f_{i-1}), \dots, f_{k-1}) \\
& \quad \text{if } t_i = 1.
\end{aligned}$$

As an example, let $w_1 : E_0 \rightarrow E_1$ lying over $\bar{w}_1 : B_0 \rightarrow B_1$ be any fibrewise map. Define w_2 to be the map showing the following diagram is fibrewise homotopy commutative.

$$\begin{array}{ccc}
E_0 \times F & \xrightarrow{w_1 \times 1} & E_1 \times F \\
\downarrow \theta_1^0 & & \downarrow \theta_1^1 \\
E_0 & \xrightarrow{w_1} & E_1
\end{array}$$

The map w_3 is to represent a homotopy between $w_1 \circ \theta_2^0$ and $\theta_2^1 \circ (w_1 \times 1 \times 1)$ which is suitably determined by w_2 at the “ends” of θ_2^0 and θ_2^1 . The remaining w_k are to be handled similarly.

Given an A_n -principal fibre space $\Sigma = \{E, p, B, F\}$ with A_n -action $\{\theta_k\}$, there exists an admissible family of maps $\{w_k\}$ of Σ into Σ with $w_1 = 1_E$, called a 1_E -admissible family.

We next define a composition of admissible families. Suppose that $\{w_k : I^{k-1} \times E_0 \times F^{k-1} \rightarrow E_1, 1 \leq k \leq n+1\}$ and $\{u_k : I^{k-1} \times E_1 \times F^{k-1} \rightarrow E_2, 1 \leq k \leq n+1\}$ are admissible families. We define the family $\{u_k w_k : I^{k-1} \times E_0 \times F^{k-1} \rightarrow E_2, 1 \leq k \leq n+1\}$ in such a way as to make it admissible. The map $u_k w_k$ is given in exactly the same form as was the map $\varphi_k \gamma_k$ when in Section 5, we defined the operation $\mu : A_n(E) \times H(n) \rightarrow A_n(E)$.

For instance $u_1 w_1 = u_1 \circ w_1$

$$\begin{aligned}
u_2 w_2(t, s, f) &= \begin{cases} u_1 \circ w_2(2t, x, f), & 0 \leq 2t \leq 1 \\ u_2(2t-1, w_1(x), f), & 1 \leq 2t \leq 2, \end{cases} \\
u_3 w_3(t, s, x, f, g) &= \begin{cases} u_1 \circ w_3(2t, 2s, x, f, g), & 0 \leq 2t \leq 1, 0 \leq 2s \leq 1 \\ u_2(2s-1, w_2(2t, x, f), g), & 0 \leq 2t \leq 1, 1 \leq 2s \leq 2 \\ u_2(2t-1, w_1(x), fg), & 1 \leq 2t \leq 2, 0 \leq 2s \leq 1 \\ u_3(2t-1, 2s-1, w_1(x), f, g) & 1 \leq 2t \leq 2, 1 \leq 2s \leq 2, \end{cases}
\end{aligned}$$

etc.

Definition 6.2. Let $B_0 = B_1$. An A_n -equivalence between Σ_0 and Σ_1 is a pair of admissible families of maps:

$$\{w_k : I^{k-1} \times E_0 \times F^{k-1} \rightarrow E_1\} \text{ and } \{u_k : I^{k-1} \times E_1 \times F^{k-1} \rightarrow E_0\}$$

such that the admissible families $\{u_k w_k\}$ and $\{w_k u_k\}$ are homotopic as admissible families to the 1_{E_0} -admissible family and 1_{E_1} -admissible family, respectively.

It is easily seen that A_n -equivalence is an equivalence relation. We will denote by $A_n F(X)$ the set of equivalence classes of A_n -principal fibre spaces $p : E \rightarrow X$ and by $[X, Y]$, the set of homotopy classes of maps of X into Y .

$A_n F(\)$ is regarded as a functor as follows. Let $p : E \rightarrow B$ be a fibre space and $g : X \rightarrow B$ be any map. We may construct a fibre space $g^*p : E_g \rightarrow X$ where

$$E_g = \{(x, y) \mid x \in X, y \in E, g(x) = p(y)\} \text{ and } g^*p(x, y) = x$$

This is the fibre space induced by g . If in addition $p : E \rightarrow B$ is an A_n -principal fibre space with A_n -action $\{\theta_i\}$, then g^*p can be given the structure of an A_n -principal fibre space by defining

$$\theta_i^g(t_1, \dots, t_{i-1}, (x, y), f_1, \dots, f_i) = (x, \theta_i(t_1, \dots, t_{i-1}, y, f_1, \dots, f_i))$$

for all $i = 1, \dots, n$. It is easily checked that the family of maps $\{\theta_i^g\}$ is an A_n -action of F on E_g making g^*p an A_n -principal fibre space. The induced function

$$A_n F(g) : A_n F(B) \rightarrow A_n F(X)$$

is given by

$$A_n F(g)[p] = [g^*p].$$

Theorem 6.3. *If h and g are homotopic maps of $X \rightarrow B$, then $A_n F(h) = A_n F(g)$.*

To prove this theorem, we will need the following lemmas:

Lemma 6.4. *Let $\Sigma_i = \{E_i, P_i, B, F\}$, $i = 0, 1$ be two fibre spaces. Suppose Σ_0 and Σ_1 are fibre homotopy equivalent. Then if Σ_0 is an A_n -principal fibre space, Σ_1 can be given the structure of an A_n -principal fibre space such that Σ_1 together with this A_n -action is A_n -equivalent to Σ_0 .*

Lemma 6.5. *If $\{\theta_i^0\}$ and $\{\theta_i^1\}$ are two A_n -actions of F on E such that, as families, the maps are fibre homotopic, then $[\Sigma, \{\theta_i^0\}]$ and $[\Sigma, \{\theta_i^1\}]$ are A_n -equivalent. (Here $[\Sigma, \{\theta_i^0\}]$ refers to the fibre space Σ together with the A_n -structure.)*

Proof of Theorem 6.3. It will be necessary to show that the induced fibre spaces $\Sigma_h = \{E_h, h^*p, X\}$ and $\Sigma_g = \{E_g, g^*p, X\}$ belong to the same A_n -equivalence class.

Since $h \simeq g$, there exists a map $r : I \times X \rightarrow B$ such that $r(0, x) = h(x)$ and $r(1, x) = g(x)$. We construct the fibre space

$$r^*p : E_r \rightarrow I \times X$$

where

$$E_r = \{(t, x, y) \mid t \in I, x \in X, y \in E, r(t, x) = p(y)\}$$

and

$$r^*p(t, x, y) = (t, x)$$

By Lemma 6.5 of [2], there exists a map

$$R : I \times E_r \rightarrow E_r$$

such that

$$(1) (r^*p) \circ R(\tau, (t, x, y)) = (\tau, x)$$

and

$$(2) T_1 : E_r \rightarrow E_r, \text{ given by } T_1(t, x, y) = R(t, (t, x, y)) \text{ is such that } T_1 \simeq_{I \times X} 1_{E_r}.$$

Defining $d(x, y) = R(1, x, y)$ and $d^1(x, y) = R(0, x, y)$, we have that $d^1 d \simeq 1_{E_g}$ and $dd^1 \simeq 1_{E_h}$. Thus, the fibre spaces Σ_h and Σ_g are fibre homotopy equivalent.

The fibre space Σ_r can be given an A_n -structure by defining an A_n -action of F on E_r , following the procedure described earlier. Over $t = 0$, this A_n -action is $\{\theta_h^1\}$, while over $t = 1$, the A_n -action is $\{\theta_g^2\}$. It happens that Σ_r and $\{I \times E_h, q, I \times X\}$, where $q(t, (x, y)) = (t, x)$ are fibre homotopy equivalent via the map $R : I \times E_h \rightarrow E_r$. The A_n -action on E_r induces an A_n -action $\{\bar{\theta}_i^r\}$ on $I \times E_h$. As the space $(r^*p)^{-1}(0 \times X)$ and E_h are clearly homeomorphic, we will identify the two spaces. Similarly, identify $(r^*p)^{-1}(1 \times X)$ with E_g .

In light of Lemma 6.4, we know that the fibre spaces $[\Sigma_r, \{\theta_i^r\}]$ and $[I \times E_h, q, I \times X, \{\bar{\theta}_i^r\}]$ are A_n -equivalent. Of course, Σ_h being an A_n -principal fibre space, it already possesses an A_n -action, namely $\{\theta_i^1\}$. We note that as families, the A_n -action $\{\bar{\theta}_i^r\}$ when restricted to $0 \times E_h$ is fibre homotopic to $\{\bar{\theta}_i^r\}$ when restricted to $1 \times E_h$. Thus by Lemma 6.5, we have that the space $[0 \times E_h, q, 0 \times X, \{\bar{\theta}_i^r\}]$ and $[1 \times E_h, q, 1 \times X, \{\bar{\theta}_i^r\}]$ are A_n -equivalent. Note that over $t = 0$, $\{\theta_i^r\}$ and $\{\theta_i^1\}$ are the same family. Thus $[\Sigma_h, \{\theta_i^1\}]$ is A_n -equivalent to $[0 \times E_h, q, 0 \times X, \{\bar{\theta}_i^r\}]$ and similarly over $t = 1$, $\{\theta_i^r\}$ and $\{\theta_i^2\}$ are the same family, so that $[\Sigma_g, \{\theta_i^2\}]$ is A_n -equivalent to $[1 \times E_h, q, 1 \times X, \{\bar{\theta}_i^r\}]$. Thus we have that $[\Sigma_h, \{\theta_i^1\}]$ is A_n -equivalent to $[\Sigma_g, \{\theta_i^2\}]$. This completes the proof of Theorem 6.3.

The proofs of Lemmas 6.4 and 6.5 are fairly straightforward but rather tedious and will not be given here.

In summary, we have shown that $A_n F(\)$ is a homotopy functor. Theorem 6.3 will be used to prove the classification theorem. We now describe the space $B_{H(n)}$ mentioned in the classification theorem and then give the proof of the theorem.

Consider the fibre space $\pi : F \rightarrow *$ (a point), where F is an associative H -space. Applying the A_n -construction, we find in this case that $A_n(F)$ is just $H(n)$ and $A_n(Ul_n(\pi))$ is a quasifibration with aspherical total space and fibre $H(n)$. We denote this quasifibration by $P_{H(n)} : E_{H(n)} \rightarrow B_{H(n)}$. We denote by $u_n : UE_n \rightarrow B_{H(n)}$ the associated Hurewicz fibre space $Hur(Ul_n(\pi))$. Under suitable conditions this space is shown to be the universal example of A_n -principal fibre spaces.

The transformation $R : [\ , B_{H(n)}] \rightarrow A_n F(\)$. Define $R(f) = A_n F(f)[u_n] = [f^* u_n : (UE_n)_f \rightarrow X]$ for any map $f : X \rightarrow B_{H(n)}$.

The transformation $T : A_n F(\) \rightarrow [\ , B_{H(n)}]$. Most of the following is due to Stasheff [16], with minor changes to reflect the A_n -structure. Let $\Sigma_i = \{E_i, p_i,$

$X_i, F\}$, $i = 0, 1$, be two A_n -principal fibre spaces and $\{w_k\}$ an admissible family of maps between Σ_0 and Σ_1 . Consider the following commutative diagram.

$$\begin{array}{ccc} E_0 & \xrightarrow{w_1} & E_1 \\ \downarrow p_0 & & \downarrow p_1 \\ X_0 & \xrightarrow{\bar{w}_1} & X_1 \end{array}$$

Carrying out the A_n -construction on p_0 and p_1 , one obtains

$$\begin{array}{ccccc} E_0 & & \xrightarrow{w_1} & & E_1 \\ & \searrow & & \swarrow & \\ & X_0 & \xrightarrow{\bar{w}_1} & X_1 & \\ & \downarrow & & \downarrow & \\ & Utl_n(X_0) & \xrightarrow{Utl_n(\bar{w}_1)} & Utl_n(X_1) & \\ & \nearrow & & \nwarrow & \\ Utl_n(E_0) & & \xrightarrow{\quad} & & Utl_n(E_1) \end{array}$$

Since $A_n(Utl_n E_0)$ and $A_n(Utl_n E_1)$ are aspherical, $Utl_n(\bar{w}_1)$ is a weak homotopy equivalence. In particular, this is true if X_0 is a point and $E_0 = F$. In this case, $Utl_n(X_0)$ can be considered as $B_{H(n)}$. We will denote the weak homotopy equivalence $B_{H(n)} \rightarrow Utl_n(X_1)$ by j . This induces a map

$$j_* : [X_1, B_{H(n)}] \rightarrow [X_1, Utl_n(X_1)]$$

for any space X_1 . In particular, if X_1 is a finite dimensional CW -complex, j_* is an isomorphism (by Corollary 23, Ch. 7 of [15]). In fact, for any point y within a given path component of X_0 , we get the same isomorphism j_* (see Stasheff's comment in [16] concerning this). On each component, therefore, we define $T(p_1)$ by $j_* T(p_1) = [g]$ where g is the inclusion $X_1 \subset Utl_n(X_1)$.

Returning to the figure for arbitrary p_0 , we have similarly

$$\begin{aligned} j^1 : B_{H(n)} &\rightarrow Utl_n(X_0), \quad g^1 : X_0 \subset Utl_n(X_0) \quad \text{and} \quad Utl_n(\bar{w}_1)j^1 \simeq j, \quad \text{so} \quad j_* T(p_0) \\ &= Utl_n(\bar{w}_1)_* j^1_* T(p_0) \\ &= Utl_n(\bar{w}_1)_* [g^1] \\ &= [Utl_n(\bar{w}_1)g^1] \\ &= [g\bar{w}_1]. \end{aligned}$$

Thus, if $X_0 = X_1$, $\bar{w}_1$ is the identity and Σ_0 is A_n -equivalent to Σ_1 , thus $T(p_0) = T(p_1)$, so T passes to A_n -equivalence classes. Since $[g\bar{w}_1] = \bar{w}_1^*[g]$, T is natural.

In order to prove the next theorem, we must first give the following definition of quasi- A_n -equivalence.

Definition 6.6. If $p_i = E_i \rightarrow B$, $i = 0, 1$ are A_n -principal quasifibrations, then a family of maps $\{w_k : I^{k-1} \times E_0 \times F^{k-1} \rightarrow E_1, 1 \leq k \leq n+1\}$ is a weak A_n -equivalence if the family is admissible and w_1 is a weak homotopy equivalence. The quasifibrations p_0 and p_1 are quasi- A_n -equivalent if there is a sequence of

quasifibrations $p_1, p_2, \dots, p_{k-1}, p_k = p_0$ such that for each i , p_i is weakly A_n -equivalent to p_{i+1} or p_{i+1} is weakly A_n -equivalent to p_i .

Theorem 6.7. *If X is a finite dimensional CW-complex, then $T : A_n F(X) \rightarrow [X, B_{H(n)}]$ is one-to-one.*

Proof. Suppose $p : E_0 \rightarrow X$ and $q : E_1 \rightarrow X$ are A_n -principal fibre spaces with fibres of the homotopy type of a finite dimensional CW-complex F . Assume that $T(p) = T(q)$, we will want to show that p and q are A_n -equivalent. Corollary 9.7 of the appendix states that two A_n -principal fibre spaces which are quasi- A_n -equivalent are A_n -equivalent if X and F are CW-complexes. Thus we must show that p and q are quasi- A_n -equivalent.

Imbed $p : E \rightarrow X$ in $Ult_n(p)$ as in defining T . The imbedding $E_0 \rightarrow Ult_n(E_0) \rightarrow Hur(Ult_n(E_0)) = E^1$ induces an admissible family of maps, say $\{w_k\}$ (see Proposition 9.4, appendix), between p and $Hur(Ult_n p) = p^1$. This induces a weak homotopy equivalence between fibres.

Construct the induced fibre spaces $\bar{w}_1^* p^1 : E^1 w_1 \rightarrow X$ where w_1 lies over $\bar{w}_1 : X \rightarrow Ult_n(X)$. The two spaces p and $\bar{w}_1^* p^1$ lie over the same base, a finite dimensional CW-complex X . Define a map $j_1 : E_0 \rightarrow E^1 w_1$ by $j_1(\epsilon) = (p(\epsilon), w_1(\epsilon))$ where ϵ is in E_0 . Since $j_1|_F$ is a weak homotopy equivalence, it follows that j_1 is a weak homotopy equivalence.

The fibre spaces p and p^1 are both A_n -principal fibre spaces. Let $\{\theta_k\}$ and $\{\theta_k^1\}$ denote their respective A_n -actions. The A_n -action of F on E^1 induces an A_n -action, say $\{\bar{\theta}_k\}$, of F on $E^1 w_1$, where $\bar{\theta}_k(t_1, \dots, t_{k-1}, (x, y), f_1, \dots, f_k) = (X, \theta_k^1(t_1, \dots, t_{k-1}, y, f_1, \dots, f_k))$ with $x \in X, y \in E^1$. We next show that $\bar{w}_1^* p^1$ and p are quasi- A_n -equivalent.

It will be necessary to show the existence of an admissible family of maps $\{j_k\}$ of p into $\bar{w}_1^* p^1$, where j_1 is a weak homotopy equivalence of E_0 into $E^1 w_1$. We already have such a j_1 . In general, define

$$j_k : I^{k-1} \times E_0 \times F^{k-1} \rightarrow E^1 w_1 \text{ by}$$

$$j_k(t_1, \dots, t_{k-1}, \epsilon, f_1, \dots, f_{k-1}) = (p(\epsilon), w_k(t_1, \dots, t_{k-1}, \epsilon, f_1, \dots, f_{k-1}))$$

Thus p and $\bar{w}_1^* p^1$ are quasi- A_n -equivalent.

Similarly, we may imbed q into $Ult_n(q)$ and get the following diagram.

$$\begin{array}{ccc} E''_1 \leftarrow E_1 & \xrightarrow{\quad} & Hur(Ult_n E_1) = E'' \\ \bar{w}_1^* q^1 \searrow \downarrow q & & \downarrow q^1 \\ X & \rightarrow & Ult_n(X) \end{array}$$

such that q and $\bar{w}_1^* q^1$ are quasi- A_n -equivalent.

Letting $T(p)$ stand for a representative of the class $T(p)$ in $[X, B_{H(n)}]$, and similarly for $T(q)$, it follows as in the above argument that p and $T(p)^* u_n$ are quasi- A_n -equivalent as are q and $T(q)^* u_n$. Since $T(p) = T(q)$, by the definition of R , $T(p)^* u_n$ and $T(q)^* u_n$ are A_n -equivalent. Thus p and q are quasi- A_n -equivalent. Since the base of these two fibre spaces is a finite dimensional CW-complex

X , and their fibres are of the homotopy type of a finite CW -complex F , it follows from Corollary 9.7 of the appendix that p and q are A_n -equivalent. Thus T is one-to-one.

Suppose now that the spaces were such that $B_{H(n)}$ itself was of the homotopy type of a finite dimensional CW -complex; then $T(u_n)$ would be defined, and we would have:

In the diagram, $T(u_n)$ is a representative of the class $T(u_n)$. The map j is a homotopy equivalence, so that $T(u_n) = h \circ g$, where h is any inverse of j , and hence $T(u_n)$ is a homotopy equivalence of $B_{H(n)}$ onto itself. Without making such an assumption about $B_{H(n)}$, we can still prove the following theorem.

Theorem 6.8. *TR is an automorphism on $[X, B_{H(n)}]$ for any finite-dimensional CW -complex X .*

Proof.

We have that $g^* : [X, B_{H(n)}] \rightarrow [X, Ultn(B_{H(n)})]$ and $(k_i)^* : [X, B_{H(n)}] \rightarrow [X, Ultn(B_{H(n)})]$ are both isomorphisms for finite-dimensional CW -complexes X . If $f : X \rightarrow B_{H(n)}$, we have that

$$(k_i)^* TR[f] = k^*[g^*] = g^*[g] : [X, B_{H(n)}] \rightarrow [X, Ultn(B_{H(n)})].$$

Since both g^* and $(k_i)^*$ are isomorphisms, it follows that TR is an isomorphism, that is, an automorphism.

Combining Theorems 6.6 and 6.7, we see that T is one-to-one and onto. This completes the proof of the classification theorem.

In the next two sections, the A -construction is related to the Dold and Lashof construction and our classification to theirs.

7. The compatibility of the A_∞ -construction and the Dold and Lashof construction. Recall that if one starts with a principal quasifibration $\Sigma = \{E, P, B, F\}$, iteration of the Dold and Lashof construction gives a sequence of principal quasifibrations $\Sigma, DL(\Sigma), DL^2(\Sigma), \dots$. By taking the limit of this sequence one obtains a universal principal quasifibration $\Sigma_\infty = \{E_\infty, p_\infty, B_\infty, F\}$. Associated with this space is a principal fibre space $Hur(\Sigma_\infty)$, the associated Hurewicz fibre space.

On the other hand, since it is clear that a principal quasifibration is an A_∞ -principal quasifibration, we obtain a sequence $\Sigma, PH(\infty)(\Sigma), PH^2(\infty)(\Sigma), \dots$, where $PH^j(\infty)(\Sigma)$ is the j^{th} iterative of the A_∞ -construction. Each $PH^j(\infty)(\Sigma)$ is an A_∞ -principal quasifibration. We can take the limit of this sequence to obtain an A_∞ -principal quasifibration $Ult_\infty(\Sigma) = \{Ult_\infty(E), Ult_\infty(p), Ult_\infty(B)\}$, where $A_\infty[Ult_\infty(E)]$ is aspherical.

In what is to follow, we show that at each stage of these parallel constructions, there exists an admissible family of maps between $DL^i(\Sigma)$ and $PH^i(\infty)(\Sigma)$.

Theorem 7.1. *Let $\Sigma^1 = \{E^1, p^1, B^1, F\}$ be a principal quasifibration and $\Sigma = \{E, p, B, F\}$ be an A_∞ -principal quasifibration with A_∞ -action $\{\theta_k\}$. If $\{w_k\}$ is an admissible family of maps of Σ^1 into Σ , then there exists an admissible family of maps $\{DL(w_k)\}$ of $DL(\Sigma^1)$ into $PH(\infty)(\Sigma)$.*

Proof. As will be recalled $DL(E^1) = CE^1 \times F \cup_m E^1$, and $P(H(\infty), E) = C[A_\infty(E)] \times F \cup_q E$. Thus we make the existence of the maps $\{DL(w_k)\}$ depend upon the map $E^1 \rightarrow A_\infty(E)$ given by

$$x \rightarrow \Phi_x = \{\varphi_1^x, \varphi_2^x, \dots\}, x \in E^1$$

where $\varphi_1^x : F \rightarrow E$ is a homotopy equivalence of F into the fibre $p^{-1}(p(w_1(x)))$ given by $\varphi_1^x(f) = \theta_1(w_1(x), f)$. The remaining maps are defined as follows:

$$\varphi_k^x(t_1, \dots, t_{k-1}, f_1, \dots, f_k) = \theta_k(t_1, \dots, t_{k-1}, w_1(x), f_1, \dots, f_k).$$

It is easily checked that Φ_x is in $A_\infty(E)$.

The procedure for defining the maps $\{DL(w_k)\}$ completely parallels the definition of the maps $\{v_k\}$ given in Theorem 5.4. In fact, we can use the same parametrization.

Define $DL(w_k) : I^{k-1} \times DL(E^1) \times F^{k-1} \rightarrow P(H(\infty), E)$ by

$$DL(w_k)(t_1, \dots, t_{k-1}, [t, x, f_1, f_1, \dots, f_{k-1}]) \\ = \begin{cases} [\mu_k(t, \dots, t_{k-1}), \Phi_x, ff_1, \dots, f_{k-1}](t, \dots, t_{k-1}) \in C_k \\ w_{k+1}(\rho_k(t, \dots, t_{k-1}), x, f_1, \dots, f_{k-1})(t, \dots, t_{k-1}) \in D_k \end{cases}$$

where $\mu_k : C_k \rightarrow I$ and $\rho_k : D_k \rightarrow I^k$ are subject to the same face conditions as

stated in Theorem 5.4. It is easily checked that all conditions are satisfied to make $\{DL(w_k)\}$ an admissible family of maps of $DL(\Sigma^1)$ into $PH(\infty)(\Sigma)$.

Since Σ^1 and Σ were respectively an arbitrary principal quasifibration and an arbitrary A_∞ -principal quasifibration, and since both constructions are iterative, we have the following corollary.

Corollary 7.2. *If Σ^1 is a principal quasifibration and Σ is an A_∞ -principal quasifibration such that $\{w_k\}$ is an admissible family of maps of Σ^1 into Σ , then there exists an admissible family $\{DL^j(w_k)\}$ of $DL^j(\Sigma^1)$ into $PH^j(\infty)(\Sigma)$, for all j , extending the admissible family $\{w_k\}$.*

Now if $E^1 = E$ and $B^1 = B$, it follows that the identity $1_E : E \rightarrow E$ induces an admissible family of maps between $DL^j(\Sigma)$ and $PH^j(\infty)(\Sigma)$ for all values of j . In particular, there exists an admissible family of maps in the limit and between the associated Hurewicz fibre spaces.

Starting with the trivial fibre space $F \rightarrow *$, where F is an associate H -space (and a finite dimensional CW -complex), the Dold and Lashof construction leads to a classifying space B_F , classifying fibre bundles. The A_∞ -construction leads to the classifying space $B_{H(\infty)}$ which classifies A_∞ -principal fibre spaces whose base space is a finite dimensional CW -complex.

Thus by Theorem 7.1 and Corollary 7.2, there exist maps ξ and $\bar{\xi}$ which make the following diagram commutative.

$$\begin{array}{ccc} E_F & \xrightarrow{\xi} & UE_\infty \\ \downarrow p_F & & \downarrow u_\infty \\ B_F & \xrightarrow{\bar{\xi}} & B_{H(\infty)} \end{array}$$

Actually (E_F, p_F, B_F, F) denotes the universal principal quasifibration, but here we are using the notation to represent the associated Hurewicz fibre space.

The map $\bar{\xi} : B_F \rightarrow B_{H(\infty)}$ induces a map

$$\bar{\xi}_* : [B, B_F] \rightarrow [B, B_{H(\infty)}]$$

given by

$$\bar{\xi}_*[h] = [\bar{\xi} \circ h] \quad \text{for } [h] \text{ in } [B, B_F]$$

Consider the following diagram:

$$\begin{array}{ccccc} F & \xrightarrow{\quad} & & & E \\ & \searrow & & & \swarrow p \\ & * & \longrightarrow & X & \\ & \downarrow & & \downarrow g^1 & \\ & B_F & \xrightarrow{i^1} & DL^\infty(X) & \\ & \nearrow & & \nwarrow & \\ E_F & \xrightarrow{\quad} & & & Hur(DL^\infty(E)) \end{array}$$

(diagram a)

Here j^1 is a weak homotopy equivalence and g^1 is the inclusion $X \subset DL^\infty(X)$. If X is a finite dimensional CW -complex, j^1 induces an isomorphism

$$j_*^1 : [X, B_F] \rightarrow [X, DL^\infty(X)].$$

Definition 7.3. We will denote by T_{DL} the transformation from $AF(\)$ to $[\ , B_F]$, where $AF(\)$ is the set of principal fibre homotopy equivalence classes of principal fibre spaces. T_{DL} is given by

$$j_*^1 T_{DL}(p) = [g^1].$$

Now consider

$$\begin{array}{ccc} F & \xrightarrow{\quad} & E \\ \downarrow & \searrow & \swarrow p \\ & * & X \\ & \downarrow & \downarrow g \\ & B_{H(\infty)} & \xrightarrow{i} & Ult_\infty(X) \\ \uparrow & \nearrow & \nwarrow \\ UE_\infty & \xrightarrow{\quad} & Hur(Ult_\infty(E)) \end{array}$$

(diagram b)

Here j is a weak homotopy equivalence and g is the inclusion $X \subset Ult_\infty(X)$; j induces an isomorphism $j_* : [X, B_{H(\infty)}] \rightarrow [X, Ult_\infty(X)]$ if X is a finite dimensional CW -complex. We recall that

$T : A_\infty F(X) \rightarrow [X, B_{H(\infty)}]$ is defined by

$$j_* T(p) = [g].$$

It follows from Corollary 7.2 that there is a map from diagram a to diagram b. The induced fibre spaces $\bar{\xi} * u_\infty$ and p_F are quasi- A_∞ -equivalent.

Theorem 7.4. $\bar{\xi} * T_{DL} = T : A_\infty F(X) \rightarrow [X, B_{H(\infty)}]$ for X a finite dimensional CW -complex.

Proof. $j_* T(p) = [g] = [\xi g^1] = \xi * [g^1] = \xi * j_*^1 T_{DL}(p) = j_* \bar{\xi} T_{DL}(p)$ where $\xi : DL^\infty(X) \rightarrow Ult_\infty(X)$ lies over $\bar{\xi} : B_F \rightarrow B_{H(\infty)}$.

8. The homotopy equivalence of B_F and $B_{H(\infty)}$. By the results of the previous section, we have a map $\bar{\xi} : B_F \rightarrow B_{H(\infty)}$. In this section, we prove that $\bar{\xi}$ is a homotopy equivalence induced from a homotopy equivalence between F and $H(\infty)$. From this it will follow that $\bar{\xi}_* : [B, B_F] \rightarrow [B, B_{H(\infty)}]$ is an isomorphism if B is a finite dimensional CW -complex.

Theorem 8.1. The spaces F and $H(\infty)$ are homotopy equivalent if F is a connected associative H -space.

Proof. Define a map $\tau : F \rightarrow H(\infty)$ as follows:

$$\tau(f) = \Phi_f = \{ \varphi_1^f, \varphi_2^f, \dots \} \quad f \in F$$

where $\varphi'_1 : F \rightarrow F$ is given by $\varphi'_1(g) = fg$, that is, a left translation of F . This is a homotopy equivalence since F is connected (see section 3 of [19]). The remaining maps are given by

$$\varphi'_k(t_1, \dots, t_{k-1}, f_1, \dots, f_k) = ff_1, \dots, f_k \quad \text{for all } k.$$

The map τ is continuous since the composition of maps

$$F \xrightarrow{\tau} M(F; F) \times M(I \times F^2; F) \times \dots \xrightarrow{p(k)} M(I^{k-1} \times F^k; F)$$

is continuous for each k . Here $p(k)$ is projection onto the corresponding factor.

On the other hand, define the map

$$\mu : H(\infty) \rightarrow F \text{ by}$$

$\mu(\Gamma) = \gamma_1(e)$, where $\Gamma = \{\gamma_1, \gamma_2, \dots\}$ is in $H(\infty)$, e is the unit of F . The map μ is continuous since it is the composition of maps

$$H(\infty) \xrightarrow{p(1)} H^2(\infty) \xrightarrow{w(e)} F$$

where $p(1)$ is projection of $H(\infty)$ onto that part of $H(\infty)$ contained in $M(F; F)$, here denoted by $H^2(\infty)$, and the map $w(e)$ is the evaluation map at e (continuous in the k -topology, see proof of Theorem 5.6 in [20]).

Note that $\mu(\tau(f)) = \mu(\Phi_f) = \varphi'_1(e) = fe = f$, so that $\mu \circ \tau = 1_F$. On the other hand $\tau \circ \mu(\Gamma) = \tau(\gamma_1(e)) = \Phi^{\gamma_1(e)} = \{\varphi_1^{\gamma_1(e)}, \varphi_2^{\gamma_1(e)}, \dots\}$. We claim that $\tau \circ \mu \simeq 1_{H(\infty)}$. That is, there exists a map $G_i : H(\infty) \rightarrow H(\infty)$ such that $G_0 =$ the identity on $H(\infty)$ and $G_1 = \tau \circ \mu$. To see this, let $H^k(\infty)$ denote the subspace of $H(\infty)$ which is contained in $M(I^{k-1} \times F^k; F)$. The map $G_i = (G_i^1, G_i^2, \dots)$ will consist of homotopies $\{G_i^k\} : H(\infty) \rightarrow H^k(\infty)$ for each k , such that $G_0^k =$ the identity on $H^k(\infty)$ and $G_1^k = \tau \circ \mu \mid H^k(\infty)$ and the G_i^k are compatible.

Define $G_i^k : H(\infty) \rightarrow H^k(\infty)$ as follows:

$$G_i^k(\Gamma)(t_1, \dots, t_{k-1}, f_1, \dots, f_k) = \gamma_{k-1}(t, t_1, \dots, t_{k-1}, e, f_1, \dots, f_k)$$

It is easily checked that $\{G_i^k\}$ is in $H(\infty)$, and that $G_1^k = \tau \circ \mu \mid H^k(\infty)$, $G_1^k(\Gamma) = \tau \circ \mu(\Gamma)$, $G_0^k =$ the identity on $H^k(\infty)$, and $\{G_0^k(\Gamma)\} = \Gamma$. This completes the proof that F and $H(\infty)$ are homotopy equivalent.

The main result of this section is the following theorem.

Theorem 8.2. *If F is a connected finite dimensional CW-complex, the homotopy equivalence $\tau : F \rightarrow H(\infty)$ induces a homotopy equivalence $\bar{\xi} : B_F \rightarrow B_{H(\infty)}$.*

Proof. Let $\Sigma = \{E, p, B, F\}$ be a principal quasifibration such that B has the homotopy type of a finite dimensional CW-complex. If we iteratively apply the Dold and Lashof construction and the A_∞ -construction, we get the following diagram.

$$\begin{array}{ccc} F & \xrightarrow{\tau} & H(\infty) \\ \searrow & & \searrow \\ DL^\infty(E) & \xrightarrow{\bar{\tau}} A_\infty(DL^\infty(E)) \xrightarrow{w} A_\infty(Ult_\infty(E)) \\ \downarrow & \swarrow & \downarrow \\ DL^\infty(B) & \xrightarrow{\bar{w}} & Ult_\infty(B) \end{array}$$

where $\bar{\tau} : DL^*(E) \rightarrow A_\infty(DL^*(E))$ is given by $\bar{\tau}(x) = \Phi_x$, x in $DL^*(E)$. Since $DL^*(E)$ and $A_\infty(Ult_\infty(E))$ are both aspherical, $w_1 \circ \bar{\tau}$ is a weak homotopy equivalence. The map w_1 is the associated map of $DL^*(E) \rightarrow Ult_\infty(E)$ induced from $1_E : E \rightarrow E$, the existence of which is guaranteed by Theorem 7.1. The map w_1 lies over the map $\bar{w}_1 : DL^*(B) \rightarrow Ult_\infty(B)$. Since both $DL^*(E)$ and $A_\infty(Ult_\infty(E))$ are aspherical, the map $\bar{w}_1$ is a weak homotopy equivalence. However, since $H(\infty)$ has the homotopy type of the finite dimensional CW -complex F , so does $Ult_\infty(B)$. Thus the map $\bar{w}_1$ is a homotopy equivalence. This is particularly true if $E = F$, $B = *$, for then $\bar{w}_1 = \bar{\xi}$ and $\bar{\xi} : B_F \rightarrow B_{H(\infty)}$ is a homotopy equivalence.

Corollary 8.3. *If F is a connected finite dimensional CW -complex, then for any finite dimensional CW -complex B the induced map*

$$\bar{\xi}_* : [B, B_F] \rightarrow [B, B_{H(\infty)}]$$

is an isomorphism.

This means that any map $g : B \rightarrow B_{H(\infty)}$ factors up to homotopy through B_F ; that is, there exists a map $h : B \rightarrow B_F$ such that $\bar{\xi} \circ h \simeq g$. Thus we can see as far as the results of principal fibre spaces are concerned, the A_∞ -construction and the Dold and Lashof construction are the same.

In addition, we have proven that given a fibre space $F \rightarrow E \rightarrow B$ which admits an A_∞ -action $\{\theta_k\}$, then, up to homotopy, there is an action $m : E \times F \rightarrow E$ which is associative. More precisely, we have Theorem 1.8 paraphrased as:

$F \rightarrow E \rightarrow B$ (where B or F is a CW -complex) is fibre homotopy equivalent to a fibre space induced from the universal fibre space $F \rightarrow E_F \rightarrow B_F$ if and only if E admits an A_∞ -action of F .

9. Appendix. Throughout this paper our constructions have been applied to quasifibrations, and our results are in terms of Hurewicz fibre spaces. In this appendix, we introduce the propositions which justify these changes.

Proposition 9.1. *If $p : E \rightarrow B$ is a Hurewicz fibre space, and if B and all the fibres have the homotopy type of CW -complexes, then so does E .*

Proof. This is proposition (0) of [16].

Definition 9.2. A map $h : X \rightarrow Y$ is said to be a weak homotopy equivalence if the induced map $h_* : \pi_i(X) \rightarrow \pi_i(Y)$ is an isomorphism for all i .

Definition 9.3. Two spaces X_1 and X_2 are said to have the same weak homotopy type if there exists a space Y and weak homotopy equivalences $f : Y \rightarrow X_1$ and $g : Y \rightarrow X_2$.

We have previously defined the notion of quasi- A_n -equivalence. We denote by $Q_n F(X)$ the set of quasi- A_n -equivalence classes of A_n -principal quasifibration $p : E \rightarrow X$ with fibres of the weak homotopy type of F and total space of the homotopy type of a CW -complex.

Definition 9.4. Two fibres $p_i : E_i \rightarrow B$, $i = 0, 1$, are said to be weakly fibre equivalent if there exists a fibrewise map $w_1 : E_0 \rightarrow E_1$ such that w_1 is a weak homotopy equivalence.

The following propositions and corollary are due to Stasheff [16] with minor modifications made to account for the A_n -actions on the fibre spaces.

Proposition 9.5. (Compare with Proposition 4 of [16]). *Each class in $Q_n F(X)$ is represented by a Hurewicz fibre space.*

Proof. Given any map $p : E \rightarrow X$, the associated Hurewicz fibre space $\text{Hur}(p) : \text{Hur}(E) \rightarrow X$ is obtained by taking $\text{Hur}(E) \subset E \times X^I$ to be $\{(x, \lambda) \mid \lambda(0) = p(x)\}$ and setting $\text{Hur}(p)(w, \lambda) = \lambda(1)$. The map $w_1 : E \rightarrow \text{Hur}(E)$ given by $w_1(x) = (x, \lambda_x)$, where $\lambda_x(t) = p(x)$ for all t , is a weak fibre equivalence if p is a quasifibration, E being a deformation retract of $\text{Hur}(E)$.

If, in addition, p is an A_n -principal quasifibration with A_n -action $\{\theta_k\}$, then $\text{Hur}(p)$ can be given an A_n -principal structure as follows. Define $\text{Hur}(\theta_k) : I^{k-1} \times \text{Hur}(E) \times I^k \rightarrow \text{Hur}(E)$, $1 \leq k \leq n$ by

$$\begin{aligned} \text{Hur}(\theta_k)(t_1, \dots, t_{k-1}, (x, \lambda), f_1, \dots, f_k) \\ = (\theta_k(t_1, \dots, t_{k-1}, x, f_1, \dots, f_k), \varphi(t_1, \dots, t_{k-1}, x) + \lambda). \end{aligned}$$

$\varphi(t_1, \dots, t_{k-1}, x)(t)$ is a homotopy from $jp(x)$ to $p(x)$ and $j : I^{k-1} \times E \rightarrow X$ is homotopic to projection on X , compatible with lower dimensional j 's on ∂I^{k-1} .

The map w_1 induces an admissible family of maps. For instance, we know $w_1 \circ \theta_1(x, f) = (xf, \lambda_{xf})$, whereas $\text{Hur}(\theta_1) \circ (w_1 \times 1)(x, f) = (xf, \varphi(x) + \lambda_x)$. Thus define $w_2(t, x, f) = (xf, \psi_t(x))$, where $\psi_t(x)$ stays at xf from 0 until t and then runs along $\varphi(x) + \lambda_x$ as far as time remaining (i.e., t until 1) permits.

The problem of w_3 can be illustrated as follows.

$$\begin{array}{ccc} & ((xf)g, \varphi(xf) + \psi_t(x)) & \\ & \boxed{} & \\ ((xf)g, \psi_t(xf)) & & (\theta_2(t, x, f, g), \varphi(t, x) + \lambda) \\ (\theta_2(t, x, f, g), \lambda\theta_2(t, x, f, g)) & & \\ & (x(fg), \psi_t(x)) & \end{array}$$

This is filled in quite easily since we assume the homotopies are compatible with the face conditions. Similar arguments would give us w_k for $k > 3$.

Proposition 9.6. (Compare with Proposition 5 of [16]). *If $p_i : E_i \rightarrow B$, $i = 0, 1$ are A_n -principal quasifibrations and $\{w_k : I^{k-1} \times E_0 \times F^{k-1} \rightarrow E_1, 1 \leq k \leq n+1\}$ is a weak A_n -equivalence, then the associated Hurewicz fibre spaces are A_n -equivalent, provided the spaces E_i have the homotopy type of CW-complexes and B is a CW-complex.*

Proof. The map $w_k : I^{k-1} \times E_0 \times F^{k-1} \rightarrow E_1$ induces $\text{Hur}(w_k) : I^{k-1} \times \text{Hur}(E_0) \times F^{k-1} \rightarrow \text{Hur}(E_1)$. Since the E_i have the homotopy type of CW -complexes, $\text{Hur}(w_1)$ is a homotopy equivalence and hence a fibre homotopy equivalence according to Dold (Theorem 6.1 of [2]). We need only observe that an admissible family with w_1 a fibre homotopy equivalence implies it is an A_n -equivalence, i.e., an inverse w'_1 to w_1 can be extended to an inverse for the whole family [cf. Fuchs [6] who does the case $B = *$].

Corollary 9.7. (Compare Corollary 6 of [16]). $Q_n F(X)$ and $A_n F(X)$ are isomorphic if X and F are CW -complexes.

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Date communicated: MAY 15, 1971