Upper Functions for Symmetric Processes with Stationary, Independent Increments

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Communicated by R. GETTOOR

1. Introduction. Let X be a one-dimensional symmetric stochastic process with stationary, independent increments, where, as is usual, we assume $X(0) = 0$, X to be right continuous and to have left limits everywhere.

Given a rather arbitrary deterministic function h , the object of this paper is to evaluate $\limsup_{t \rightarrow 0} |X(t)|/h(t)$ and $\limsup_{t \rightarrow \infty} |X(t)|/h(t)$. Unfortunately we shall have to make an additional assumption about the stochastic process.

This paper was motivated by Feller's paper ([1]) discussing a similar question for sums of independent random variables.

Except for Section 5, we shall deal only with the case where $t \rightarrow 0$; and, thus, the notation $t \rightarrow 0$ will be omitted from some limit notations. For "most" processes the behavior as t approaches 0 will be governed by the largest sample function jump occurring before time t . The exceptional cases are roughly those processes whose Levy measures are almost too big near 0 to be Levy measures—in these cases a "law of the iterated logarithm" type result holds. In any case it should be noted that the random variable $\limsup |X(t)|/h(t)$ is a constant (possibly infinite) with probability one.

2. Notation and Assumptions. We assume that the process has no Gaussian part, for if it did then a result of Khinchin ([3]) shows that the usual (*i.e.* as if the non-Gaussian part were 0) law of the iterated logarithm holds. Thus we have the following canonical formula:

$$\begin{aligned} (1) \quad \log Ee^{iuX(t)} &= -t \int (1 - \cos uy)\nu(dy) \\ &= -2t \int_0^\infty (1 - \cos uy)\nu(dy) \end{aligned}$$

where ν (called the Levy measure) is a symmetric measure on $R - \{0\}$ such that $\int y^2(1 + y^2)^{-1}\nu(dy) < \infty$. In our notation we have suppressed " ω " a

typical member of the underlying probability space. We shall continue to do so except in the following paragraph.

Let $J(t, \omega) = X(t, \omega) - X(t-, \omega)$. Then we have the following canonical representation for X :

$$(2) \quad X(t, \omega) = \lim_{\epsilon \rightarrow 0} \sum \{J(s) : s \leq t, |J(s)| > \epsilon\}.$$

Let λ denote Lebesgue measure. Let B be a Borel subset of $[0, \infty) \times (0, \infty)$ and define $N(B, \omega)$ to equal the number of distinct t such that $(t, J(t, \omega)) \in B$. Then $N(B, \cdot)$ is a Poisson random variable with mean $(\lambda \times \nu)(B)$ —it being understood that $N(B, \omega) = \infty$ with probability one if $(\lambda \times \nu)(B) = \infty$. If $\{B_\alpha\}$ is a family of disjoint Borel sets the corresponding family of Poisson random variables is independent.

We define a function σ by the formula

$$(3) \quad \sigma^{-1}(x) = \int_0^x \frac{2y \, dy}{H(y/\log |\log y|)}, \quad x > 0$$

where

$$(4) \quad H(x) = 2 \int_0^x y^2 \nu(dy).$$

If $\sigma^{-1}(x) = \infty$ for all $x > 0$, we say $\sigma(t) \equiv 0$. We shall not worry here or elsewhere about the problems of definition away from 0. We introduce some more notation:

$$(5) \quad \rho(t) = \sigma(t) [\log |\log \sigma(t)|]^{-1}$$

$$(6) \quad g(t) = \sigma(t) [2 \log |\log \sigma(t)|]^{1/2}.$$

From definition (3) we conclude

$$(7) \quad \sigma(t)^2 = \int_0^t H(\rho(s)) \, ds.$$

We shall need to make a further assumption about the process. The first step is to define

$$(8) \quad a_n = e^{-n \log (\log n)^2}.$$

Wherever $\int_0^1 g^{-1}(y) \, \nu(dy) < \infty$, or equivalently, whenever

$$\int_0^1 \nu\{y : |y| > g(s)\} \, ds < \infty,$$

we shall also assume

$$(9) \quad \sum (\log n)^{12} \left[\int_{a_{n+1}}^{a_n} g^{-1}(y) \nu(dy) \right]^2 < \infty.$$

This last assumption is obviously contrived in order to make the proof of the coming theorem valid. We investigate more intuitive assumptions that would

imply assumption (9). First, we note that if $\sum b_n < \infty$ and $b_n \downarrow 0$ as $n \rightarrow \infty$, then $\sum nb_n^2 < \infty$; so if

$$\int_{a_{n+1}}^{a_n} g^{-1}(y) \nu(dy) \rightarrow 0$$

monotonically as $n \rightarrow \infty$, then condition (9) is valid with $(\log n)^{12}$ replaced by n . We can also get by by assuming

$$\int_{e^{-(n+1)}}^{e^{-n}} y^2 \nu(dy) \rightarrow 0$$

monotonically as $n \rightarrow \infty$; for the fact that we have not replaced $(\log n)^{12}$ by n gives us enough freedom to replace a_n by e^{-n} and $g^{-1}(y)$ by y^2 in the light of the following calculations:

$$\begin{aligned} \int_{a_{n+1}}^{a_n} g^{-1}(y) \nu(dy) &\leq g^{-1}(a_n) a_{n+1}^{-2} \int_{a_{n+1}}^{a_n} y^2 \nu(dy), \\ \int_{a_n}^{a_{n-1}} g^{-1}(y) \nu(dy) &\geq g^{-1}(a_n) a_{n-1}^{-2} \int_{a_n}^{a_{n-1}} y^2 \nu(dy). \end{aligned}$$

By dividing $\sigma^2(t)$ by t and then differentiating using equation (7) one concludes that

$$(10) \quad \frac{\sigma(t)^2}{t} \downarrow 0 \quad \text{as} \quad t \downarrow 0$$

from which follows, for example,

$$(11) \quad \frac{g(t)^3}{t} \downarrow 0 \quad \text{as} \quad t \downarrow 0.$$

When $\int_0^1 g^{-1}(y) \nu(dy) = \infty$ we shall be considering an "arbitrary" function h such that $\int_0^1 h^{-1}(y) \nu(dy) < \infty$. We shall then assume that h has certain properties:

$$(12) \quad \sum (\log n)^{12} \left[\int_{a_{n+1}}^{a_n} h^{-1}(y) \nu(dy) \right]^2 < \infty,$$

$$(13) \quad \frac{h(t)^3}{t} \downarrow 0 \quad \text{as} \quad t \downarrow 0,$$

$$(14) \quad (\tau^{-1})' \left(\frac{x}{c} \right) \leq \frac{2x}{H(x/(\log |\log x|))}$$

for sufficiently small x , depending on c , where $\tau(t) = h(t)/(2 \log |\log h(t)|)^{1/2}$. Assumption (14) is quite natural when one notes definition (3) and the first sentence of this paragraph. (We can change the power on $h(t)$ in condition (13) if we are willing to change it appropriately on $\log n$ in condition (12).)

3. Basic Result.

Theorem 1. *Let g be defined by formula (6). If assumption (9) is satisfied and if $\int_0^1 g^{-1}(y) \nu(dy) < \infty$, then*

$$(15) \quad \limsup_{t \rightarrow 0} \frac{|X(t)|}{g(t)} = 1 \quad a.s.$$

If $\int_0^1 g^{-1}(y) \nu(dy) = \infty$, then

$$(16) \quad \limsup_{t \rightarrow 0} \frac{|X(t)|}{g(t)} = \infty \quad a.s.$$

In the latter case, suppose that h is a function which satisfies conditions (12), (13), and (14) and $\int_0^1 h^{-1}(y) \nu(dy) < \infty$. Then

$$(17) \quad \limsup_{t \rightarrow 0} \frac{|X(t)|}{h(t)} = 0.$$

If $\int_0^1 h^{-1}(y) \nu(dy) = \infty$ and condition (13) is satisfied, then

$$(18) \quad \limsup_{t \rightarrow 0} \frac{|X(t)|}{h(t)} = \infty.$$

Proof.

Part A. Assume $\int_0^1 h^{-1}(y) \nu(dy) = \infty$. Integration by parts shows that $\int_0^1 \nu\{y: |y| > h(t)\} dt = \infty$. Note, by property (13), that if $k > 1$, then $kh(t) < h(tk^3)$; hence $\int_0^1 \nu\{y: |y| > kh(t)\} dt = \infty$. Therefore, there are infinitely many $t \leq 1$ such that $|J(t)| > kh(t)$, and for each such t either $|X(t)| > kh(t)/2$ or $|X(t-)| > kh(t)/2$. Hence, $\limsup_{t \rightarrow 0} |X(t)|/h(t) = \infty$. This same argument holds, of course, with h replaced by g , with property (11) being used.

Part B. Assume that $\int_0^1 g^{-1}(y) \nu(dy) < \infty$. Let $0 < \delta < 1$. We write $X = X_1 + X_2 + X_3$ where

$$(19) \quad X_1(t) = \sum \{J(s): s \leq t, \delta g(s) < |J(s)|\}$$

$$(20) \quad X_2(t) = \sum \{J(s): s \leq t, \rho(s) < |J(s)| \leq \delta g(s)\}$$

$$(21) \quad X_3(t) = \lim_{\epsilon \rightarrow 0} \sum \{J(s): s \leq t, \epsilon < |J(s)| \leq \rho(s)\}.$$

At this point it is still conceivable that the series in equations (19) and (20) might not be absolutely convergent, but the formula for $X_3(t)$ certainly does make sense, since it is the canonical representation, similar to (2), for the process X_3 with independent, not necessarily stationary, increments. We proceed to deal with X_1 , X_2 , and X_3 separately.

Part B1. By hypothesis and an argument contained in Part A it follows that $\int_0^1 \nu\{y: |y| > \delta g(s)\} ds < \infty$ so $X_1(t)$ equals a finite sum; in fact, $X_1(t) = 0$ for t sufficiently small (depending on ω).

Part B2. We define a sequence $s_n \rightarrow 0$ by

$$\delta g(s_n) = e^{-n \log (\log n)^2} = a_n .$$

We shall show that for sufficiently large n , $J(s) \neq 0$ for at most one value of $s \in (s_{n+1}, s_n]$. The probability that $J(s) \neq 0$ for more than one value of $s \in (s_{n+1}, s_n]$ is bounded by

$$\begin{aligned} \left[\int_{s_{n+1}}^{s_n} \nu \{y: \rho(s) < |y| \leq \delta g(s)\} ds \right]^2 &\leq s_n^2 [\nu \{y: \delta g(s_{n+2}) < |y| \leq \delta g(s_n)\}]^2 \\ &\leq [s_n^2 / \delta^6 s_{n+2}^2] \left[2 \int_{a_{n+2}}^{a_n} g^{-1}(y) \nu(dy) \right]^2 \end{aligned}$$

where we have used property (11). We use property (11) again to conclude

$$\delta^3 s_n \leq (\log n)^6 s_{n+2} ,$$

and, thus, we are in position to use the basic assumption (9) together with a Borel–Cantelli argument to prove our announced assertion.

It follows that if $t \in (s_{n+1}, s_n]$ and if n is sufficiently large (depending on ω), then

$$|X_2(t)| \leq \delta g(t) + \sum_{k=n+1}^{\infty} \delta g(s_k) < 3 \delta g(t)$$

Therefore, it follows that

$$\limsup |X_2(t)|/g(t) < 3\delta .$$

We note, that although the definition of X_2 depends on δ , the definition of $X_1 + X_2$ does not; hence

$$\limsup |X_1(t) + X_2(t)|/g(t) = 0$$

Part B3. We plan to use the classical law of the iterated logarithm in order to prove

$$(22) \quad \limsup X_3(t)/g(t) = 1 .$$

Symmetry will enable us to then replace $X_3(t)$ by $|X_3(t)|$. Let $\{t_n\}$ be a sequence with $t_n \rightarrow 0$ monotonically as $n \rightarrow \infty$. Let

$$\sigma_n^2 = \sigma(t_n)^2 = \text{Var } X_3(t_n)$$

and

$$Y_n = \frac{X_3(t_n)}{\sigma_n^2} - \frac{X_3(t_{n-1})}{\sigma_{n-1}^2} .$$

An easy calculation shows that

$$\text{Var } Y_n = \sigma_n^{-2} - \sigma_{n-1}^{-2}$$

and

$$\text{Var } (Y_1 + \cdots + Y_n) = \sigma_n^{-2} - \sigma_1^{-2} .$$

If the law of the iterated logarithm were applicable we could conclude

$$\begin{aligned} 1 &= \limsup (Y_1 + \cdots + Y_n)/(2\sigma_n^{-2} \log \log \sigma_n^{-1})^{1/2} \\ &= \limsup X_3(t_n)/(2\sigma_n^2 \log \log \sigma_n^{-1})^{1/2}. \end{aligned}$$

Since $\{t_n\}$ is arbitrary, equation (22) would follow.

The above arguments would hold even if we were to ignore all $J(s)$ with $|J(s)| < \epsilon(s)$ providing we choose $\epsilon(s) \rightarrow 0$ sufficiently fast as $s \rightarrow 0$. Thus we can choose the numbers t_n sufficiently close together so that for sufficiently large n (depending on ω since a Borel–Cantelli argument is needed) $J(s) \neq 0$ for at most one value of s between t_{n+1} and t_n . Therefore, by choosing $\{t_n\}$ arbitrarily, but with the numbers t_n sufficiently close together, we have

$$Y_n = o(\sigma_n^{-2} (\log \log \sigma_n^{-1})^{-1/2})$$

and our previous use of the law of the iterated logarithm is justified.

Part C. Now we assume that $\int_0^1 h^{-1}(y) \nu(dy) < \infty$ and that h satisfies assumptions (12), (13), and (14). The decomposition of Part B together with the arguments of Parts B1 and B2 are valid with g replaced by h .

By assumption (14) we have

$$\begin{aligned} \int_0^t H(\tau(s)/(\log |\log \tau(s)|)) ds &= \int_0^{\tau(t)} H(x/(\log |\log x|)) d\tau^{-1}(x) \\ &\leq \int_0^{\tau(t)} 2x dx = \tau(t)^2 \end{aligned}$$

We shall use this fact to show

$$(23) \quad \limsup |X(t)|/h(t) \leq 1$$

But if we can do this we can also do it with h replaced by ch where $c < 1$ is arbitrary and thus we would obtain

$$\limsup |X(t)|/h(t) = 0.$$

To prove (23) all we need is the argument of Part B3 together with the following lemma.

Lemma. *Let $\{Y_n\}$ be a sequence of independent random variables with mean zero and variances $\{b_n\}$. Let $\{c_n\}$ be a sequence with*

$$c_n^2 \geq \sum_{k=1}^n b_k^2.$$

Suppose that $Y_n = o(c_n (\log \log c_n)^{-1/2})$ and that $c_{n+1}/c_n \rightarrow 1$ as $n \rightarrow \infty$. Then

$$\limsup (Y_1 + \cdots + Y_n)/c_n (2 \log \log c_n)^{1/2} \leq 1.$$

Proof of lemma. The appropriate part of the proof of the law of the iterated logarithm as given, for example, in Chapter 5 of [5] is easily adapted to give a proof.

4. Examples and remarks.

Example 1. Consider the stable case; *i.e.* $\nu(dy) = |y|^{-1-\alpha} dy$, $0 < \alpha < 2$. Then $\sigma^{-1}(x) > x^\alpha(2 - \alpha)/\alpha$ so $\int_0^1 g^{-1}(y) \nu(dy) = \infty$ and Theorem 1 checks with Khinchin's result ([4]) for stable processes.

Example 2. Let

$$\nu(dy) = (\beta/2) |y|^{-3} |\log y|^{-1-\beta} dy$$

for $\beta > 0$ (and $|y|$ small). Then

$$\begin{aligned} \sigma^{-1}(x) &\sim x^2 |\log x|^\beta \Rightarrow \sigma(t) \sim t^{1/2} |\log t|^{-\beta/2} \\ &\Rightarrow g(t) \sim t^{1/2} |\log t|^{-\beta/2} (2 \log |\log t|)^{1/2} \\ &\Rightarrow g^{-1}(x) \sim x^2 |\log x|^\beta (2 \log |\log x|)^{-1}. \\ &\Rightarrow \int_0^1 g^{-1}(y) \nu(dy) = \infty. \end{aligned}$$

Example 3. Let

$$\nu(dy) = (\beta/2) |y|^{-3} |\log y|^{-1} (\log |\log y|)^{-1-\beta} dy$$

for $\beta > 0$ (and $|y|$ small.) Then

$$\begin{aligned} \sigma^{-1}(x) &\sim x^2 (\log |\log x|)^\beta \Rightarrow g^{-1}(x) \sim x^2 (\log |\log x|)^{\beta-1} 2^{-1} \\ &\Rightarrow \int_0^1 g^{-1}(y) \nu(dy) < \infty \\ &\Rightarrow \limsup \frac{|X(t)|}{(2t \log |\log t|)^{1/2} (\log |\log t|)^{-\beta/2}} = 1. \end{aligned}$$

Remark 1. It would be nice if assumption (9) could be removed, although one probably can not avoid some assumptions about the function h . Even if assumptions (9) and (12) are removed, Theorem 1, by itself, would still not be a completely satisfactory description of the situation. The rather innocuous appearing assumption (13) is unsatisfactory for some processes since if the Levy measure is sufficiently "uneven", it is possible, using a function h not satisfying assumption (13), to obtain the following

$$\begin{aligned} \int_0^1 \nu\{y: |y| > h(t)\} dt &= \infty, \\ \int_0^1 \nu\{y: |y| > 2h(t)\} dt &< \infty. \end{aligned}$$

The "0 or ∞ " dichotomy in the second part of Theorem 1 would no longer be valid.

Remark 2. The symmetry assumption does not play a role in the application of the classical law of the iterated logarithm except to guarantee that the

truncated mean is zero. In the non-symmetric case, therefore, one might hope to first assume for convenience only that there are no jumps bigger than 1 so that $EX(t)$ exists and then to add an appropriate linear drift to make $EX(t) = 0$. Although Part B3 of the proof of Theorem 1 still is valid, problems arise in adapting Parts B1 and B2 to the new situation. Indeed, without some separation into cases the non-symmetric case has inherent problems; for, in the case of increasing processes ([2]), if one adds a drift in order that $EX(t) = 0$, then the drift will constitute the dominant aspect of the local sample function behavior. Nevertheless, it is conceivable that Theorem 1 is valid in the case where $\int_0^1 g^{-1}(y) \nu(dy) < \infty$ and the process is not necessarily symmetric.

5. Large times. There is a great amount of similarity between the situation for large times and the situation for small times. Thus, rather than treat the "large time" case completely we shall only point out the significant differences from the small time case.

One might as well assume $\text{Var } X(t) = \infty$, for otherwise the classical law of the iterated logarithm holds; and once one does that, one can, with no loss of generality, assume that there is no Gaussian part and that there are no jumps of magnitude less than one.

Zero should be replaced by some finite number as the lower limit of integration in formula (3) in order to guarantee that $\sigma^{-1}(x) < \infty$. In place of formula (8) we should use

$$(24) \quad a_n = e^{n \log(\log n)^2}$$

Instead of property (10) we get

$$(25) \quad \frac{\sigma(t)^2}{t} \uparrow \infty \quad \text{as } t \uparrow \infty$$

and instead of property (11) we actually get

$$(26) \quad \frac{g(t)^2}{t} \uparrow \infty \quad \text{as } t \uparrow \infty.$$

Assumption (13) should be changed in the obvious manner and assumption (14) should hold for large x .

Theorem 2. This theorem is the same as Theorem 1 except for the changes in assumptions indicated above, the fact that " $t \rightarrow 0$ " is replaced by " $t \rightarrow \infty$ ", and the fact that 0 and 1 as limits of integration are replaced, respectively, by 1 and ∞ .

Proof. The proof is similar to the proof of Theorem 1 and, in particular, Part B3 of that proof becomes easier for one can simply define

$$Y_n = X_3(t_n) - X_3(t_{n-1}).$$

Theorem 2 is closely related to Feller's results ([1]).

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This research was supported by NSF Grant No. GP-21188 and by the United States Army under Contract No. DA-31-124-ARO-D-462.

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and
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Date communicated: MAY 10, 1971