Local Isometric Embeddings of Surfaces into Euclidean Four Space

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Communicated by S. S. CHERN

An old unsolved problem in differential geometry concerns the local isometric embedding of surfaces into three dimensional Euclidean space. Let g be a metric defined in a neighborhood of a point $p \in \mathbf{R}^2$ and let E^3 denote $\mathbf{R}^3$ with the usual Euclidean metric. We ask if there exists a possibly smaller neighborhood U of p and a map $f: U \rightarrow E^3$, $f(x_1, x_2) = (f^1(x_1, x_2), f^2(x_1, x_2), f^3(x_1, x_2))$ such that the induced metric on U agrees with the original metric g . That is, we seek to solve the system of equations

$$\sum_{r=1}^3 \frac{\partial f^r}{\partial x_i} \frac{\partial f^r}{\partial x_j} = g_{ij}(x_1, x_2); \quad 1 \leq i \leq j \leq 2,$$

where g_{ij} are the components of the metric g with respect to local coordinates x_1, x_2 . If such a map f exists, the metric g is called *realizable*. There are several reasons for excluding the case where the map f is only in C^1 —any continuous metric can be realized by a C^1 map [1] and very little differential geometry can be done with such embeddings.

It is not known if all metrics are realizable (*i.e.*, smoothly realizable). Affirmative results occur only in special cases; for instance, if the metric is analytic or the Gaussian curvature K of the metric is nonzero at p . Both of these results can be proved using a device introduced by Weingarten [2]. In this note we make the simple observation that this same device yields a proof that all two dimensional surfaces have local isometric embeddings into E^4 . We do this by a reduction to the case of a metric with positive Gaussian curvature. We also provide a new proof that such metrics are realizable.

Pogorelov has recently constructed a $C^{2,1}$ metric with no C^2 realization in E^3 (Soviet Math. Dokl. 12(1971), 729–731). This example may be modified to provide a $C^{3+\alpha}$ metric with no such $C^{2+\beta}$ realization, $1 > \beta > 2\alpha > 0$. This demonstrates the validity of our conjecture, contained in the Remark below, that additional smoothness occurs upon allowing an extra dimension.

Denote by C^{l+a} , l an integer and $0 < a < 1$, the space of functions with derivatives up to order l all of which are Hölder continuous with exponent a .

Theorem. *For each Riemannian metric g of class C^{l+a} , $l \geq 3$ and $0 < a < 1$, there is a neighborhood U and an isometric embedding $f: U \rightarrow E^4$ which is also of class C^{l+a} .*

Remark. A simple modification of the argument in [2] shows that f in general is not any smoother than C^{l+a} . The existence theorem with E^3 as a target space and $K < 0$ only asserts that $f \in C^{l+a-1}$. The author believes this is also best possible. If this is so, then additional smoothness is possible upon allowing an extra dimension.

Let g be the given C^{l+a} metric. The curvature can be written

$$(*) \quad K(x, y) = -\frac{1}{4}\Delta^{-4}D - \frac{1}{2}\Delta^{-1}\{[\Delta^{-1}(g_{11y} - g_{12x})]_y - [\Delta^{-1}(g_{12y} - g_{22x})]_x\}$$

where

$$\Delta = (g_{11}g_{22} - g_{12}^2)^{1/2} \quad \text{and} \quad D \quad \text{is det} \quad \begin{pmatrix} g_{11} & g_{12} & g_{22} \\ g_{11x} & g_{12x} & g_{22x} \\ g_{11y} & g_{12y} & g_{22y} \end{pmatrix}.$$

Let us first seek a function $w(x, y)$ for which $\tilde{g}_{ij}(x, y) = g_{ij}(x, y) - (\partial w / \partial x_i) \cdot (\partial w / \partial x_j)$ is a metric with curvature $\tilde{K}$ positive at p , which we take to be the origin. We claim that this will be the case for any function w with $w_x(0, 0) = w_y(0, 0) = w_{xx}(0, 0) = w_{yy}(0, 0) = 0$ and $w_{xy}(0, 0)$ large. For, using (*) to calculate $\tilde{K}$, we obtain for such a $w(x, y)$

$$\tilde{K}(0, 0) = K(0, 0) + \left[\frac{w_{xy}(0, 0)}{\Delta(0, 0)} \right]^2.$$

So $\tilde{K}(x, y) > 0$ in some neighborhood U of the origin. Since $\tilde{g}(0, 0) = g(0, 0)$ we can also assume that $\tilde{g}$ is a Riemannian metric (*i.e.*, positive definite) in that neighborhood. We might as well take w to be analytic. If one can now find a map $f: U \rightarrow E^3$ such that f induces the metric $\tilde{g}$, then the map $f \oplus w: U \rightarrow E^4$ realizes the metric g .

Since $\tilde{K} > 0$, it is known that $\tilde{g}$ has a isometric embedding in E^3 . One can prove this by now seeking a function $\varpi(x, y)$ such that the metric $\tilde{g}_{ij} - (\partial \varpi / \partial x_i) \cdot (\partial \varpi / \partial x_j)$ has $K = 0$ in a neighborhood of the origin. This idea is due to Weingarten. This involves solving a nonlinear elliptic equation, see [3] where however the embedding is only shown to be in $C^{l+a-\epsilon}$. Note that in [3] the proof and possibly the statement is incorrect for $K < 0$, [4]. One can of course also obtain the local result for positive curvature by appealing to the powerful global theorem of [5]. Here we shall sketch a new proof of this simple result with stronger differentiability results than [3] and weaker assumptions than [5]. The formal similarity with [6] should be noted. Nash assumes that the target space has large dimension in order to be able to invert the appropriate linearized

equations. We do not need these extra dimensions; the condition $K > 0$ is used to guarantee invertibility.

We first show that if $f_0 : U \rightarrow E^3$ is an analytic map inducing a metric g_0 with positive Gaussian curvature, then any metric close to g_0 is also realizable. Choose local coordinates x_1, x_2 such that U is a neighborhood of the origin and f_0 has the form $f_0(x_1, x_2) = (x_1, x_2, z(x_1, x_2)) \in \mathbb{R}^3$. Here and subsequently we may have to replace U by a smaller neighborhood. Let $F(f) = \sum_{i=1}^3 (\partial f'' / \partial x_i) \cdot (\partial f'' / \partial x_i)$ be the metric induced by a map $f: U \rightarrow E^3$. To solve $F(f) = g$ given $F(f_0) = g_0$ and g close to g_0 , it is natural to consider the operator $dF(f_0)u \equiv \sum_{i=1}^3 (\partial f_0'' / \partial x_i) (\partial u'' / \partial x_i) + (\partial f_0'' / \partial x_i) (\partial u'' / \partial x_i)$. Let us write $x = x_1, y = x_2$. From the special form of $f_0(x, y)$ we have

$$(1) \quad dF(f_0)u = \begin{pmatrix} 2\left(\frac{\partial u^1}{\partial x} + z_x \frac{\partial u^3}{\partial x}\right) \\ \frac{\partial u^1}{\partial y} + \frac{\partial u^2}{\partial x} + z_x \frac{\partial u^3}{\partial y} + z_y \frac{\partial u^3}{\partial x} \\ 2\left(\frac{\partial u^2}{\partial y} + z_y \frac{\partial u^3}{\partial y}\right) \end{pmatrix}.$$

We are using the notation $u: U \rightarrow E^3$ is given by $u = (u^1(x, y), u^2(x, y), u^3(x, y))$. A metric h will similarly be written $h = (h^1(x, y), h^2(x, y), h^3(x, y))$ ($= (E, F, G)$ in classical notation).

Lemma. *If g_0 has positive curvature, then for any metric $h \in C^{l+a}(U)$, $l \geq 3$ and $0 < a < 1$, $dF(f_0)u = h$ has a solution $u: U \rightarrow E^3$ which satisfies $u \in C^{l+a}(U)$ and*

$$(2) \quad |u^3|_{l+a} \leq c \left| \frac{1}{2}(h_{yy}^1 + h_{xx}^3) - h_{xy}^2 \right|_{l+a-2}.$$

The constant c depends on U and $z(x, y)$; if U is a ball of radius r centered at the origin, and if r is small, then c is independent of r .

Proof. Set (1) equal to

$$\begin{pmatrix} h^1 \\ h^2 \\ h^3 \end{pmatrix}.$$

Upon differentiating the first equation twice with respect to y , the third twice with respect to x , and the middle one with respect to x and then y , and adding suitable multiples of the new equations, we obtain the following equation for u^3

$$(3) \quad z_{yy} u_{xx}^3 - 2z_{xy} u_{xy}^3 + z_{xx} u_{yy}^3 = -\frac{1}{2}(h_{yy}^1 + h_{xx}^3) + h_{xy}^2.$$

The fact that $K > 0$ in U for the metric g_0 means that $z_{yy} z_{xx} - z_{xy}^2 > 0$, i.e., the equation is elliptic. We solve this with boundary condition $u^3 = 0$ on ∂U .

Thus, by the Schauder estimates, $|u^3|_{l+a}$ can be estimated by (2). It is easy to see that c can be taken independent of r for r small. For if r is small enough, then one is really considering the case of constant coefficients and here a change of scale allows one to just treat the unit sphere. The constant c is then some large multiple of the constant which occurs for $r = 1$; the size of the multiplication factor depends on the values of the derivatives of the coefficients of the operator near the origin.

Using the uniquely determined u^3 , we can integrate the first and third equations of (1)

$$(4) \quad \begin{aligned} u^1(x, y) &= \int_0^x \left\{ \frac{h^1}{2}(\tau, y) - z_x(\tau, y)u_x^3(\tau, y) \right\} d\tau + H(y) \\ u^2(x, y) &= \int_0^y \left\{ \frac{h^3}{2}(x, \tau) - z_y(x, \tau)u_y^3(x, \tau) \right\} d\tau + G(x) \end{aligned}$$

with

$$\begin{aligned} -H(y) &= \int_0^y \left\{ \int_0^{y'} \left(\frac{h^3}{2}(0, \tau) - z_y(0, \tau)u_y^3(0, \tau) \right) d\tau \right. \\ &\quad + z_x(0, y')u_x^3(0, y') + z_y(0, y')u_y^3(0, y') - h^2(0, y') \\ &\quad \left. - \frac{1}{2}(z_x(0, 0)u_y^3(0, 0) + z_y(0, 0)u_x^3(0, 0) - h^2(0, 0)) \right\} dy' \quad \text{and} \\ -G(x) &= \int_0^x \left\{ \int_0^{x'} \left(\frac{h^1}{2}(\tau, 0) - z_x(\tau, 0)u_x^3(\tau, 0) \right) d\tau \right. \\ &\quad + z_x(x', 0)u_y^3(x', 0) + z_y(x', 0)u_x^3(x', 0) - h^2(x', 0) \\ &\quad \left. - \frac{1}{2}(z_x(0, 0)u_y^3(0, 0) + z_y(0, 0)u_x^3(0, 0) - h^2(0, 0)) \right\} dx' \end{aligned}$$

A calculation shows that indeed u^1, u^2, u^3 satisfy $dF(f_0)u = h$, provided $u^3 \in C^3$. We do this by setting $s(x, y) = (\partial u^1/\partial y) + (\partial u^2/\partial x) + z_x(\partial u^3/\partial y) + z_y(\partial u^3/\partial x) - h^2$ and showing $s_{xy} = 0$, $s(x, 0) = s(0, y) = 0$.

Now let U be a small ball so that the constant in (2) is independent of the size of U . Again take $l \geq 3$, $0 < a < 1$.

Lemma. *There exists some $\nu > 0$ depending only on $z(x, y)$ such that if $g \in C^{l+a}$ and ϵ satisfies $0 < \epsilon < a$ then $|g - g_0|_{C^{l+a-\epsilon}(U)} < \nu$ implies the existence of some $u \in C^{l+a}(U)$ with $F(u) = g$.*

Proof. Set $u_0 = 0$ and define u_n by

$$dF(f_0)u_n = g - g_0 - F(u_{n-1})$$

where u_n is that solution given by the previous lemma. h^1 is now the first component of $g - g_0 - F(u_{n-1})$, etc. By induction one can obtain the estimates

$$|u_n^3|_{l+a} \leq L_1 ; \quad |u_n^i|_{l+a} \leq L_1 L_2 , \quad i = 1, 2.$$

$$\sum_{i=1}^3 |u_n^i|_{l+a-\epsilon} \leq \delta,$$

$$\sum_{i=1}^3 |u_n^i - u_{n-1}^i|_{l+a-\epsilon} \leq \frac{1}{2} \sum_{i=1}^3 |u_{n-1}^i - u_{n-2}^i|_{l+a-\epsilon}$$

for some constants δ , L_1 , L_2 provided $|g - g_0|_{l+a-\epsilon}$ is sufficiently small. The constant L_1 depends on $|g - g_0|_{l+a}$. To indicate how the reasoning proceeds, let us assume the estimates for n and obtain for example the estimate for $|u_{n+1}^i - u_n^i|_{l+a-\epsilon}$. Let $w = u_{n+1}^3 - u_n^3$ be the unique solution vanishing on the boundary of U of

$$\begin{aligned} z_{\nu\nu} w_{xx} - 2z_{x\nu} w_{x\nu} + z_{xx} w_{\nu\nu} &= -\frac{1}{2}((F^1(u_n) - F^1(u_{n-1}))_{\nu\nu} + (F^3(u_n) - F^3(u_{n-1}))_{xx}) \\ &\quad + (F^2(u_n) - F^2(u_{n-1}))_{x\nu} \\ &= -\sum_{i=1}^3 \{ (u_{n,x\nu}^i)^2 - u_{n,x}^i u_{n,\nu\nu}^i - (u_{n-1,x\nu}^i)^2 + u_{n-1,x}^i u_{n-1,\nu\nu}^i \}. \end{aligned}$$

Hence by the Schauder estimates

$$|w|_{l+a-\epsilon} \leq C_1 \sum_i |u_{n,x\nu}^i (u_{n,x\nu}^i - u_{n-1,x\nu}^i)|_{l+a-\epsilon-2} + \text{similar terms}.$$

Thus

$$|w|_{l+a-\epsilon} \leq C_2 (|u_n|_{l+a-\epsilon} + |u_{n-1}|_{l+a-\epsilon}) |u_n - u_{n-1}|_{l+a-\epsilon},$$

and by the inductive assumption

$$|u_{n+1}^3 - u_n^3|_{l+a-\epsilon} \leq 2C_2 \delta \sum_i |u_n^i - u_{n-1}^i|_{l+a-\epsilon}.$$

We now go to (4)

$$\begin{aligned} u_{n+1}^1 - u_n^1 &= \int_0^x (\frac{1}{2} \sum_i (u_{n,x}^i u_{n,x}^i - u_{n-1,x}^i u_{n-1,x}^i) - z_x (u_{n+1,x}^3 - u_n^3)) d\tau \\ &\quad + H_{n+1}(y) - H_n(y). \end{aligned}$$

It is clear what we mean by H_n and H_{n+1} . Also

$$\begin{aligned} |H_{n+1} - H_n|_{l+a-\epsilon} &\leq C_3 (|u_n|_{l+a-\epsilon} + |u_{n-1}|_{l+a-\epsilon}) |u_n - u_{n-1}|_{l+a-\epsilon} \\ &\quad + |u_{n+1}^2 - u_n^2|_{l+a-\epsilon}. \end{aligned}$$

Now $\|u_{n+1}^1 - u_n^1\|_{l+a-\epsilon}$ is less than the same right hand side where we use $\|\cdots\|$ to mean only l^h order derivatives containing at least one differentiation with respect to x enter into the norm. However if we perform an integration by parts to write

$$\frac{\partial^2(u_{n+1}^1 - u_n^1)}{\partial y^2} = (u_{n,x}(t, y) u_{n,\nu\nu}(t, y) - u_{n-1,x}(t, y) u_{n-1,\nu\nu}(t, y))|_{t=0}^t$$

$$\begin{aligned}
& - \int (u_{nxx} u_{nvv} - u_{n-1xx} u_{n-1vv}) \\
& + \int (u_{nxv} u_{nxv} - u_{n-1xv} u_{n-1xv}) \\
& + \text{similar terms}
\end{aligned}$$

(summation over suppressed i understood), then we see that the estimate

$$\begin{aligned}
& |u_{n+1}^1 - u_n^1|_{l+a-\epsilon} \\
& \leq C_3(|u_n|_{l+a-\epsilon} + |u_{n-1}|_{l+a-\epsilon}) |u_n - u_{n-1}|_{l+a-\epsilon} + |u_{n+1}^3 - u_n^3|_{l+a-\epsilon}
\end{aligned}$$

indeed holds.

For δ small enough this implies that

$$\sum_{i=1}^3 |u_{n+1}^i - u_n^i|_{l+a-\epsilon} \leq \frac{1}{2} \sum_{i=1}^3 |u_n^i - u_{n-1}^i|_{l+a-\epsilon}.$$

The other estimates are handled similarly.

Thus for $|g - g_0|_{l+a-\epsilon}$ small, the sequence $\{u_n\}$ converges in $C^{l+a-\epsilon}$ to some u . Further $u \in C^{l+a}$. Since $F(f_0 + u) = \lim F(f_0 + u_n) = \lim (F(f_0) + dF(f_0)u_n + F(u_n)) = \lim (g_0 + (g - g_0 - F(u_{n-1})) + F(u_n)) = g$, the lemma is established.

Theorem. Every C^{l+a} metric, $l \geq 3$ and $0 < a < 1$, with positive Gaussian curvature, is locally realizable by a C^{l+a} map into E^3 .

Proof. Given such a metric g we define g_0 by the truncated Taylor series containing terms up to and including order l . Any analytic metric can be realized. This is a consequence of the Cauchy-Kowalewski theorem. Take U to be a ball of radius r . $|g - g_0|_{l+a-\epsilon} \rightarrow 0$ as r goes to zero. Since the constants of the previous lemmas do not depend on r for r small, the theorem follows immediately.

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This research was supported by a grant from the National Science Foundation (GP-7952X2) made to the Institute for Advanced Study.

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Date communicated: APRIL 29, 1971