The Strong Szegő Limit Theorem for Circular Arcs

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The strong Szegő limit theorem states that if f is a positive function on the unit circle whose derivative satisfies a Lipschitz condition (these conditions have been considerably relaxed in recent years) and $D_n[f]$ is the Toeplitz determinant

$$\det \left(\frac{1}{2\pi} \int_0^{2\pi} e^{i(j-k)\theta} f(\theta) d\theta \right), \quad j, k = 0, \dots, n-1,$$

then as $n \rightarrow \infty$ one has the asymptotic formula

$$D_n[f] \sim E[f]G[f]^n$$

where, if

$$\log f(\theta) \sim \sum_{k=-\infty}^{\infty} f_k e^{ik\theta},$$

one defines

$$G[f] = \exp f_0, \quad E[f] = \exp \frac{1}{4} \sum_{k=1}^{\infty} k f_k f_{-k}.$$

The analogue of this is obtained here for a function f which satisfies $f(\theta) = f(2\pi - \theta)$, which is supported on a closed arc $\alpha \leq \theta \leq 2\pi - \alpha$, and which when restricted to that arc is positive and has derivative satisfying a Lipschitz condition with exponent greater than $1/2$ (these conditions also can be relaxed). The formula reads

$$D_n[f] \sim 2^{1/12} e^{3\pi'(-1)} \left(\sin \frac{\alpha}{2} \right)^{-1/4} E[F]^2 n^{-1/4} G[F]^n \left(\cos \frac{\alpha}{2} \right)^n,$$

where E and G have the same meanings as before and

$$F(\theta) = f \left(2 \cos^{-1} \cos \frac{\alpha}{2} \cos \theta \right).$$

We always take $0 \leq \cos^{-1} x \leq \pi$.

Here is an outline of the derivation. It is first shown that

$$(1) \quad (2\pi)^{2n+1} D_{n+1}[f] D_n[f] = 2^{2n(n+1)+1} \Delta_{n+1}[u] \Delta_n[v]$$

where

$$\begin{aligned} u(x) &= (1 - x^2)^{-1/2} f(2 \cos^{-1} x) \\ v(x) &= (1 - x^2)^{1/2} f(2 \cos^{-1} x) \end{aligned}$$

and $\Delta_n[u]$ is the Hankel determinant

$$\det \left(\int_{-1}^1 x^{j+k} u(x) dx \right), \quad j, k = 0, \dots, n-1.$$

This is a consequence of formulas connecting orthogonal polynomials on the circle with orthogonal polynomials on the unit interval.

If f is supported on $[\alpha, 2\pi - \alpha]$ then u and v are supported on $[-\cos(\alpha/2), \cos(\alpha/2)]$. Thus if we write

$$U(x) = u\left(x \cos \frac{\alpha}{2}\right), \quad V(x) = v\left(x \cos \frac{\alpha}{2}\right),$$

then U and V are supported on $[-1, 1]$ and it is trivial that

$$\Delta_{n+1}[u] = \left(\cos \frac{\alpha}{2}\right)^{(n+1)^2} \Delta_{n+1}[U], \quad \Delta_n[v] = \left(\cos \frac{\alpha}{2}\right)^{n^2} \Delta_n[V].$$

Next it is shown that for any U supported in $[-1, 1]$ one has

$$(2) \quad \Delta_n[U] = A_n \mathfrak{D}_n[U]$$

where

$$\mathfrak{D}_n[U] = \det \left(\int_{-1}^1 P_i(x) P_k(x) U(x) dx \right), \quad j, k = 0, \dots, n-1,$$

(here the P_i are the normalized Legendre polynomials) and

$$A_n = \prod_{i=0}^{n-1} \left(i + \frac{1}{2}\right)^{-1} \left[\frac{2^i (i!)^2}{(2i)!} \right]^2.$$

A routine computation gives

$$(3) \quad A_n \sim 2^{1/12} e^{3\pi'(-1)} n^{-1/4} (2\pi)^n 2^{-n^2},$$

and so at this point one has established

$$(4) \quad D_{n+1}[f] D_n[f] \sim 2^{1/6} e^{6\pi'(-1)} n^{-1/2} \Delta_{n+1}[U] \Delta_n[V] \left(\cos \frac{\alpha}{2}\right)^{2n^2+2n+1}.$$

A theorem of Hirschman [3] gives an asymptotic formula for $\mathfrak{D}_n[U]$ which can be written

$$\mathfrak{D}_n[U] \sim E[U(\cos \theta)]^2 G[U(\cos \theta)].$$

Our assumptions on f imply that Hirschman's theorem applies to U and V . Finally, the ratio $D_{n+1}[f]/D_n[f]$ is easily taken care of and the asymptotic formula for $D_n[f]$ follows after a little more computation.

Let us now work out the details. Theorem 11.5 of [4] gives formulas expressing $p_n(x)$ and $q_n(x)$, the orthonormal polynomials associated with u, v respectively, on $-1 \leq x \leq 1$, in terms of $\varphi_n(z)$, the orthonormal polynomials on the unit circle associated with

$$u(\cos \theta) |\sin \theta| = f(2|\theta|), \quad -\pi \leq \theta \leq \pi.$$

If $\psi_n(z)$ are the orthonormal polynomials associated with $f(\theta)$, $0 \leq \theta \leq 2\pi$, then it is easy to see that

$$\varphi_{2n}(z) = \psi_n(z^2), \quad \varphi_{2n+1}(z) = z\psi_n(z^2).$$

Here the fact $f(\theta) = f(2\pi - \theta)$ is used. The second and third assertions of the above-mentioned theorem give

$$\begin{aligned} p_n(x) &= (2\pi)^{-1/2} \left\{ 1 - \frac{\psi_n(0)}{k_n} \right\}^{-1/2} \{ z^{-n+2} \psi_{n-1}(z^2) + z^{n-2} \psi_{n-1}(z^{-2}) \} \\ q_{n-1}(x) &= \left(\frac{2}{\pi} \right)^{-1/2} \left\{ 1 - \frac{\psi_n(0)}{k_n} \right\}^{-1/2} \frac{z^{-n} \psi_n(z^2) - z^n \psi_n(z^{-2})}{z - z^{-1}} \end{aligned}$$

where $z = x + i(1 - x^2)^{1/2}$ and k_n is the (positive) highest coefficient of ψ_n . Let k'_n, k''_n denote the highest coefficients of p_n, q_n respectively. Then comparing highest coefficients of x gives

$$\begin{aligned} k'_n &= 2^n (2\pi)^{-1/2} \{ k_n - \psi_n(0) \}^{-1/2} k_n^{1/2} k_{n-1} \\ k''_{n-1} &= 2^n (2\pi)^{-1/2} \{ k_n - \psi_n(0) \}^{1/2} k_n^{1/2} \end{aligned}$$

and so

$$(5) \quad 2\pi k'_n k''_{n-1} = 2^{2n} k_n k_{n-1}.$$

Since

$$k_n^2 = \frac{D_n[f]}{D_{n+1}[f]}, \quad k_n'^2 = \frac{\Delta_n[u]}{\Delta_{n+1}[u]}, \quad k_n''^2 = \frac{\Delta_n[v]}{\Delta_{n+1}[v]}$$

(which hold also for $n = 0$ with the convention that determinants of order 0 equal 1), squaring (5), setting $n = 1, \dots, n$, and multiplying give

$$(2\pi)^{2n} \frac{\Delta_1[u]}{\Delta_{n+1}[u] \Delta_n[v]} = 2^{n(n+1)} \frac{D_1[f]}{D_{n+1}[f] D_n[f]}.$$

Since

$$\Delta_1[u] = \int_0^\pi f(2\theta) d\theta = \pi D_1[f],$$

(1) is established.

To obtain (2) we use the fact that if

$$P_i(x) = \sum_{k \leq i} a_{ik} x^k,$$

then

$$\det \left(\int_{-1}^1 P_i(x) P_k(x) U(x) dx \right) = \left(\prod_{i=0}^{n-1} a_{ii} \right)^2 \det \left(\int_{-1}^1 x^{i+k} U(x) dx \right).$$

This results from the observation that the matrix of the determinant on the left is, as seen by expanding P_i and P_k , the product of three matrices; one is the matrix of the determinant appearing on the right and the other two are triangular matrices with diagonal elements a_{ii} . Since

$$a_{ii} = \left(i + \frac{1}{2} \right)^{1/2} \frac{(2i)!}{2^i (i!)^2},$$

(2) follows.

To compute A_n asymptotically, observe that

$$\begin{aligned} \prod_{i=1}^n \left(i + \frac{1}{2} \right)^{-1} &= \frac{2^{2n+1} n!}{(2n+1)!}, \\ \prod_{i=1}^n i! &= \prod_{k=1}^n k^{n-k+1} = (n!)^{n+1} \prod_{k=1}^n k^{-k}, \\ \prod_{i=1}^n (2i)! &= \prod_{k=1}^{2n} k^{\lfloor n-k/2+1 \rfloor} = \prod_{k=1}^{2n} k^{n-k/2+1/2} \prod_{j=1}^n (2j)^{1/2} \\ &= [(2n)!]^{n+1/2} 2^{n/2} \left[\prod_{k=1}^{2n} k^k \right]^{-1/2}, \end{aligned}$$

and so

$$A_{n+1} = 2^{n^2+2n} (n + \frac{1}{2})^{-1} (n!)^{4n+4} [(2n)!]^{-2n-2} \prod_{k=1}^{2n} k^k \left(\prod_{k=1}^n k^k \right)^{-4}.$$

Stirling's formula gives

$$\frac{(n!)^2}{(2n)!} = (2\pi)^{1/2} 2^{-n-1} n^{-1/2} \left[1 + \frac{1}{8n} + O(n^{-2}) \right].$$

Now it follows from the Euler-Maclaurin summation formula that

$$\zeta(s) = \lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^n k^{-s} - \frac{n^{1-s}}{1-s} + \frac{1}{2} n^{-s} + \frac{1}{12} s n^{-s-1} \right\}$$

uniformly for $Rs \geq -3 + \delta$ ($\delta > 0$). See, for example, [2, p. 333]. One may differentiate under the limit sign and set $s = -1$. This gives

$$\sum_{k=1}^n k \log k = \left(\frac{1}{2} n^2 + \frac{1}{2} n + \frac{1}{12} \right) \log n - \frac{1}{4} n^2 + \frac{1}{12} - \zeta'(-1) + o(1)$$

and so

$$\prod_{k=1}^{2n} k^k \left(\prod_{k=1}^n k^k \right)^{-4} \sim 2^{2n^2+n+1/12} n^{-n-1/4} e^{-1/4} e^{3\zeta'(-1)}.$$

Hence

$$A_{n+1} \sim 2^{-(n+1)^2+1/12} (2\pi)^{n+1} n^{-1/4} e^{3\zeta'(-1)}$$

which is (3) with n replaced by $n + 1$.

From (4) and Hirschman's theorem it follows that

$$(6) \quad D_{n+1}[f] D_n[f] \sim 2^{1/6} e^{6\zeta'(-1)} E[U(\cos \theta)]^2 E[V(\cos \theta)]^2 \\ \cdot G[U(\cos \theta)] n^{-1/2} G[F]^{2n} \left(\cos \frac{\alpha}{2} \right)^{2n^2+2n+1}.$$

Here the simple fact

$$G[U(\cos \theta)] G[V(\cos \theta)] = G[F]^2$$

was used.

Now the arc $\alpha \leq \theta \leq 2\pi - \alpha$ of the unit circle has transfinite diameter (capacity) $\cos(\alpha/2)$ and so it follows from quite general considerations [1, p. 85] that

$$\lim_{n \rightarrow \infty} \left(\cos \frac{\alpha}{2} \right)^{-2n-1} \frac{D_{n+1}[f]}{D_n[f]}$$

exists. It is therefore equal to the limit of the n^{th} root of

$$\left(\cos \frac{\alpha}{2} \right)^{-n^2} D_n[f]$$

which is, by (6), just $G[F]$. Hence

$$(7) \quad D_n[f] \sim 2^{1/12} e^{3\zeta'(-1)} E[U(\cos \theta)] E[V(\cos \theta)] \\ \cdot G[U(\cos \theta)]^{1/2} n^{-1/4} G[F]^{n-1/2} \left(\cos \frac{\alpha}{2} \right)^{n^2}.$$

Since

$$U(\cos \theta) = \left(1 - \cos^2 \frac{\alpha}{2} \cos^2 \theta \right)^{-1/2} F(\theta), \\ V(\cos \theta) = \left(1 - \cos^2 \frac{\alpha}{2} \cos^2 \theta \right)^{1/2} F(\theta),$$

we have

$$(8) \quad E[U(\cos \theta)] E[V(\cos \theta)] = E[F]^2 E \left[\left(1 - \cos^2 \frac{\alpha}{2} \cos^2 \theta \right)^{1/2} \right]^2,$$

$$(9) \quad G[U(\cos \theta)]^{1/2} = G[F]^{1/2} / G \left[\left(1 - \cos^2 \frac{\alpha}{2} \cos^2 \theta \right)^{1/2} \right].$$

From the Fourier series expansion

$$\frac{1}{2} \log (1 - a^2 \cos^2 \theta) = -\log s - \sum_{k=1}^{\infty} \frac{r^{2k}}{k} \cos k\theta,$$

where

$$r = \frac{1 - (1 - a^2)^{1/2}}{a}, \quad s = 2r/a,$$

we compute

$$\frac{E[(1 - a^2 \cos^2 \theta)^{1/2}]^2}{G[(1 - a^2 \cos^2 \theta)^{1/2}]} = s^{1/2}(1 - r^4)^{-1/4}$$

which for $a = \cos(\alpha/2)$ is just $(\sin(\alpha/2))^{-1/4}$. Putting this, (8), and (9) into (7) completes the derivation.

A few words are in order concerning the relaxation of conditions mentioned at the beginning. The Lipschitz condition may be relaxed by requiring only that f' satisfy a Lipschitz condition with some positive exponent and that

$$\frac{\log f(\theta) - \log f(\alpha)}{\left(\cos \frac{\alpha}{2} - \cos \frac{\theta}{2}\right)^{3/2}}, \quad \frac{\log f(\theta) - \log f(2\pi - \alpha)}{\left(\cos \frac{\alpha}{2} + \cos \frac{\theta}{2}\right)^{3/2}}$$

be integrable near $\theta = \alpha$, $\theta = 2\pi - \alpha$ respectively. These conditions are sufficient to apply Hirschman's theorem to U and V .

The derivation of (1), which appealed to formal properties of the polynomials orthogonal with respect to positive weight functions, can be carried out for arbitrary complex integrable weight functions. Instead of the orthogonal polynomials p_n , for example, one uses the polynomials defined by

$$\int_{-1}^1 p_n(x) x^k u(x) dx = \delta_{kn}, \quad k = 0, \dots, n,$$

with q_n and ψ_n defined similarly. One can then deduce (1) under the assumption that none of the determinants up to order $n + 1$ vanish, and a simple limiting argument removes this assumption.

Since Hirschman's theorem applies not only to positive functions, but arbitrary nonzero functions, satisfying the appropriate smoothness conditions, it would appear that we can also drop the positivity requirement. However, recall our derivation of (7). What was really obtained, and what can be obtained similarly without the assumption of positivity, was the square of this relationship. Of course for f positive, (7) itself follows because $D_n[f] > 0$. But in the more general case one can only deduce (7) modified with a sequence of factors ± 1 on the right side. As a way around this, introduce a complex parameter t and replace f by $e^{t \log f}$, where any continuous logarithm is used. It is virtually certain that Hirschman's limit theorem holds uniformly for any family of functions satisfying the appropriate smoothness and integrability conditions

uniformly. This has, admittedly, not been checked. However, if so, we could conclude that the quotient of the left side of (7) by the right, denoted say by $Q_n(t)$, is a normal family of analytic functions of t . We know that $|Q_n(t)| \rightarrow 1$ as $n \rightarrow \infty$ and that $Q_n(0)$, corresponding to the case of $f = 1$, converges to 1. It follows that $Q_n(t) \rightarrow 1$ for all t and so in particular for $t = 1$. This gives (7) for f .

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