Best Approximation by Compact Operators

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§1. Introduction. A closed linear subspace M of a Banach space X is said to be *proximal* (or, an *existence set*) if every element in X has a best approximation out of M , that is, if $\text{dist}(x, M)$ is attained for each $x \in X$. Finite dimensional subspaces are clearly proximal, and, as is well known, every subspace of X is proximal exactly when X is reflexive [4, p. 253]. In contrast, M is said to be *antiproximal* if the only elements of X which have best approximations out of M are those in M itself. The standard example of such a subspace is the kernel of a bounded linear functional on X which fails to attain its norm on the unit ball of X .

J. Blatter has observed in his thesis [1] that for a number of standard Banach spaces X , the canonical image of X is proximal in the second conjugate space X^{**} . As a test of the conjecture that this property is true of Banach spaces in general he suggested the following question. Is the space $\mathcal{C}(X)$ of compact operators on a Hilbert space X proximal in the space $\mathcal{B}(X)$ of bounded operators on X ? The point here is of course that $\mathcal{B}(X)$ may be identified with $\mathcal{C}(X)^{**}$ [6, p. 48]. We show below in Section 3 that this question has an affirmative answer.

The remaining two sections contain counter-examples which demonstrate in particular that this result and Blatter's conjecture are generally false. More precisely, we first exhibit a (separable rotund) isomorph X of Hilbert space for which $\mathcal{C}(X)$ is antiproximal in $\mathcal{B}(X)$. We then prove that this same space has the property that $\mathcal{C}(X)^{**} = \mathcal{B}(X)$. In this way we see that a Banach space may be antiproximal in its second conjugate. Finally, we observe that a simplified example of such behavior is possible by similar reasoning. Namely, the sequence space c_0 , suitably renormed, has this property.

Finally we should emphasize that the two principal questions investigated in this work remain open in general. That is, let X be a given Banach space.

- a) Is $\mathcal{C}(X)$ proximal in $\mathcal{B}(X)$?
- b) Is X proximal in X^{**} ?

Apropos the first question we note that an affirmative answer to it is not characteristic of Hilbert space. For example, suppose that the Banach space X has a finite codimensional subspace which is a Hilbert space and which is the range of a norm decreasing projection on X . Then, taking into account the result in Section 3 below, it can be shown that $\mathfrak{C}(X)$ is proximal in $\mathfrak{B}(X)$.

§2. Preliminaries. Let X and Y be Banach spaces, and let $\mathfrak{B}(X, Y)$ be the space of all bounded linear operators from X to Y , with the usual operator norm. Let $\mathfrak{C}(X, Y)$ be the subspace of $\mathfrak{B}(X, Y)$ consisting of those operators which carry sequences weakly convergent to θ in X into sequences norm convergent to θ in Y . When X is reflexive, $\mathfrak{C}(X, Y)$ is the space of compact operators from X to Y . The distance from an operator $T \in \mathfrak{B}(X, Y)$ to $\mathfrak{C}(X, Y)$ is (by definition) $d(T, \mathfrak{C}) = \inf \{ \|T - L\| : L \in \mathfrak{C}(X, Y) \}$. When X and Y are the same Hilbert space, $d(T, \mathfrak{C})$ is the reciprocal of the Fredholm radius of T [5, p. 205]. Finally let $\mathfrak{S}(X)$ be the set of all sequences of unit vectors in X which converge weakly to θ .

Lemma 1. Let $T \in \mathfrak{B}(X, Y)$ and define

$$\Delta(T) = \sup \{ \limsup_{n \rightarrow \infty} \|T(s_n)\| : s \in \mathfrak{S}(X) \}.$$

Then $\Delta(T) \leq d(T, \mathfrak{C})$. Consequently, if there exists $L \in \mathfrak{C}(X, Y)$ for which $\|T - L\| \leq \Delta(T)$, then $\|T - L\| = \Delta(T) = d(T, \mathfrak{C})$, and hence L is a best approximation to T out of $\mathfrak{C}(X, Y)$.

Proof. Given any $s \in \mathfrak{S}(X)$ and any $L \in \mathfrak{C}(X, Y)$, we have $\lim \|L(s_n)\| = 0$. It follows that $\limsup \|(T - L)(s_n)\| = \limsup \|T(s_n)\|$, and so $\Delta(T) = \Delta(T - L) \leq \|T - L\|$; thus $\Delta(T) \leq d(T, \mathfrak{C})$. If also $\|T - L\| \leq \Delta(T)$, then $d(T, \mathfrak{C}) \leq \|T - L\| \leq \Delta(T) \leq d(T, \mathfrak{C})$. Q.E.D.

Lemma 2. Let X and Y be Hilbert spaces and suppose that $T \in \mathfrak{B}(X, Y)$ fails to attain its norm, in the sense that $\|T(x)\| = \|T\| \|x\|$ only if $x = \theta$. If s is a sequence of unit vectors in X such that $\lim \|T(s_n)\| = \|T\|$, then $s \in \mathfrak{S}(X)$.

Proof. The unit ball in X is weakly sequentially compact. Hence if s does not converge weakly to θ , we may find a subsequence t which converges weakly to some non-zero $x \in X$. Since T is weakly continuous, $T \circ t$ converges weakly to $T(x)$. In particular, $\lim (x, t_n - x) = 0$ and $\lim (T(x), T(t_n - x)) = 0$, where $(\cdot, \cdot)$ is the inner product. It follows that $\lim \|t_n - x\|^2 = \lim \|t_n\|^2 - \|x\|^2 - 2 \lim \operatorname{Re} (x, t_n - x) = 1 - \|x\|^2$, and then that

$$\begin{aligned} \|T\|^2 (1 - \|x\|^2) &\geq \lim \|T(t_n - x)\|^2 \\ &= \lim \|T(t_n)\|^2 - \|T(x)\|^2 - 2 \lim \operatorname{Re} (T(x), T(t_n - x)) \\ &= \|T\|^2 - \|T(x)\|^2. \end{aligned}$$

But $\|T(x)\| < \|T\| \|x\|$ by hypothesis, so

$$\begin{aligned}\|T\|^2 - \|T(x)\|^2 &> \|T\|^2 - \|T\|^2 \|x\|^2 \\ &= \|T\|^2 (1 - \|x\|^2).\end{aligned}$$

This contradiction completes the proof.

§3. Approximation by compact operators. In this section we prove the following theorem which shows in particular that $\mathfrak{C}(X, Y)$ is always proximal in $\mathfrak{B}(X, Y)$ when X and Y are Hilbert spaces.

Theorem. *If X and Y are Hilbert spaces and $T \in \mathfrak{B}(X, Y)$, there exists $L \in \mathfrak{C}(X, Y)$ such that $\Delta(T) = d(T, \mathfrak{C}) = \|T - L\|$. L is a best approximation to T out of $\mathfrak{C}(X, Y)$.*

Proof. The proof is divided into several cases.

Case 1. If X is finite dimensional, then $T \in \mathfrak{C}(X, Y)$ and $\Delta(T) = d(T, \mathfrak{C}) = 0$.

Case 2. Suppose there is an orthogonal projection P of X onto a subspace of finite codimension such that $TP (= T \circ P)$ does not attain its norm. Let t be a sequence of unit vectors in X such that $\lim \|TP(t_n)\| = \|TP\|$. Then $t \in \mathfrak{S}(X)$ by Lemma 2, hence $\Delta(TP) \geq \lim \|TP(t_n)\| = \|TP\|$. By Lemma 1, $d(TP, \mathfrak{C}) = \Delta(TP) = \|TP\|$. Now if $L \in \mathfrak{C}(X, Y)$ then $L - (T - TP) \in \mathfrak{C}(X, Y)$ also, since $T - TP$ has finite rank. Therefore, $\|T - (T - TP)\| = \|TP\| = d(TP, \mathfrak{C}) \leq \|TP - (L - (T - TP))\| = \|T - L\|$. This shows that $T - TP$ is a best approximation to T out of $\mathfrak{C}(X, Y)$, and that $d(T, \mathfrak{C}) = \|TP\| = \Delta(TP) = \Delta(TP - (T - TP)) = \Delta(T)$.

Case 3. Suppose that X is infinite dimensional and that TP attains its norm whenever P is an orthogonal projection of X onto a finite codimensional subspace. We construct an orthonormal sequence in X as follows. We let e_1 be any unit vector in X for which $\|T(e_1)\| = \|T\|$. Having chosen $e_1, \dots, e_n$, we let P_n be the orthogonal projection of X onto $\{e_1, \dots, e_n\}^\perp \equiv E_n$. If $TP_n = 0$, let e_{n+1} be any unit vector in E_n . Otherwise, choose a unit vector e_{n+1} such that $\|TP_n(e_{n+1})\| = \|TP_n\|$. Since $\|TP_n(P_n(e_{n+1}))\| = \|TP_n(e_{n+1})\| = \|TP_n\|$, $\|P_n(e_{n+1})\| = 1$. This implies that $P_n(e_{n+1}) = e_{n+1}$, that is, $e_{n+1} \in E_n$.

By the choice of e_n ,

$$\begin{aligned}|(P_{n-1}T^*T(e_n), x)| &= |(TP_{n-1}(e_n), TP_{n-1}(x))| \\ &\leq \|TP_{n-1}\|^2 = \|TP_{n-1}(e_n)\|^2 \\ &= (P_{n-1}T^*T(e_n), e_n),\end{aligned}$$

for any unit vector $x \in X$. Setting $\alpha = \|P_{n-1}T^*T(e_n)\|$, we thus have $P_{n-1}T^*T(e_n) = \alpha e_n$. Therefore, for any $x \in X$,

$$\begin{aligned}(T(e_n), TP_n(x)) &= (T(e_n), TP_{n-1}P_n(x)) \\ &= (P_nP_{n-1}T^*T(e_n), x) = \alpha(P_n(e_n), x) = 0.\end{aligned}$$

In particular, if $m > n$, $e_m = P_n(e_m)$, and so

$$(3.1) \quad (T(e_n), T(e_m)) = 0.$$

If $s \in \mathcal{S}(X)$, $\lim_{m \rightarrow \infty} (I - P_n)(s_m) = 0$. Therefore,

$$\begin{aligned} \limsup ||T(s_m)|| &= \limsup_{m \rightarrow \infty} ||TP_n(s_m)|| \\ &\leq ||T(e_n)||, \end{aligned}$$

whence $\Delta(T) \leq ||T(e_n)||$. Now evidently $||T(e_{n+1})|| \leq ||T(e_n)||$, and since $e \in \mathcal{S}(X)$, we see that

$$(3.2) \quad \lim ||T(e_n)|| = \Delta(T).$$

For a vector $y \in Y$, we use the notation

$$\text{sgn}(y) = \begin{cases} \theta, & y = \theta \\ y/||y||, & y \neq \theta. \end{cases}$$

We now define an operator $L \in \mathcal{C}(X, Y)$ by

$$L = \sum_{n=1}^{\infty} (||T(e_n)|| - \Delta(T)) \text{sgn}(T(e_n)) \otimes \bar{e}_n.$$

This means that for $x \in X$,

$$L(x) = \sum_{n=1}^{\infty} (||T(e_n)|| - \Delta(T)) \langle x, e_n \rangle \text{sgn}(T(e_n)).$$

This series converges for every x by (3.1) and L is compact because of (3.2) [6, p. 8, 13].

We are going to show that $||T - L|| \leq \Delta(T)$; this will complete the proof in view of Lemma 1. To this end we express any $x \in X$ as $x = u + v$, where $u \perp v$ and $v \in \{e_1, e_2, \dots\}^\perp$. Then

$$\begin{aligned} ||(T - L)(x)||^2 &\leq \sum_{n=1}^{\infty} \Delta(T)^2 |\langle u, e_n \rangle|^2 + ||T(v)||^2 \\ &= \Delta(T)^2 ||u||^2 + ||T(v)||^2 \\ &\leq \Delta(T)^2 (||u||^2 + ||v||^2) = \Delta(T)^2 ||x||^2. \quad \text{Q.E.D.} \end{aligned}$$

Remark. Although not needed in the present work, it may be useful for other purposes to know that the following alternative (and simpler) formula for $d(T, \mathcal{C})$ is valid. Namely,

$$d(T, \mathcal{C}) = \sup \{ \lim_{n \rightarrow \infty} \sup ||T(e_n)|| \},$$

where the supremum is taken over all orthonormal sequences $\{e_n\}$ in X .

§4. Antiproximinal spaces of compact operators. In this section we show that there exist Banach spaces Y , which are strictly convex isomorphs of Hilbert

space, and which have the property that the subspace $\mathcal{C}(Y)$ of compact operators on Y is antiproximinal in the space $\mathfrak{B}(Y)$ of all bounded operators on Y . This fact is a particular consequence of the following theorem.

Theorem. *Let $(Y, \|\cdot\|)$ be a separable infinite dimensional Hilbert space. Then for every $\epsilon > 0$ there is a norm $|\cdot|$ on Y such that*

- a) $\|y\| \leq |y| \leq (1 + \epsilon)\|y\|$;
- b) *if X is any infinite dimensional Hilbert space, the subspace $\mathcal{C}(X, Y)$ is $|\cdot|$ -antiproximinal in $\mathfrak{B}(X, Y)$.*

Proof. Let $\{e_1, e_2, \dots\}$ be an orthonormal basis for Y . We define

$$(4.1) \quad |y| = \|y\| + \epsilon \sum_{n=1}^{\infty} 2^{-n} |(y, e_n)|$$

If s is any sequence of vectors in Y weakly convergent to θ , then s is bounded and $\lim (\|s_n\| - \|s_n\|) = 0$. It follows that if $T \in \mathfrak{B}(X, Y)$ and $s \in \mathcal{S}(X)$, then $\limsup \|T(e_n)\| = \limsup |T(e_n)|$. This shows that

$$\begin{aligned} \Delta(T) &= \sup \{ \limsup \|T(s_n)\| : s \in \mathcal{S}(X) \} \\ &= \sup \{ \limsup |T(s_n)| : s \in \mathcal{S}(X) \}. \end{aligned}$$

Let P_n be the orthogonal projection of Y onto $\text{span}\{e_1, \dots, e_n\}$. Let L be a $\|\cdot\|$ -best approximation to T out of $\mathcal{C}(X, Y)$. Then

$$\begin{aligned} \|T - L\| &\leq \|T - P_n T\| \\ &= \|(I - P_n)(T - L + L)\| \\ &\leq \|(I - P_n)(T - L)\| + \|(I - P_n)L\| \\ &\leq \|T - L\| + \|L - P_n L\|. \end{aligned}$$

Since L is a compact operator, $\lim \|L - P_n L\| = 0$. Thus

$$\lim \|T - P_n T\| = \|T - L\| = \Delta(T).$$

But

$$|T - P_n T| \leq \|T - P_n T\| + \epsilon 2^{-n+1} \|T\|,$$

and so we also have

$$\lim |T - P_n T| = \Delta(T).$$

This shows that the distance from T to $\mathcal{C}(X, Y)$ is $\Delta(T)$ with respect to both the norms $\|\cdot\|$ and $|\cdot|$ on Y . It remains to show that, with respect to $|\cdot|$, this distance is attained only when T is compact.

Let T be a non-compact operator in $\mathfrak{B}(X, Y)$. The proof in Section 3 shows that we can choose a sequence $s \in \mathcal{S}(X)$ such that $\lim \|T(s_n)\| = \Delta(T) > 0$ (in Case 3 of that proof, this assertion is immediate from (3.2); in Case 2 we take $S_n =$

$\text{sgn } (P(t_n))$). If L is any compact operator then $\lim \|L(s_n)\| = 0$. In particular we can choose an n so that $\|(T - L)(s_n)\| > 0$. Then we can choose an m so that

$$A \equiv \epsilon 2^{-m} |((T - L)(s_n), e_m)| > 0.$$

We define

$$\alpha^2 = A^2 + \Delta(T)^2,$$

$$\lambda = A/\alpha,$$

$$\mu = \Delta(T)/\alpha,$$

$$x_k = \lambda s_n + \mu s_k.$$

Since $s \in \mathcal{S}(X)$,

$$\lim_{k \rightarrow \infty} (s_n, s_k) = 0,$$

and therefore $\lim \|x_k\|^2 = \lambda^2 + \mu^2 = 1$. Since $T - L$ is weakly continuous,

$$\lim_{k \rightarrow \infty} ((T - L)(s_k), e_m) = 0 = \lim_{k \rightarrow \infty} ((T - L)(s_k), (T - L)(s_n)).$$

It follows that

$$\lim_{k \rightarrow \infty} |((T - L)(x_k), e_m)| = \lambda |((T - L)(s_n), e_m)|.$$

Further,

$$\begin{aligned} \| (T - L)(x_k) \|^2 &= \lambda^2 \| (T - L)(s_n) \|^2 + \mu^2 \| (T - L)(s_k) \|^2 \\ &\quad + 2\lambda\mu \operatorname{Re} ((T - L)(s_k), (T - L)(s_n)), \end{aligned}$$

which approaches the limit

$$\lambda^2 \| (T - L)(s_n) \|^2 + \mu^2 \Delta(T)^2 > \mu^2 \Delta(T)^2,$$

as $k \rightarrow \infty$. Thus

$$\lim \| (T - L)(x_k) \| > \mu \Delta(T).$$

But

$$\begin{aligned} \limsup_{k \rightarrow \infty} | (T - L)(x_k) | &\geq \limsup_{k \rightarrow \infty} \{ | (T - L)(x_k) | + \epsilon 2^{-m} | ((T - L)(x_k), e_m) | \} \\ &> \mu \Delta(T) + \lambda A = \alpha > \Delta(T), \end{aligned}$$

so that $|T - L| > \alpha > \Delta(T)$.

Q.E.D.

§5. Spaces antiproximal in their second conjugates. We are now going to apply the result of the preceding section to exhibit some Banach spaces whose canonical images in their second conjugate spaces are antiproximal. The interest in such spaces was discussed in the Introduction.

Let X and Y be separable infinite-dimensional Hilbert spaces. We denote the Hilbert norm on both X and Y by $\|\cdot\|$. Let $\{e_1, e_2, \dots\}$ be an orthonormal basis for Y and define $|\cdot|$ to be the equivalent norm on Y given by (4.1). For $T \in \mathfrak{B}(X, Y)$, let $|T| = \sup \{|T(x)| : \|x\| = 1\}$. This norm is equivalent to the original operator norm on $\mathfrak{B}(X, Y)$ given by $\|T\| = \sup \{\|T(x)\| : \|x\| = 1\}$.

Theorem. *The second conjugate of the space $\mathfrak{C}(X, Y)$, normed by $|\cdot|$, can be identified with the space $\mathfrak{B}(X, Y)$, normed by $|\cdot|$. Consequently, $\mathfrak{C}(X, Y)$, under the norm $|\cdot|$, is antiproximinal in its second conjugate.*

Proof. Only the first assertion requires proof, since the second then follows from the theorem of Section 4. The most immediate proof of the first assertion is by way of a recent result of Cc [2, p. 391]. According to this proposition, we have $\mathfrak{C}(X, Y)^{**} = \mathfrak{B}(X, Y)$ whenever X and Y are reflexive and X has a shrinking Schauder basis. Because X and Y are both isomorphic to Hilbert spaces these hypotheses are clearly satisfied in the present situation. Q.E.D.

In order to have this paper reasonably self-contained (in particular to avoid the use of deep results on topological tensor products made by Cc), we provide an alternate proof of the present theorem below. Before giving this proof we wish to observe that an analogous (but easier) argument will establish the following simpler example of a space antiproximinal in its second conjugate.

In the theorem above let us set $X = Y$ and consider only diagonal operators, that is, operators T of the form

$$T = \sum_{n=1}^{\infty} t_n e_n \otimes \bar{e}_n.$$

Now we have essentially the problem of approximating a bounded sequence $t \in \ell^\infty$ by a null sequence, that is, by elements of the space c_0 .

Corollary. *With respect to the norm*

$$|x| = \|x\|_\infty + \epsilon \sum_{n=1}^{\infty} 2^{-n} |x_n|,$$

on ℓ^∞ , c_0 is an antiproximinal subspace of ℓ^∞ and $c_0^{**} = \ell^\infty$.

In this case, $x \in \ell^\infty$ implies $d(x, c_0) = \limsup |x_n|$ with respect to both the norm introduced above and the usual ℓ^∞ norm. However, it is clear that for any $s \in c_0$, $|x - s| = \limsup |x_n|$ only if $x = s$. As indicated above, the formula $c_0^{**} = \ell^\infty$ follows the lines of the subsequent argument, *mutatis mutandis*.

Alternate proof of the Theorem. Consider for a moment that both X and Y are equipped with their original (Hilbert) norms. Then $\mathfrak{C}(X, Y)^*$ can be identified with the space of all nuclear operators from X to Y with the trace norm. If N is such an operator then there is an orthonormal basis $\{b_1, b_2, \dots\}$ for X such that

$$(5.1) \quad N(x) = \sum_{n=1}^{\infty} (x, b_n) N(b_n),$$

and $\sum \|N(b_n)\| < \infty$. The formula

$$(5.2) \quad \Phi_N(L) = \sum_{n=1}^{\infty} (L(b_n), N(b_n))$$

then defines a bounded linear functional on $\mathcal{C}(X, Y)$, with $\|\Phi_N\| = \text{trace } ((N^*N)^{1/2})$. Every element of $\mathcal{C}(X, Y)^*$ is of this form, and every element of $\mathcal{C}(X, Y)^{**}$ is of the form

$$i_T(\Phi_N) = \Phi_N(T),$$

where $T \in \mathcal{B}(X, Y)$ and $\|i_T\| = \|T\|$. These facts are proved in [3] and [6] for the case $X = Y$; the general case can be deduced from this special one by applying it to the Hilbert direct sum of X and Y .

When we renorm $\mathcal{C}(X, Y)$ with the equivalent norm $|\cdot|$, we induce a new norm on $\mathcal{C}(X, Y)^*$ equivalent to the original one:

$$|\Phi_N| = \sup \{ |\Phi_N(L)| : L \in \mathcal{C}(X, Y), |L| = 1 \};$$

$$|\Phi_N| \leq \|\Phi_N\| \leq (1 + \epsilon) |\Phi_N|.$$

This in turn induces a new norm on $\mathcal{C}(X, Y)^{**}$ equivalent to the original one:

$$|i_T| = \sup \{ |i_T(\Phi_N)| : |\Phi_N| = 1 \}.$$

The proof is now reduced to showing that $|i_T| = |T|$.

We first show that $|i_T| \geq |T|$, for any given $T \in \mathcal{B}(X, Y)$. Let $\epsilon > 0$ and choose a unit vector $x \in X$ such that $|T(x)| > |T| - \epsilon$. Then choose $f \in Y^*$ such that $|f| = 1$ and $fT(x) = |T(x)|$. The formula $\Phi(L) = fL(x)$ defines an element of $\mathcal{C}(X, Y)^*$ and clearly $|\Phi| \leq 1$. Hence

$$\begin{aligned} |i_T| &\geq i_T(\Phi) = \Phi(T) = fT(x) \\ &= |T(x)| > |T| - \epsilon. \end{aligned}$$

For the reverse inequality, let Φ_N be any element of $\mathcal{C}(X, Y)^*$ and let $\{b_1, b_2, \dots\}$ be an orthonormal basis for X , chosen so that (5.1) and (5.2) hold. Let P_n be the orthogonal projection of Y onto $\text{span } \{e_1, \dots, e_n\}$. Then $|P_n| = \|P_n\| = 1$. Hence for each integer k and n , $\|P_n N(b_k)\| \leq \|N(b_k)\|$. By the Dominated Convergence Theorem, we then have

$$\begin{aligned} \Phi_N(P_n T) &= \sum_{k=1}^{\infty} (P_n T(b_k), N(b_k)) \\ &= \sum_{k=1}^{\infty} (T(b_k), P_n N(b_k)) \\ &\rightarrow \sum_{k=1}^{\infty} (T(b_k), N(b_k)) = \Phi_N(T), \end{aligned}$$

as $n \rightarrow \infty$, since for each k , $P_n N(b_k) \rightarrow N(b_k)$, as $n \rightarrow \infty$. Now $P_n T \in \mathcal{C}(X, Y)$ and so

$$\begin{aligned} |i_T(\Phi_N)| &= |\Phi_N(T)| \\ &= \lim |\Phi_N(P_n T)| \leq |\Phi_N| |P_n T| \\ &\leq |\Phi_N| |T|. \end{aligned}$$

This shows that $|i_T| \leq |T|$ and completes the proof.

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