Interpolating Sequences for Bounded Harmonic Functions

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Introduction. A sequence $\{z_i\}_{i=1}^\infty$ in the open unit disc Δ is an *interpolating sequence* if for every bounded sequence $\{a_i\}_{i=1}^\infty$ there is a bounded analytic function f on Δ such that $f(z_i) = a_i$ for all i . If for every such $\{a_i\}_{i=1}^\infty$ there is a bounded harmonic function u on Δ with $u(z_i) = a_i$, we call $\{z_i\}_{i=1}^\infty$ a *harmonic interpolating sequence*. It is clear that every interpolating sequence is a harmonic interpolating sequence. Our purpose here is to prove the converse.

Theorem. *Every harmonic interpolating sequence is an interpolating sequence.*

The proof uses Hall's lemma and the construction from the proof of the corona theorem given in [2]. The theorem given here had also been proved by Carleson by similar means (unpublished).

§1. Two elementary lemmas. Fix a harmonic interpolating sequence $\{z_i\}_{i=1}^\infty$.

Lemma 1. *There are constants $M > 1$ and $\eta = \eta(M) > 0$ such that*

$$(1) \quad \left| \frac{z_j - z_k}{1 - \bar{z}_k z_j} \right| \geq \eta \quad \text{if} \quad j \neq k,$$

(2) *If $|a_i| \leq 1$ for all i , there is u harmonic with $u(z_i) = a_i$ and $\|u\| = \sup |u(z)| \leq M$.*

Proof. The open mapping theorem for Banach spaces gives (2), and (1) is an easy consequence of (2). Indeed, let u be real and harmonic with $u(z_k) = 0$, $u(z_i) = 1$, $\|u\| \leq M$. If v is the conjugate harmonic function vanishing at z_k , then $f(z) = e^{u+iv} - 1$ satisfies $f(z_k) = 0$, $|f(z_i)| \geq e - 1$, and $\|f\| \leq e^M + 1$. By Schwarz's lemma,

$$\left| \frac{z_i - z_k}{1 - \bar{z}_k z_i} \right| \geq \frac{|f(z_i)|}{e^M + 1} \geq \frac{e - 1}{e^M + 1}.$$

For any sequence $\{z_i\}_{i=1}^\infty$ in Δ , let μ be the measure with mass $1 - |z_i|^2$ at z_i . Function theoretic properties of $\{z_i\}_{i=1}^\infty$ can often be expressed in terms of μ .

For example, μ is finite if and only if $\{z_i\}_{i=1}^\infty$ is a Blaschke sequence [4]. Similarly we have

Lemma 2. *Let $\{z_i\}_{i=1}^\infty$ be a sequence in Δ satisfying (1). Then $\{z_i\}_{i=1}^\infty$ is an interpolating sequence if and only if there is a constant A such that for every set of the form $S = \{r > 1 - \delta, |\theta - \theta_0| < \delta\}$,*

$$(3) \quad \mu(S) \leq A \cdot \delta.$$

Proof. By a theorem of Carleson [2], [3], related to the Hardy–Littlewood maximal theorem, condition (3) implies there is a constant c_1 such that

$$(4) \quad \sum_k (1 - |z_k|^2) |g(z_k)| = \int |g(z)| d\mu \leq \frac{c_1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})| d\theta$$

for all $g \in H^1$. Taking $g(z) = (1 - |z|^2)/(1 - \bar{z}_j z)^2$ in (4), we have

$$(5) \quad \sum_k \frac{(1 - |z_k|^2)(1 - |z_j|^2)}{|1 - \bar{z}_j z_k|^2} \leq c_1.$$

Since

$$1 - \left| \frac{z_j - z_k}{1 - \bar{z}_j z_k} \right|^2 = \frac{(1 - |z_k|^2)(1 - |z_j|^2)}{|1 - \bar{z}_j z_k|^2},$$

(1) and (5) yield

$$\log \prod_{k, k \neq j} \left| \frac{z_j - z_k}{1 - \bar{z}_k z_j} \right|^{-2} \leq c_2 \sum_{k, k \neq j} 1 - \left| \frac{z_k - z_j}{1 - \bar{z}_k z_j} \right|^2 \leq c_1 c_2,$$

so that for all j ,

$$(6) \quad \prod_{k, k \neq j} \left| \frac{z_j - z_k}{1 - \bar{z}_k z_j} \right| \geq \gamma > 0.$$

Now (6) is a necessary and sufficient condition for interpolation ([1], [3], or [4]). On the other hand, it is well known that any interpolation sequence satisfies (4) (see for example p. 72 of [5]), and (4) is equivalent to (3) ([2], [3]).

We will prove the theorem by verifying (3) directly.

§2. Proof of the Theorem. We return to our fixed harmonic interpolating sequence $\{z_i\}_{i=1}^\infty$, and verify condition (3). By Lemma 2 that will prove the theorem. Fix $S = \{r > 1 - \delta, |\theta - \theta_0| < \delta\}$ and map Δ to the half plane $H = \{y > 0\}$ so that S falls inside the square $Q = \{0 < y < \delta, 0 < x < \delta\}$ but inside no smaller such square. Retain the notation $\{z_i\}_{i=1}^\infty$ for the image sequence, but replace μ by the measure ν on H with mass $\text{Im } z_i$ at z_i . Condition (3) is equivalent to

$$(7) \quad \nu(Q) \leq A_1 \cdot \delta.$$

We will show A_1 depends only on M and thus we suppose $\delta = 1$.

Let R^n denote the strip in Q with $2^{-n-1} < y \leq 2^{-n}$. By Lemma 1 and Harnack's

theorem there is an integer $K = K(M)$ such that partitioning each R^n into dyadic squares S^n of side 2^{-n-K} we have:

(a) No S^n contains more than one z_i .

(b) If u is harmonic on H and $\|u\| \leq 2M$, then the oscillation of u over any S^n is at most 1.

For $z_i \in R^n$ let $I(z_i)$ be the horizontal segment of length $|I(z_i)| = 2^{-K-n}$ through z_i and inside some S^n . By (a) we have

$$(8) \quad \nu(Q) \leq 2^K \sum |I(z_i)|.$$

Reindex the sum in (8) as follows: Say z_i is in the first generation if no $I(z_k)$ lies above an interior point of $I(z_i)$ ¹.

Let $\{z_p\}$ denote the first generation. For each z_p take a square Q_p with base the projection $I(z_p)^*$ of $I(z_p)$ onto $\{y = 0\}$. The Q_p are shaded in the figure and their interiors are pairwise disjoint. Let T_p be the rectangle between $I(z_p)$ and Q_p ; it contains at most $A_2 = A_2(M)$ intervals $I(z_i)$ by (a). So if $W^0 = Q \setminus \bigcup Q_p$, then

$$(9) \quad \nu(W^0) = \sum_p \nu(T_p) \leq \sum_p A_2 |I(z_p)| \leq A_2.$$

Repeat the above process inside each Q_p , obtaining a second generation $\{z_{p,q}\}$ of z_i 's and new shaded squares $Q_{p,q} \subset Q_p$ ².

¹ If it is raining from the top of Q , z_i is first generation precisely when $I(z_i)$ gets wet.

² Of course some z_i in T_p may fall in no generation, but then $|I(z_i)|$ has already been counted.

Letting $W_p = Q_p \setminus \bigcup_a Q_{p,a}$ we have by (9) and a dilation, $\nu(W_p) \leq A_2 |I(z_p)|$. So if $W^1 = \bigcup W_p$ and $E_1 = \bigcup I(z_p)^*$ then

$$(10) \quad \nu(W^1) \leq A_2 |E_1|.$$

Now repeat the process with the $Q_{p,a}$ and with the corresponding squares for every later generation. We get sets $\{W^k\}_{k=0}^\infty$ and $\{E_k\}_{k=1}^\infty$ such that $E_{k+1} \subset E_k \subset \{y = 0\}$, and by (9) and (10)

$$(11) \quad \nu(Q) = \sum_{k=0}^\infty \nu(W^k) \leq A_2 \left(1 + \sum_{k=1}^\infty |E_k|\right).$$

Lemma 3. *There exists $n = n(M)$ such that $|E_k| \leq 1/2$ for $k \geq n$.*

Assuming the lemma for a moment, note that the E_k are constructed iteratively so that then $|E_k| < 2^{-m}$ if $k > n \cdot m$. Then by (11) we have

$$\nu(Q) \leq 2nA_2$$

which proves the theorem.

Proof of Lemma 3. Let D be the strip $\{0 < y < \pi\}$ and let $L_k = \bigcup \{I(z_i) : z_i \text{ is in the } k\text{-th generation}\}$. Thus its projection is $L_k^* = E_k$. Fix n , and set $G_n = D \setminus \bigcup_{k=1}^n L_k$, $z^* = \pi i/2 \in G_n$. Let λ_n be the harmonic measure on ∂G_n for z^* , let $a_n = \lambda_n(\bigcup_{k \leq n} L_k)$, and reindexing the L_i so that $\lambda_n(L_n)$ is largest, let $b_n = \lambda_n(L_n) \geq a_n/n$. Define G_{n-1} , a_{n-1} , b_{n-1} , $\dots$, G_1 , a_1 , b_1 similarly. By the hypothesis there is a real harmonic function u on D such that $\|u\| \leq 2M$, $u(z_i) = -2$ if $z_i \in L_n$, $u(z_i) = 2$ if $z_i \in L_k$, $k < n$. By (b), $u < -1$ on L_n and $u > 1$ on L_k , $k < n$. Letting λ_ζ be the harmonic measure for $\zeta \in L_n$ on ∂G_{n-1} , we have by the maximum principle

$$-1 > \lambda_\zeta\left(\bigcup_{k < n} L_k\right) - 2M(1 - \lambda_\zeta\left(\bigcup_{k < n} L_k\right))$$

so that

$$\lambda_\zeta\left(\bigcup_{k < n} L_k\right) < \frac{2M - 1}{2M + 1} = \alpha.$$

Hence

$$a_{n-1} \leq \alpha b_n + (a_n - b_n) \leq \frac{(n-1) + \alpha}{n}.$$

Repeating this gives

$$(12) \quad a_1 \leq \prod_{k=1}^{n-1} \frac{k + \alpha}{k + 1} < \epsilon$$

provided n is large enough.

Map D to H by $z \mapsto e^z$. Then L_1 falls on a collection of radial slits in $0 < x < e$, and by Hall's lemma (see [3] p. 208) these slits project angularly onto a set of small length. Mapping this projection back to $\bar{D}$, we see $|E_n| < 1/2$ if ϵ is sufficiently small.

§3. Other interpolations. The corresponding problem for L_p , $1 \leq p < \infty$ does not require the construction and is thus much easier. Here one considers the mapping T_p on $L^p[0, 2\pi]$ defined by the Poisson kernel

$$T_p f(z_i) = \frac{1}{2\pi} \int f(e^{i\theta}) \frac{1 - |z_i|^2}{|e^{i\theta} - z_i|^2} d\theta.$$

Now if $f \in H^p$, then

$$(1 - |z_i|)^{1/p} |T_p f(z_i)| \rightarrow 0$$

(see [3]), so it is natural to define an H^p interpolating sequence as one for which $T_p(H^p) = L^p(\mu)$. The converse of the Carleson theorem used to show (3) implies (4) shows that (3) holds if $T_p: L^p[0, 2\pi] \rightarrow L^p(\mu)$ is bounded when $1 < p < \infty$ or if T_1 is weak type $(1, 1)$. Variants of Lemmas 1 and 2 then show that (6) holds if $T_p(L^p) = L^p(\mu)$, and this means $T_p(H^p) = L^p(\mu)$.

Finally, since the interpolation problem is local, the theorem still holds on any finite open Riemann surface. We now show by example that it fails on an infinitely connected plane domain. Let $D = \Delta \setminus \bigcup_{i=1}^{\infty} \Delta_i$, where the Δ_i are pairwise disjoint open discs with centers on $x > 0$ clustering at 0. See [6] for information on such domains. The Δ_i can be chosen so that 0 is a regular point of ∂D but so that the measure dz/z on ∂D has finite total variation. It follows that $\lim_{x \rightarrow 0^-} f(x)$ exists for every bounded analytic function f on D . However, we can choose inductively a sequence $x_1 < x_2 < \dots$ with $\lim x_n = 0$ such that if λ_n is the harmonic measure for $x_n \in D$ on ∂D , there is a constant M such that

$$(13) \quad \sum |c_i| \leq M \left| \left| \sum c_i \lambda_i \right| \right|$$

for every sequence $\{c_i\}$. But (13) is the dual way of saying that $\{x_i\}$ is an interpolating sequence for the bounded harmonic functions on D . This example was first noticed by S. Fisher.

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