The Sharp Maximum Principle for Nonlinear Inequalities

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Let $u(x)$ satisfy a second-order differential inequality with semidefinite quadratic form in a given region Ω . For a given point $P \in \Omega$, it is required to construct a set $S(P, \Omega)$ relative to which the maximum principle holds in the following sense: If $u(P) = \sup u(x)$ for $x \in S(P, \Omega)$, then necessarily $u(x) = u(P)$ for $x \in \tilde{S}(P, \Omega)$. Description of a suitable set $S(P, \Omega)$ and statement of the above fact about it constitute the *sharp maximum principle*. When $\tilde{S}(P, \Omega) = \Omega$ the sharp maximum principle reduces to the strong maximum principle of Bernstein-Hopf, and a different set $S(P, \Omega)$ gives the maximum principle of Nirenberg. The formulation above, which is vastly more general than either of these, was recently introduced by Hill. Hill's characterization of the set $S(P, \Omega)$ is retained here. The objective of this paper is to prove Hill's theorem (his proof is not quite complete) and at the same time extend it to broad classes of nonlinear inequalities. The analysis has many applications, some of which are indicated in the accompanying references. However, no applications are presented here.

1. The propagation set of Bony and Hill. Throughout this paper we assume

$$\alpha_{ij}(x) \in C^1, \quad \beta_i(x) \in C^1, \quad u(x) \in D^2, \quad v(x) \in D^2, \quad c(x) \in D^2$$

in a given region Ω of real Euclidean space E_n , though these assumptions could be weakened. All functions are real. A summation convention is used in which, unless otherwise stated, the indices i, j run from 1 to n while k and l run from 1 to m . Associated with α_{ij} and β_i are vector fields

$$X_k(x) = (\alpha_{ik}(x)), \quad X_0(x) = (\beta_i(x))$$

and differential operators

$$D_k = \alpha_{ik}(x) \frac{\partial}{\partial x_i}, \quad D_0 = \beta_i(x) \frac{\partial}{\partial x_i}.$$

According to Hill [2] the *diffusion trajectories* are curves C_k satisfying

$$\frac{dx}{dt} = X_k(x), \quad x(t) \in \Omega, \quad x'(t) \neq 0$$

on a given interval $t_1 \leq t \leq t_2$, and these curves can be traversed in either direction. The *drift trajectories* are defined similarly, with X_0 instead of X_k , but are oriented in the direction of increasing t . Hill's *propagation set* $S(P, \Omega)$ consists of P itself together with all points that can be joined to P by a finite number of diffusion and drift trajectories, the latter being traversed in the prescribed direction. The closure $\bar{S}(P, \Omega)$ is relative to Ω .

A somewhat different view of the propagation set is obtained when Hill's paper is read in conjunction with a related paper of Bony [1]. Besides the fields X_k , Bony introduces the Lie algebra

$$\mathfrak{L}(X_k) = \mathfrak{L}(X_1, X_2, \dots, X_m).$$

He shows in the C^∞ case that if $X \in \mathfrak{L}(X_k)$ then the trajectories of $dx/dt = X(x)$ can be uniformly approximated by Hill's diffusion trajectories. Hence, $\bar{S}(P, \Omega)$ contains all trajectories belonging to the Lie algebra. Bony's proof goes through in the C^1 case, provided iteration of the Poisson brackets is not carried further than the differentiability allows.

The reason for calling $S(P, \Omega)$ or $\bar{S}(P, \Omega)$ the *propagation set* is given by the following theorem:

Theorem 1. (Bony and Hill). *Let $u(P) = 0$ at a point $P \in \Omega$, let $u(x) \leq 0$ on $S(P, \Omega)$, and let*

$$D_k D_k u + D_0 u \geq 0, \quad x \in S(P, \Omega).$$

If the Lie algebra associated with (X_k) has constant rank throughout Ω , then $u(x) = 0$ for $x \in \bar{S}(P, \Omega)$.

The theorem as stated by Hill omits the hypothesis on $\mathfrak{L}(X_k)$ but his proof is not quite complete.

2. Purpose of this paper. Our first objective is to show that Hill's theorem is correct, under the above conditions on α_{ik} and β_i ; in other words, the assumption about the rank of $\mathfrak{L}(X_k)$ really is superfluous. Our second objective is to generalize the result to broad classes of quasilinear inequalities.

We use A_{kl} to denote a symmetric m by m matrix function of the three variables x , u and p , where x is an n vector, u a scalar, and p an m vector. It is convenient to write

$$pAp = A_{kl}p_kp_l, \quad \|A\|^2 = A_{kl}A_{kl}, \quad \|A\| \geq 0$$

and to consider $A(x, u, Du)$ where

$$Du = (D_1u, D_2u, \dots, D_mu).$$

Similarly, $B(x, u, Du)$ is a numerical function of the indicated arguments, and likewise for $C(x, u, Du)$. These three functions are subject locally to the following

restrictions: For each $x_0 \in \Omega$ there is a positive constant $\epsilon = \epsilon(x_0)$ such that the inequalities

$$|x - x_0| < \epsilon, \quad -\epsilon < u < 0, \quad |p| < \epsilon$$

together imply

$$A(x, u, p) \geq 0, \quad pA(x, u, p)p \geq \epsilon |p|^2, \quad ||A(x, u, p)|| \leq \frac{1}{\epsilon}$$

$$\epsilon \leq B(x, u, p) \leq \frac{1}{\epsilon}, \quad C(x, u, p) \leq \frac{1}{\epsilon} (|u| + |p|).$$

The above conditions would follow from

$$A(x, 0, 0) > 0, \quad B(x, 0, 0) > 0, \quad C(x, 0, 0) \leq 0, \quad x \in \Omega$$

provided A and B are continuous and C is Lipschitzian. However they can also hold without any assumption of continuity.

The hypothesis involving $pA(x, u, p)p$ goes back to early work of Bernstein and has recently been revived by Serrin [3]. Since A can be discontinuous, this hypothesis is much more general than the condition

$$pA(x, u, q)p \geq \epsilon |p|^2.$$

For instance, our results apply to

$$(D_1 u)^2 D_1^2 u + 2(D_1 u)(D_2 u)D_1 D_2 u + (D_2 u)^2 D_2^2 u + |Du|^3 \geq 0$$

although the associated quadratic form is only semidefinite.

We shall establish the following:

Theorem 2. *Let $u(P) = 0$ at a point $P \in \Omega$, let $u(x) \leq 0$ on $S(P, \Omega)$, and let*

$$A_{ki}(x, u, Du)D_k D_i u + B(x, u, Du)D_0 u + C(x, u, Du) \geq 0$$

on $\bar{S}(P, \Omega)$. Then $u(x) = 0$ on $\bar{S}(P, \Omega)$.

3. Reformulation. If the differential inequality is multiplied through by ϵ the result is an inequality in which A and B satisfy the same sort of hypothesis as before (with a different ϵ) but C satisfies the corresponding hypothesis with $\epsilon = 1$. Since all arguments used in the proof are local, and since the above modification can be done in the neighborhood of any given x_0 , we can therefore assume that

$$C(x, u, Du) = |Du| + |u|.$$

Further simplification is achieved by setting

$$A(x, p) = A(x, u(x), p), \quad B(x) = B(x, u(x), Du(x)).$$

The differential inequality is thus reduced to

$$(1) \quad Tu + |Du| + |u| \geq 0$$

where

$$Tu = A_{kl}(x, Du)D_k D_l u + B(x)D_0 u.$$

It should be observed that we have a usable hypothesis on u , A and B only at points x satisfying

$$|x - x_0| < \epsilon, \quad -\epsilon < u(x) < 0, \quad |Du(x)| < \epsilon, \quad x \in \bar{S}(P, \Omega)$$

where ϵ is sufficiently small. These are referred to as the *side conditions*. As seen above, the side conditions contain a major part of the nonlinearity. When they hold, the coefficients A and B satisfy

$$A(x, p) \geq 0, \quad pA(x, p)p \geq \epsilon |p|^2, \quad ||A(x, p)|| \leq \frac{1}{\epsilon}, \quad \epsilon \leq B(x) \leq \frac{1}{\epsilon}$$

where $p = Du$. These are referred to as the ϵ hypothesis for T .

The differential inequality will be used only at points where the side conditions hold. In particular, A will be bounded, and hence a term

$$A_{kl}(x, u, Du)[D_k, D_l]u$$

could be incorporated into the term $|Du|$. This shows that the assumption that A is symmetric involves no loss of generality.

4. Lemmas of Bony and Hill. Since Bony considers $\sup u$ relative to the whole region Ω , his results do not quite contain Theorem 1, or even the familiar maximum principle for parabolic inequalities. The following lemma is used by Hill to treat the case of a maximum which is only relative to $S(P, \Omega)$, and will be used in the same way here:

Lemma 1. (Hill). *If $P^* \in \bar{S}(P, \Omega)$, then $\bar{S}(P^*, \Omega) \subset \bar{S}(P, \Omega)$.*

We also use a special case of Bony's result concerning the relation of Hill's trajectories to the Lie algebra, which was already mentioned above. Bony actually assumes C^∞ instead of Lip 1, but his proof applies to the general case:

Lemma 2. (Bony). *Let $X_1, X_2, \dots, X_m$ be vector fields belonging to Lip 1 and let $X = \lambda_k X_k$ where λ_k are constant. Then each integral curve associated with X can be approximated uniformly by piecewise differentiable curves, of which each differentiable arc is an integral curve associated with one of the fields X_k .*

In our use of Lemma 2 we want the differentiable arcs of the approximating curve, Γ^* , to be diffusion trajectories in the sense of Hill, and hence, the condition $X^*(x) \neq 0$ is needed. To justify this added restriction, let one of the differentiable arcs of Γ^* be

$$\frac{dx}{dt} = X_k(x), \quad t_1 \leq t \leq t_2.$$

If $X_k[x(t_0)] = 0$ for some t_0 , $t_1 \leq t_0 \leq t_2$, then $x(t) = x(t_0)$ on this interval by uniqueness, and hence the trace of Γ^* is unchanged when this arc is dropped.

Therefore $dx/dt \neq 0$ can be assumed with no loss of generality. The trajectories of Lemma 2 are called the *generalized diffusion trajectories*.

The next lemma replaces the analysis on pages 225–226 of [2], which is not entirely convincing, because the constant C of the argument might depend on t . However, a lemma of Bony, also of interest in its own right, supplies what is needed.

To state Bony's lemma let F be any closed subset of Ω . According to Bony, a vector ν is *normal* to F at one of its points, P_0 , if there exists an open ball contained in $\Omega - F$, centered at P_1 , such that P_0 is on the boundary of this ball, and such that $\nu = s(P_1 - P_0)$ with $s > 0$.

Lemma 3. (Bony). *Let Ω be an open set in E_n and let F be a closed subset of Ω . Let $x(t)$ satisfy $dx/dt = X(x)$ where $X(x) \in \text{Lip } 1$. Suppose the vector field X has the following special property: For every point P_0 of F and every vector ν normal to F at P_0 the scalar product satisfies*

$$(2) \quad \nu \cdot X(P_0) \leq 0.$$

Then if $x(t_0) \in F$, it follows that $x(t) \in F$ for $t \geq t_0$.

As is clear from Bony's analysis, the condition is considered to be vacuously fulfilled if there is no normal ν at P_0 . In other words, the hypothesis is required only for each P_0 which possesses a normal in the sense of the previous definition.

If equality holds in (2), for each P_0 , we can replace t by $-t$ and deduce that $x(t) \in F$ for all t , not just for $t \geq t_0$. This is Bony's Theorem 2.1. (The more general form is given on page 284.) Both forms are needed here, and will be used in the same way as in [1].

Another lemma needed here is Hill's Lemma 4. Unfortunately, as noted in [2], the proof is not quite complete. The difficulty is that R must be chosen smaller than the radius of the Hopf sphere, to satisfy the geometric requirements at the beginning of the argument, but it must be chosen large to satisfy the differential inequality at the end of the argument. A proof of Hill's Lemma 4 is contained in Sections 7–11 below. This proof also gives Theorem 3.2 of Bony without any assumption concerning the rank of $\mathcal{L}(X_k)$.

5. Conditions at a maximum. The following lemma generalizes the familiar fact that the gradient is 0 and the matrix of second derivatives is nonpositive at an interior maximum:

Lemma 4. *Let v have a local maximum relative to $\bar{S}(P, \Omega)$ at a point $P^* \in \bar{S}(P, \Omega)$. Then, at P^* ,*

$$D_0 v \leq 0, \quad D_k v = 0, \quad (D_i D_k v) \leq 0.$$

By Lemma 1, the point P^* gives a maximum relative to $\bar{S}(P^*, \Omega)$. Let Γ be any curve satisfying

$$\frac{dx_i}{dt} = \lambda_k \alpha_{ik}(x), \quad x(0) = P^*$$

where λ_k are constant. If $v \leq m$ in $\bar{S}(P^*, \Omega)$, then Lemma 2 shows $v \leq m$ on this curve Γ and hence, the function

$$\varphi(t) = v[x_1(t), x_2(t), \dots, x_n(t)]$$

has a local maximum at $t = 0$. Thus $\varphi'(0) = 0$, $\varphi''(0) \leq 0$. Clearly

$$\varphi'(t) = \lambda_k u_i \alpha_{ik} = \lambda_k D_k v, \quad \varphi''(t) = (\lambda_i D_i) \lambda_k D_k v.$$

Since λ_k is arbitrary, we conclude that $D_k v = 0$ and $(D_i D_k v) \leq 0$ at P^* . The fact that $D_0 v \leq 0$ at P^* follows from $\varphi'(0) \leq 0$, where φ is as above but where $x(t)$ is now associated with D_0 .

The special case of Lemma 4 in which $(D_i D_k)$ is diagonal is due to Hill. This case does not require Lemma 2.

Lemma 4 indicates that many familiar results for differential inequalities continue to hold when $\partial/\partial x_k$ are replaced by D_k and when the maximum is only relative to $S(P, \Omega)$ rather than Ω . For instance one can get a new form of the weak maximum principle which, in turn, leads to a new class of operators monotone in the sense of Collatz. The monotony is relative to $S(P, \Omega)$.

Lemma 4 can also be used in conjunction with Theorem 2 to increase the scope of the latter. Suppose, for example, that $\bar{S}(P, \Omega)$ contains a point x at which $C(x, 0, 0) < 0$. Then it follows that $u(P) < 0$. Indeed, in the contrary case $u(x) = 0$ by Theorem 2, and Lemma 4 leads to a contradiction.

For purposes of this paper the main use of Lemma 4 is to establish the following:

Lemma 5. *Let $\sup u = 0$ in $\bar{S}(P, \Omega)$, let ρ and σ be positive constants, and suppose the function*

$$v(x) = u(x) + \rho e^{\sigma u(x)}$$

attains its maximum relative to $\bar{S}(P, \Omega)$ at a point P^ where $u(P^*) < 0$. Then if ρ is sufficiently small we have*

$$Tu + |Du| + |u| \leq T(-\omega c) - \omega \sigma \epsilon |Dc|^2 + \omega |Dc| + \frac{\omega}{\sigma}$$

at P^ , where $\omega = \rho \sigma \epsilon^\sigma$ and where ϵ is the ϵ for which*

$$pA(x, p)p \geq \epsilon |p|^2,$$

both evaluated at P^ .*

Before giving the proof we comment on the way in which this result will be used. In connection with the diffusion trajectories the primary term comes from $pA(x, p)p$ and the other terms are estimated, but for the drift trajectories the primary term comes from $B(x)D_0 u$. In either case the estimated terms are small throughout a neighborhood of P^* . As a consequence, it turns out that the hypothesis of Theorem 2 is needed only in $S(P, \Omega)$ rather than $\bar{S}(P, \Omega)$, provided $A(x, p)$ is continuous for $p \neq 0$. Continuity for $p \neq 0$ is verified in

most cases of interest; the main reason for allowing discontinuity is to allow it at $p = 0$, as in the example preceding Theorem 2. Some remarks in the sequel pertain to this refinement of Theorem 2, though the subject is not emphasized.

Proof of Lemma 5. Lemma 4 gives

$$D_k u = -\omega D_k c, \quad D_0 u \leq -\omega D_0 c$$

where $\omega = \rho \sigma e^{\sigma c}$ evaluated at P^* , and also

$$(D_l D_k u) \leq -(D_l D_k \rho e^{\sigma c}) = -\omega (D_l D_k c) - \sigma \omega [(D_l c)(D_k c)].$$

Furthermore, since $\sup u = 0$ it follows that $\sup v > 0$ and hence $v(P^*) > 0$. Together with $u \leq 0$ this gives

$$0 < v \leq \rho e^{\sigma c}$$

at P^* , and subtracting $\rho e^{\sigma c}$ from both sides gives an estimate for u . Thus we get

$$(3) \quad |Du| \leq \omega |Dc|, \quad D_0 u \leq -\omega D_0 c, \quad -\frac{\omega}{\sigma} \leq u \leq 0$$

at P^* . If ρ is sufficiently small we can assume $|Du| < \epsilon$, so that the ϵ -hypothesis for T gives

$$A(x, p) \geq 0, \quad pA(x, p)p \geq \epsilon |p|^2, \quad B(x) \geq 0$$

with $p = Du = -\omega Dc$. Since $A(x, p)$ is symmetric we can apply the lemma of Schur and Fejér, even though $(D_l D_k u)$ may not be symmetric. The result is

$$(4) \quad A_{lk}(x, p) D_l D_k u \leq -\omega A_{lk}(x, p) D_l D_k c - \sigma \omega A_{lk}(x, p) (D_l c)(D_k c)$$

where $x = P^*$ and p is as above. The choice $p = -\omega Dc$ in $pA(x, p)p$ gives

$$(5) \quad (D_l c) A_{lk}(x, p) (D_k c) \geq \epsilon |Dc|^2$$

after division by ω^2 , and the fact that $B(x) \geq 0$ gives also

$$(6) \quad B(x) D_0 u \leq -\omega B(x) D_0 c.$$

Upon computing Tu and $T(-\omega c)$ from the definition of T , we see that (4), (5) and (6) together give

$$Tu \leq T(-\omega c) - \sigma \omega \epsilon |Dc|^2$$

at P^* . Since (3) gives also

$$|Du| + |u| \leq \omega |Dc| + \frac{\omega}{\sigma},$$

Lemma 5 follows by addition.

6. Lemmas of Hopf type. We shall establish:

Lemma 6. *Let u satisfy the hypothesis of the theorem, and let H denote a Hopf sphere relative to the set $\tilde{S}(P, \Omega)$. That is, let $u < 0$ in $H \cap \tilde{S}(P, \Omega)$ but $u = 0$ at some point $x_0 \in \partial H \cap \tilde{S}(P, \Omega)$. Then the unit normal ν to H at x_0 is perpendicular to each of the vectors $X_k = (\alpha_{ik})$ at x_0 .*

Proof. As usual, the radius r_0 of H can be diminished, if necessary, so that x_0 is the only point of $\partial H \cap \tilde{S}(P, \Omega)$ at which $u = 0$. Let

$$c(x) = -\frac{|x - \bar{x}_0|^2}{2r_0}, \quad \bar{x}_0 = \text{center},$$

so that $c'(x_0) = \nu$, the unit normal to H at x_0 . If $|Dc| \neq 0$ at x_0 then by continuity there is a sphere $H_\epsilon : |x - x_0| < \epsilon$ in which $\inf |Dc| \geq \eta > 0$. We can suppose also that the ϵ hypothesis for T at the point x_0 holds throughout H_ϵ . Since $u = 0$, $Du = 0$, $D_0u \leq 0$ at x_0 , it can be assumed, finally, that

$$-\epsilon < u < 0, \quad |Du| < \epsilon, \quad D_0u < \epsilon$$

holds at points of $S(P, \Omega)$ that are in H_ϵ . Hence, at such points the differential inequality for u is available.

Let ρ and σ be positive constants. If ρ is sufficiently small (the smallness depending on σ) then the function $v(x)$ considered in Lemma 5 has a maximum relative to $\tilde{S}(P, \Omega)$ at some point P^* in $H_\epsilon \cap H$. This follows as in the familiar argument of Hopf. The boundedness of A and B then gives

$$|T(-\omega c)| \leq M_0 \omega, \quad M_0 = \text{constant}, \quad x \in H_\epsilon$$

and hence, by Lemma 5,

$$Tu + |Du| + |u| \leq \left(M_0 + |Dc| + \frac{1}{\sigma} - \sigma \epsilon \eta^2 \right) \omega.$$

We first choose σ so large that this expression is negative throughout H_ϵ , and then choose ρ so v attains an interior maximum as explained above. The resulting contradiction shows that the hypothesis $|Dc| > 0$ at x_0 was not tenable. Hence $D_k c = 0$ at x_0 for each k , and since $c'(x_0) = \nu$ we get

$$\alpha_{ik}(x_0)\nu_i = 0$$

as desired. This completes the proof.

Lemma 6 generalizes the familiar maximum principle of Bernstein-Hopf and Nirenberg. It also generalizes Lemma 3 of Hill and Proposition 3.1 of Bony.

As in [1] we can combine Lemma 6 with Lemma 3, taking F to be the subset of $\tilde{S}(P, \Omega)$ in which $u = 0$. The result is the following:

Lemma 7. *Let u satisfy the hypothesis of the theorem, let $u = 0$ at a point P^**

of $\bar{S}(P, \Omega)$ and let Γ be a curve satisfying

$$\frac{dx_i}{dt} = \lambda_k \alpha_{ik}(x), \quad x(t_0) = P^*$$

where λ_k are constant. Then $u = 0$ on Γ ; in other words, the relative maximum propagates along the generalized diffusion trajectories.

Lemma 7 remains valid if the condition $B(x, u, p) \geq \epsilon$ in our initial hypothesis is replaced by $B(x, u, p) \geq 0$. However the main difficulty in the proof consists in exploitation of the condition $B(x, u, p) \geq \epsilon$, as described next.

7. Geometric interpretation. If the conclusion of Theorem 2 fails there will be a Hopf sphere, H , relative to the set $\bar{S}(P, \Omega)$; see the small shaded sphere in Figure 1. By choice of coordinates the Hopf point can be assumed at the origin, so that

$$0 \in \bar{S}(P, \Omega), \quad 0 \in \partial H, \quad u(0) = 0$$

but $u < 0$ at other points of $\bar{S}(P, \Omega)$ that are in H . Lemma 6 shows that the diffusion trajectories are tangent to H at $x = 0$, and by choice of coordinates they can be assumed perpendicular to the x_n axis there. The points of $\bar{S}(P, \Omega)$ where $u(x) = 0$ propagate along the generalized diffusion trajectories, and hence the points where $u < 0$ also propagate along them. Thus we have $u(x) < 0$ not only in $H \cap \bar{S}(P, \Omega)$, but on every generalized diffusion trajectory belonging to $\bar{S}(P, \Omega)$ that meets this set.

The dotted curve in Figure 1 shows the projection of such a trajectory on the plane of the paper. It is assumed that the corresponding vector field X is tangent to the plane of the paper at $x = 0$, as well as perpendicular to the x_n axis there. Hence, the trajectory is well approximated by its projection.

The solid curve in Figure 1 shows the intersection of a certain quadric surface $c(x) = 0$ with the plane of the paper. This has a slightly larger curvature than

FIGURE 1

has the dotted curve, and hence one can surmise that the dotted curves would fill up the bottom of the solid curve as shown in Figure 2. These considerations form the geometric basis for the following analysis, insofar as the latter pertains to the space spanned by the fields X_k at $x = 0$.

For points in the orthogonal complement of this space, the diffusion trajectories are not available as a means of propagating the set $\{x \mid u(x) < 0\}$ and a different procedure is needed. In Figure 3, the dot represents the drift trajectory, which appears only in cross section, since it is approximately perpendicular to the plane of the paper. The desired relationship is achieved by increasing the curvature of the solid curve, which is the cross section of the boundary of the quadric surface. It turns out that if this curvature is large enough, the generalized diffusion trajectories again meet the Hopf sphere, so that the set $\{x \mid u(x) < 0\}$ is enlarged as before. Then net result is that $u(x) < 0$ at all points of $\bar{S}(P, \Omega)$ that are near the bottom of the quadric surface, and a contradiction is obtained as in the familiar argument of Hopf; cf. Figure 4. The role usually taken by a spherically symmetric comparison function is here taken by $\rho e^{c(x)}$, where $c(x)$ is the above quadratic form. Some of the principal curvatures could even be negative, as shown in Figure 5.

Although it does not enter our plan to go beyond the proof of Theorem 2, we do want to point out the relevance of the geometric interpretation to the problem of boundary estimation. When the following lemmas are used to estimate interior derivatives of u at the boundary (as they clearly can be), the geometric interpretation indicates that the principal curvatures of the boundary must be suitably restricted in directions lying in the space spanned by the X_k , but are unrestricted in directions in the orthogonal complement. The quadric surface $c(x) = 0$ replaces the internally tangent sphere that is traditionally used for estimates of this kind.

When the restrictions hold, one gets a form of Hill's boundary maximum principle, [2], Theorem 2, that applies to broad classes of quasilinear differential inequalities. As in [2], the boundary-maximum principle in turn leads to a uniqueness theorem for quasilinear problems of Fichera type. It should be

FIGURE 2

FIGURE 3

stressed, however, that the statement is not quite the same as that in [2], even for the linear case, since the latter omits the geometric conditions on boundary curvature as sketched above. For *noncharacteristic* boundary points it is sufficient just to have an internally tangent sphere, as in [2], but further conditions are needed at points where the main estimate is obtained from $D_\phi u$. The nature of these conditions will be clear from the above discussion and from the sequel.

8. The shadow curvature. Let Γ be a smooth curve $x = x(t)$ through the origin and perpendicular to the x_n axis there. Thus

$$x(0) = 0, \quad x'(0) = (\theta_1, \theta_2, \dots, \theta_{n-1}, 0)$$

where, we assume, $|\theta| > 0$. We want to project Γ on the two-dimensional flat

FIGURE 4

FIGURE 5

(ξ, η) given by

$$(7) \quad x_1 = \xi \frac{\theta_1}{|\theta|}, \quad x_2 = \xi \frac{\theta_2}{|\theta|}, \quad \dots, \quad x_{n-1} = \xi \frac{\theta_{n-1}}{|\theta|}, \quad x_n = \eta,$$

and find the curvature at $(0, 0)$ of the plane curve so obtained. The plane curve is thought to be the shadow of Γ on the 2-flat which contains θ and the x_n axis, and its curvature is here called the shadow curvature.

The meaning of the term "projection" is that (ξ, η) is chosen so as to minimize $|x - \bar{x}|^2$, where the coordinates of $\bar{x}$ are the expressions on the right side of equalities (7). This gives

$$\xi = \frac{x_i \theta_i}{|\theta|}, \quad \eta = x_n.$$

The shadow curvature is therefore

$$\kappa = \frac{\eta''(0)}{[\xi'(0)]^2} = \frac{x_n''(0)}{|x'(0)|^2}.$$

Suppose next that curves $\Gamma_1, \Gamma_2, \dots, \Gamma_m$ pass through the origin, are perpendicular to the x_n axis there, and satisfy the respective differential equations

$$\frac{dx_i}{dt} = \alpha_{ik}(x).$$

Let θ be in the space spanned by their tangents at the origin, so that

$$(8) \quad \theta_i = \lambda_k \alpha_{ik}(0), \quad \theta_n = 0,$$

where the λ_k are constant. We assume $|\theta| > 0$. Associated with θ is a curve Γ

satisfying

$$\frac{dx_i}{dt} = \lambda_k \alpha_{ik}(x), \quad x(0) = 0, \quad x'(0) = \theta.$$

By the above, the shadow curvature of Γ is

$$\kappa(\lambda) = \frac{\beta_{jk} \alpha_{jl} \lambda_k \lambda_l}{\alpha_{jk} \alpha_{jl} \lambda_k \lambda_l}$$

where, for short,

$$(9) \quad \alpha_{ik} = \alpha_{ik}(0), \quad \beta_{jk} = \beta_{jk}(0) = \left. \frac{\partial \alpha_{nk}}{\partial x_j} \right|_{x=0}.$$

This function of $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ is called the shadow curvature of the set of m vector fields, $X_k = (\alpha_{ik})$, associated with the Γ_k .

On geometric grounds it is clear that $\kappa(\lambda)$ is invariant under rigid motions that leave x_n fixed. For an analytic proof, suppose all vectors, including x , are referred to a new orthonormal basis $\bar{e}_i$ related to the old basis e_i by a non-singular affine transformation

$$\bar{e}_i = h_{ij} e_j, \quad e_i = H_{ij} \bar{e}_j, \quad \bar{e}_n = e_n.$$

Thus the vectors $X = \alpha_j e_j = \bar{\alpha}_j \bar{e}_j$ and $x = x_j e_j = \bar{x}_j \bar{e}_j$ satisfy

$$\bar{\alpha}_i = H_{ij} \alpha_j, \quad x_i = h_{ij} \bar{x}_j$$

and also $\bar{\alpha}_n = \alpha_n$, $\bar{x}_n = x_n$. This gives

$$\bar{\beta}_{jk} = \frac{\partial \bar{\alpha}_{nk}}{\partial \bar{x}_j} = \frac{\partial \alpha_{nk}}{\partial \bar{x}_j} = \frac{\partial \alpha_{nk}}{\partial x_i} h_{ij} = h_{ij} \beta_{ik}.$$

By linearity, the relations

$$\theta_i = \lambda_k \alpha_{ik}, \quad \bar{\theta}_i = \bar{\lambda}_k \bar{\alpha}_{ik}$$

imply $\bar{\lambda}_k = \lambda_k$. It follows that the formula for $\bar{\kappa}(\bar{\lambda})$ in the new coordinates is reducible to

$$\frac{h_{ij} \beta_{ik} H_{ip} \alpha_{pl} \lambda_k \lambda_l}{H_{ip} \alpha_{pk} H_{iq} \alpha_{ql} \lambda_k \lambda_l}$$

in the old coordinates. Clearly

$$h_{ij} H_{ip} = \delta_{jp}, \quad H_{ip} H_{iq} = \delta_{pq}$$

where the first equation holds if (h_{ij}) is nonsingular and the second if (h_{ij}) is orthogonal. In the latter case the above expression reduces to $\kappa(\lambda)$, as expected.

We want to inquire, next, whether the shadow curvature can be duplicated by the sectional curvature of a suitable quadric surface. Let

$$(10) \quad Q(x) = x_n - \frac{1}{2} \sum_{i,j=1}^{n-1} c_{ij} x_i x_j$$

where the c_{ij} are constant. (One can use a summation convention if it is understood that $c_{ij} = 0$ when $i = n$ or $j = n$.) The surface $Q(x) = 0$ intersects the two-flat associated with θ in a two-dimensional curve whose equation is

$$\eta = \frac{1}{2} \frac{c_{ij} \theta_i \theta_j}{|\theta|^2} \xi^2.$$

If θ satisfies (8) then the curvature at the origin is

$$\kappa_0(\lambda) = \frac{c_{ij} \alpha_{ik} \alpha_{jl} \lambda_k \lambda_l}{\alpha_{jk} \alpha_{il} \lambda_k \lambda_l}$$

and is here called the *sectional curvature* of the quadric surface. This agrees with the shadow curvature $\kappa(\lambda)$ for all λ if

$$c_{ij} \alpha_{ik} = \beta_{jk}, \quad i, j = 1, 2, \dots, n-1.$$

Finally, we want to investigate the relation of these conditions to the differential inequality. By a short calculation

$$D_l D_k Q = \alpha_{il} (\beta_{jk} - \alpha_{ik} c_{ij}), \quad x = 0,$$

where the constants β_{jk} are the same as those occurring in $\kappa(\lambda)$. The results are summarized as follows:

Lemma 8. *Let α_{ik} and β_{jk} be given by (9) and $Q(x)$ by (10). Then the shadow curvature $\kappa(\lambda)$ of the vector fields (α_{ik}) at 0 agrees with the sectional curvature $\kappa_0(\lambda)$ of the quadric surface $Q = 0$ at 0 if $c_{ij} \alpha_{ik} = \beta_{jk}$, that is, if $c' \alpha = \beta$ holds as a matrix equation. The same condition ensures*

$$D_l D_k Q = 0 \quad \text{at} \quad x = 0$$

for every choice of k and l .

9. A system of differential inequalities. In the notation of the last section, we want to study the trajectories of

$$\frac{dx_i}{dt} = \lambda_k \alpha_{ik}(x), \quad x_i(0) = 0,$$

where $|\lambda| \leq 1$ and where the shadow curvature $\kappa(\lambda)$ is known. The estimates involve two functions $\tau(\delta)$ and $L(\delta)$ that describe the continuity properties of $\alpha_{ik}(x)$ near 0. Namely, let $\beta_{ik}(x) = \partial \alpha_{nk} / \partial x_i$ and let $\tau > 0$ be given. Then there is a neighborhood $|x| < \delta$ throughout which

$$\|\alpha_{ik}(x) - \alpha_{ik}(0)\| < \tau, \quad \|\beta_{ik}(x) - \beta_{ik}(0)\| < \tau,$$

$$\|\alpha_{ik}(x) - \alpha_{ik}(\bar{x})\| \leq L |x - \bar{x}|$$

where L is a constant. Since $|\lambda| \leq 1$ we can suppose that the same conditions are satisfied by the function $\lambda_k \alpha_{ik}(x)$ considered above. It is important that τ and L are independent of λ ; thus $L = L(\delta)$, $\tau = \tau(\delta)$.

By rotation of coordinates, keeping x_n fixed, the x_1 axis is directed along the tangent to the curve, so that θ has the form

$$\theta = (1, 0, 0, \dots, 0).$$

For brevity we write α_i for $\lambda_k \alpha_{ik}$ in the new coordinates, and we also change the parameter t , if necessary, so that $\alpha_i(0) = 1$. Thus the equation being considered is

$$\frac{dx_i}{dt} = \alpha_i(x), \quad x(0) = \theta$$

with θ as above. Although $\alpha_i(x)$ depends on λ , the constants τ and L for $\alpha_i(x)$ do not.

Results of the preceding section give

$$\kappa = \frac{\partial \alpha_n}{\partial x_1}$$

where $\kappa = \kappa(\lambda)$. This equation together with the initial conditions $\alpha_i(0) = \theta_i$ suggests that

$$\alpha_1 \doteq 1, \quad \alpha_2 \doteq 0, \quad \dots, \quad \alpha_{n-1} \doteq 0, \quad \alpha_n \doteq \kappa x_1$$

hold as approximate equalities in the neighborhood of 0 and hence that the trajectory through a given point y , near 0, has the approximate equation

$$x_1 \doteq y_1 + t, \quad x_2 \doteq y_2, \quad \dots, \quad x_{n-1} \doteq y_{n-1}, \quad x_n \doteq y_n + \kappa y_1 t + \frac{1}{2} \kappa t^2.$$

We shall estimate the error in this approximation when $t = -y_1$.

In the above neighborhood $|x| < \delta$, clearly

$$(11) \quad |\alpha_1(x) - 1| < \tau, \quad |\alpha_i(x)| < \tau, \quad 2 \leq i \leq n-1.$$

The equation

$$\left| \frac{\partial \alpha_n}{\partial x_1} - \kappa \right| < \tau$$

together with $\alpha_n(0) = 0$ gives

$$|\alpha_n(x_1, 0, 0, \dots, 0) - \kappa x_1| \leq \tau |x_1|$$

and hence

$$(12) \quad |\alpha_n(x) - \kappa x_1| < \tau |x_1| + L(|x_2| + |x_3| + \dots + |x_n|).$$

Lemma 9. *Let $0 < \tau < 1$ and let*

$$\frac{dx_i}{dt} = \alpha_i(x), \quad x_i(0) = y_i,$$

where $\alpha(x)$ is a continuous vector-valued function satisfying (11) and (12) for

$|x| < \delta$. Let $|\kappa| \leq K$. Then there exists a constant M , depending only on K, L and n , such that the following holds: If t is between 0 and $-y_1$, and if

$$|y| \leq \min \left(\frac{\delta}{M}, \frac{\tau}{M} \right),$$

then $|x(t)| < \delta$ and, at $t = -y_1$,

$$|x_1| \leq \tau |y_1|, \quad |x_i| \leq |y_i| + \tau |y_1|, \quad x_n = y_n - \frac{1}{2}\kappa y_1^2 + z$$

where

$$|z| \leq M |y_1| (\tau |y_1| + |y_2| + |y_3| + \cdots + |y_n|).$$

Proof. The hypothesis (11) gives

$$(13) \quad |x_1(t) - y_1 - t| \leq \tau |t|, \quad |x_i(t) - y_i| \leq \tau |t|, \quad 2 \leq i \leq n-1,$$

provided $|y|$ is so small that the trajectory remains in the sphere $|x| < \delta$. Since $\tau < 1$, Eq. (11) shows that $x_1(t)$ is monotone, and hence $|x_1| \leq |y_1|$. Together with the other relations (13) this gives

$$(14) \quad |x_1| \leq |y_1|, \quad |x_i| \leq |y_i| + \tau |y_1|, \quad 2 \leq i \leq n-1$$

for the specified interval of t .

To estimate x_n , let z be defined by

$$(15) \quad x_n = y_n + \kappa y_1 t + \frac{1}{2}\kappa t^2 + z.$$

Then, by (13)

$$\left| \frac{dz}{dt} - \left(\frac{dx_n}{dt} - \kappa x_1 \right) \right| = |\kappa| |x_1 - y_1 - t| \leq K\tau |y_1|$$

where we have used the fact that $|t| \leq |y_1|$. Upon recalling that $dx_n/dt = \alpha_n(x)$ and using (12) we get

$$(16) \quad \left| \frac{dz}{dt} \right| \leq K\tau |y_1| + \tau |x_1| + L(|x_2| + \cdots + |x_n|).$$

The terms $|x_i|$ for $1 \leq i \leq n-1$ on the right can be estimated by (14), and (15) gives also

$$(17) \quad |x_n| \leq |y_n| + 2K |y_1|^2 + |z|$$

since $|t| \leq |y_1|$. The result of substitution into (16) is

$$(18) \quad \left| \frac{dz}{dt} \right| \leq N + L |z|, \quad z(0) = 0,$$

where

$$N = \tau |y_1| (K + 1 + nL) + 2KL |y_1|^2 + L(|y_2| + \cdots + |y_n|).$$

According to (18)

$$|z| \leq \frac{N}{L} (e^{L|t|} - 1) \leq 2N |t|$$

provided $L |t| \leq 1$. Since $|t| \leq |y_1|$ this condition is assured if $|y| \leq 1/L$, which we now assume. We also assume

$$2KL |y_1|^2 \leq \tau |y_1|$$

which follows if $|y| \leq \tau/(2KL)$. Then the estimate $|z| \leq 2N |y_1|$ gives the result desired in Lemma 9, with an explicit value of M . It is easily checked that (17), (14), and the above estimate for $|z|$ makes $|x(t)| \leq (\text{const}) |y|$ if $|y_1| \leq 1$, and hence $|x| < \delta$ if $|y|$ is a sufficiently small multiple of δ .

10. A lemma concerning rank. In the rest of this paper the condition involving $pA(x, p)p$ will not be needed, and the problem can be fully linearized except for the side conditions. Given the particular function $u = u(x)$ for which the theorem is to be verified, we define

$$A(x) = A(x, Du),$$

and a linear operator T_u whose action is described by

$$T_u v = A_{ki}(x) D_k D_i v + B(x) D_0 v.$$

Clearly $T_u u = Tu$. Furthermore, in the neighborhood of any point x_0 where $Du = 0$ we have $|Du| < \epsilon$, and hence the ϵ -hypothesis on $A(x, p)$ and $B(x, p)$ gives

$$(19) \quad ||A(x)|| \leq \frac{1}{\epsilon}, \quad A(x) \geq 0, \quad B(x) \geq \epsilon.$$

We shall use T_u only near such a point, so these conditions are available.

Determination of (c_{ij}) in the preceding discussion is facilitated when (α_{ik}) has rank m at the point $x = 0$. Suppose, then, that the m vectors $X_k(0)$ are dependent. Without loss of generality it can be assumed, more explicitly, that

$$X_m(0) = \mu_k X_k(0)$$

the summation being from 1 to $m - 1$. We define a symmetric $m - 1$ by $m - 1$ matrix $\tilde{A}$ by requiring that

$$pA(x)p = q\tilde{A}(x)q$$

whenever

$$p_1 = q_1, p_2 = q_2, \dots, p_{m-1} = q_{m-1}, p_m = \mu_k p_k.$$

(Thus, $p_m = \mu_k p_k$ is substituted on the left and the resulting quadratic form in the variables $p_1, \dots, p_{m-1}$ has matrix $\tilde{A}$.) The above conditions on $A(x)$ lead to similar conditions for $\tilde{A}(x)$ in the same neighborhood, $|Du| < \epsilon$.

In such a neighborhood,

$$(20) \quad A_{ki}D_kD_iv = \tilde{A}_{ki}D_kD_iv + O(|Dv|) + o(1)$$

where $o(1)$ represents a term that tends to 0 as $x \rightarrow 0$. For proof, define

$$\tilde{X}_m(x) = X_m(x) - \mu_k X_k(x), \quad \tilde{D}_m = D_m - \mu_k D_k$$

the summation being from 1 to $m - 1$. Then

$$\begin{aligned} A_{ki}D_kD_iv &= A_{ki}D_kD_iv + A_{mi}(\tilde{D}_m + \mu_k D_k)D_iv \\ &\quad + A_{km}D_k(\tilde{D}_m + \mu_l D_l)v + A_{mm}(\tilde{D}_m + \mu_k D_k)(\tilde{D}_m + \mu_l D_l)v \end{aligned}$$

where all repeated indices k, l on the right are summed from 1 to $m - 1$, rather than from 1 to m as on the left.

Since $\tilde{X}_m(0) = 0$ it follows that terms of form $\tilde{D}_m D_k v$ in the above sum are $o(1)$ as $x \rightarrow x_0$. Since

$$[D_k, D_l]v = O(|Dv|)$$

any troubles caused by noncommutativity of the D_k can be absorbed in the term $o(|Dv|)$ and, in particular,

$$D_k \tilde{D}_m v = o(1) + O(|Dv|).$$

Upon recalling the definition of $\tilde{A}$ we get (20).

If one of the remaining vectors $X_k(0)$ is dependent on the others, we can assume that $X_{m-1}(0)$ is dependent and repeat the above process. The result is the following:

Lemma 10. *Let $x_0 = 0$ be a point where $Du = 0$ and suppose the matrix (α_{ik}) has rank $r \geq 1$ at 0. Let the D_k be renumbered, if necessary, so that $X_1, \dots, X_r$ are independent at 0. Then there is a neighborhood $|x| < \delta$ in which*

$$T_u v = \sum_{k,l=1}^r \tilde{A}_{kl}(x) D_k D_l v + B(x) D_0 v + O(|Dv|) + o(1)$$

where the r by r symmetric matrix $\tilde{A}$ satisfies conditions similar to those satisfied by the original m by m matrix A , where $o(1)$ represents a term that tends to 0 as $x \rightarrow 0$, and where the r by n matrix $[\alpha_{ik}(0)]$ has rank r . In case $r = 0$ the same holds with the sum in the expression for $T_u v$ replaced by 0.

11. Proof of the Theorem. Let F be the set of points of $\tilde{S}(P, \Omega)$ at which $u = 0$, and let ν be normal to F at one of its points, x_0 , in the sense of Bony. The drift trajectory through x_0 satisfies

$$\frac{dx_i}{dt} = \beta_i(x), \quad x(t_0) = x_0$$

and is oriented in the direction of increasing t . If $\nu_i \beta_i(x_0) \leq 0$ for every such x_0 , then Lemma 3 shows that the drift trajectory through x_0 stays in F and, together with Lemma 7 above, this gives the theorem. Throughout this section, therefore, we assume

$$(21) \quad \nu_i \beta_i(x_0) > 0.$$

Without loss of generality, the coordinates are chosen so that $x_0 = 0$ and so that ν is directed along the positive x_n axis. By Lemma 6 the diffusion trajectories are all perpendicular to ν at x_0 . If the space spanned by their tangents at $x = 0$ has dimension r , we can assume that the first r vectors $X_k(0)$ are independent. The coordinates are now rotated, keeping x_n fixed, so that the first r basis vectors e_i span the same space as is spanned by the $X_k(0)$. In the new coordinate system $\alpha_{ik}(0) = 0$ for $i > r$. Also Lemma 10 gives

$$T_u v = \tilde{A}_{ki}(x) D_k D_i v + B(x) D_0 v + o(|Dv|) + o(1)$$

near 0, where $(\tilde{A}_{ki})$ is a symmetric r by r matrix. By the above, the r by r matrix

$$(\alpha_{ik}), \quad 1 \leq i \leq r, \quad 1 \leq k \leq r$$

is nonsingular at $x = 0$.

Since Lemma 4 gives $|Du| = 0$ at 0, there is a neighborhood $|x| < \delta$ in which the hypothesis of Lemma 10 holds, and hence $\tilde{A}$ has the properties described there. It can be supposed that the same neighborhood $|x| < \delta$ permits the use of Lemma 9. Furthermore we can assume, just as in the proof of Lemma 6, that the side conditions

$$u > -\epsilon, \quad |Du| < \epsilon, \quad D_0 u < \epsilon$$

hold for $|x| < \delta$. On $\tilde{S}(P, \Omega)$ only $u \leq 0$ is given *a priori*. However the differential inequality will in fact be used only at points of $\tilde{S}(P, \Omega)$ where $u < 0$, so that all side conditions are fulfilled.

As a final restriction on δ , let the radius of the Hopf sphere associated with ν be R , where, without loss of generality, $R \leq 1$. It is required, then, that $\delta < R$. This ensures that the trajectory through a point y , as described in Lemma 9, will satisfy $|x| < R$, and hence,

$$(22) \quad |x_n(t)| < R, \quad t = -y_1.$$

We determine constants c_{ij} by

$$c_{ij} \alpha_{ik}(0) = \beta_{ik}(0),$$

for $1 \leq i \leq r$, $1 \leq j \leq r$, and by $c_{ij} = 0$ if $i > r$ or $j > r$. (The case $r = 0$ is simpler and is left to the reader.) A function $c(x)$ is defined by

$$c(x) = x_n - \frac{1}{2} \sum_{i,j=1}^r c_{ij} x_i x_j - \gamma \sum_{i=r+1}^{n-1} x_i^2 - \eta \sum_1^n x_i^2$$

where γ is a large positive constant and η is a small positive constant. Values for these constants are assigned next.

By Lemma 8 the only contribution to $\|D_i D_k c\|$ at $x = 0$ comes from the coefficient of η . Since $\|D_i D_k c\|$ is continuous and $\tilde{A}$ bounded, it follows that

$$\|\tilde{A}_{ki} D_k D_i c\| \leq O(\eta) + o(1)$$

near $x = 0$. Similarly,

$$|Dc| = o(1), \quad B(x)D_0c \geq \epsilon\beta_n(0) - o(1).$$

The hypothesis about $\nu_i\beta_i$ makes $\beta_n(0) > 0$ in the coordinate system being used here, and hence the following is true: Let η be sufficiently small compared to $\epsilon\beta_n(0)$. Then there exists a positive constant ϵ_1 , and a neighborhood $|x| < \delta_1$ depending on γ , such that

$$(23) \quad T_{\nu}c - |Dc| \geq \epsilon_1, \quad \text{for } |x| < \delta_1.$$

This argument does not require continuity of $\tilde{A}$ or B , and the needed smallness of η does not depend on γ .

With η chosen as above, it remains to define γ . Let $\tau > 0$ be chosen so that

$$\frac{1}{2}\eta R > (M + 3n\tau)$$

where M is as in Lemma 9. Then γ is chosen so that

$$\frac{1}{2}\gamma R > 2 + \frac{M}{\tau}.$$

It is also convenient to define a constant η_0 by

$$\eta_0 = \frac{\eta}{K + 2\eta}$$

where, as above, K is a bound for $\kappa(\lambda)$.

Let the equation of the Hopf sphere for ν be

$$(24) \quad x_1^2 + x_2^2 + \cdots + x_{n-1}^2 < x_n(2R - x_n).$$

It is a major objective here to show that $u < 0$ in a region of the form

$$0 < |y| \leq \delta_0, \quad c(y) \geq 0, \quad y \in \tilde{S}(P, \Omega).$$

If y is a point of this region, then y can be decomposed as the sum of a vector

$$(y_1, y_2, \cdots, y_r, 0, 0, \cdots, 0)$$

in the space spanned by the $X_k(0)$ and another vector in the orthogonal complement. Since all quantities involved in our estimates are invariant under rigid motions which leave X_n fixed, we can rotate coordinates so that e_1 is parallel to the above vector. This shows that the only case that actually needs consideration is the one for which

$$y = (y_1, 0, 0, \cdots, 0, y_{r+1}, y_{r+2}, \cdots, y_n).$$

The equation $c(y) \geq 0$ gives

$$(25) \quad y_n \geq \frac{1}{2}\kappa_0(\lambda)y_1^2 + \gamma y_0^2 + \eta y_1^2$$

where, for short,

$$y_0^2 = \sum_{i=1}^{n-1} y_i^2.$$

Here $\kappa_0(\lambda)$ is the sectional curvature of the quadric associated with $c_{i,i}$ and hence $\kappa_0(\lambda) = \kappa(\lambda)$. This shows that $|\kappa_0(\lambda)| \leq K$. If $y_n \geq 0$ the inequality (25) is multiplied by $1 - \eta_0$ and if $y_n < 0$ it is multiplied by $1 + \eta_0$. In either case the result is

$$y_n > \frac{1}{2}\kappa_0(\lambda)y_1^2 + \frac{1}{2}\gamma y_0^2 + \frac{1}{2}\eta y_1^2 + \eta_0 |y_n|$$

with strict inequality for $y \neq 0$.

We now consider the trajectory through the point y in a direction parallel to the x_1 axis, as in Lemma 9. At the point $t = -y_1$ the result of that lemma together with the above inequality, with $\kappa_0(\lambda) = \kappa(\lambda)$, gives

$$(26) \quad x_n > \frac{1}{2}\eta y_1^2 + \frac{1}{2}\gamma y_0^2 + \eta_0 |y_n| - |z|$$

where z is as in the lemma. Since

$$|y_1| |y_i| \leq \tau |y_1|^2 + \frac{|y_i|^2}{\tau}$$

and since $y_i = 0$ for $i \leq 2 \leq r$, it is seen that

$$(27) \quad |z| \leq \tau(Mn)y_1^2 + \frac{M}{\tau} y_0^2 + M |y_1| |y_n|.$$

The lemma also gives

$$(28) \quad |x_1|^2 \leq \tau |y_1|^2, \quad |x_i|^2 \leq 2 |y_i|^2 + 2\tau |y_1|^2.$$

To show that the point (x_i) so obtained lies in the sphere (24) it suffices to show that (24) holds with Rx_n on the right, since our conditions ensure $|x_n| \leq R$. By the estimates (28) it suffices to show that

$$3n\tau y_1^2 + 2y_0^2 + |z| < R(x_n + |z|)$$

where we have added $|z|$ on the left and $R|z| \leq |z|$ on the right. Upon substituting the lower bound (26) for $x_n + |z|$ and the upper bound (27) for $|z|$ one finds that the inequality does hold, provided

$$|y_1| \leq \frac{\eta_0 R}{M}.$$

This condition is assured by requiring a similar inequality for $|y|$. Thus, δ_0 in the condition $0 < |y| < \delta_0$ is defined by

$$\delta_0 = \min \left(\delta, \frac{\delta}{M}, \frac{\tau}{M}, \frac{\eta_0 R}{M} \right).$$

The first condition pertains to the δ introduced at the beginning of this discussion, the next two come from Lemma 9, and the last ensures the needed bound for $|y_1|$ as explained above.

If H_0 denotes the region $0 < |x| \leq \delta_0$, $c(x) \geq 0$ considered above, then clearly

$$u(x) < 0 \quad \text{in} \quad H_0 \cap \bar{S}(P, \Omega).$$

This follows from Lemma 7, since otherwise the value $u(y) = 0$ would propagate along the diffusion trajectory through y and would give $u = 0$ at a point of $H \cap \bar{S}(P, \Omega)$. If ρ and σ are positive constants, with ρ sufficiently small, the function

$$v(x) = u(x) + \rho e^{\sigma c(x)}$$

attains a maximum relative to $\bar{S}(P, \Omega)$ at an interior point, P^* , of H_0 . This follows as in the familiar argument of Hopf, since

$$D_0 c > 0, \quad D_0 u = 0 \quad \text{at} \quad x = 0.$$

We choose $\sigma = 2/\epsilon_1$ where ϵ_1 is given by (23), and then we choose ρ sufficiently small. Lemma 5 applied to the linear operator T_u instead of T gives

$$T_u u + |Du| + |u| \leq -\omega \left(T_u c + \sigma \epsilon |Dc|^2 - |Dc| - \frac{1}{\sigma} \right) \leq -\frac{1}{2} \omega \epsilon_1,$$

at P^* , and a similar inequality holds at nearby points of $S(P, \Omega)$. Since $T_u u = Tu$ this is a contradiction.

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