Growth and Continuity of Functions Satisfying Quadratic Integral Inequalities

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Communicated by M. ROSENBLATT

1. Introduction. Let f be a real measurable function on $[0, 1]$. We are concerned with conditions on f of the form

$$(1.1) \quad I_p(f) = I(f) = \int_0^1 \int_0^1 \left[\frac{f(x) - f(y)}{p(x-y)} \right]^2 dx dy < \infty.$$

Here, p is a continuous function on $[-1, 1]$ such that

$$(1.2) \quad p(x) = p(-x) > 0 \text{ for } x > 0, \quad p \uparrow \text{ on } [0, 1], \quad p(0) = 0.$$

A. Garsia, E. Rodemich, and H. Rumsey [3, 4] considered a more general condition

$$(1.3) \quad I_{\Psi, p}(f) = \int_0^1 \int_0^1 \Psi \left(\frac{f(x) - f(y)}{p(x-y)} \right) dx dy < \infty,$$

where Ψ is a function on $(-\infty, \infty)$ satisfying

$$(1.4) \quad \Psi(x) = \Psi(-x), \quad \Psi \uparrow \text{ on } [0, \infty), \quad \lim_{x \rightarrow \infty} \Psi(x) = \infty.$$

Their theorem is

Theorem 1.1. *Assume that*

$$(1.5) \quad I_{\Psi, p}(f) \leq B,$$

$$(1.6) \quad \int_0^1 \Psi^{-1} \left(\frac{B}{t^2} \right) dp(t) < \infty.$$

Then we may make f continuous by modifying it on a set of measure 0. The modified f satisfies

$$(1.7) \quad |f(x) - f(y)| \leq 8 \int_0^{|x-y|} \Psi^{-1} \left(\frac{4B}{t^2} \right) dp(t)$$

for all x, y in $[0, 1]$.

If the integral in (1.6) is infinite then the theorem says nothing. In this case, however, there is another theorem [2], which estimates the rate of growth of f :

Theorem 1.2. *Suppose f is decreasing on $(0, 1)$ and satisfies (1.5). Then*

$$(1.8) \quad f(x) - f(\tfrac{1}{2}) \leq 4\Psi^{-1}\left(\frac{B}{x^2}\right)p(x) + 4 \int_x^{1/2} \Psi^{-1}\left(\frac{B}{t^2}\right) dp(t)$$

for $0 < x \leq \frac{1}{2}$.

Obviously the same estimate holds for $f(\frac{1}{2}) - f(1 - x)$.

The proofs of these theorems are easy, but the extension of Theorem 1.2 to non-monotonic functions requires a more difficult theorem of Garsia, Rodemich, and H. Taylor [2, 8]. Let f^* be a decreasing rearrangement of f , i.e., $f^* \downarrow$ on $(0, 1)$ and the sets $\{f > \lambda\}$, $\{f^* > \lambda\}$ have equal measure, for all λ .

Theorem 1.3. *Let Ψ and p satisfy (1.4) and (1.2). Assume also that $\Psi(e^x)$ is convex. Then*

$$I_{\Psi, p}(f^*) \leq I_{\Psi, p}(f).$$

Therefore, provided $\Psi(e^x)$ is convex, we can apply Theorem 1.2 to f^* and obtain: *If f is measurable and $I(f) \leq B$, then (1.8) holds with f replaced by f^* .*

The results in the present paper are also of this nature—we prove them for decreasing f and use Theorem 1.3 to extend them to general f . Our ultimate aim is to get best possible versions of Theorems 1.1 and 1.2 for as wide a variety of Ψ and p as possible. For the present, we develop a theory for $\Psi(x) = x^2$ and obtain concrete results for $p(x) = x^\alpha$ or $x \log^\beta(1/x)$.

Some conventions are in order. First, when we say $p(x) = x^\alpha \log^\beta(1/x)$, we mean this to hold only on some unspecified interval $(0, \delta]$. Secondly, when we say that $\theta(x)$ is like $\varphi(x)$ for x in some domain, we mean that $\theta(x)/\varphi(x)$ is bounded away from 0 and ∞ . The bounds may or may not be uniform in others parameters involved. Thirdly, $\theta(x) \sim \varphi(x)$ as $x \rightarrow 0$ means $\theta(x)/\varphi(x) \rightarrow 1$ as $x \rightarrow 0$.

From now on, $\Psi(x) = x^2$, and p satisfies (1.2). If $0 < a < b < 1$ define

$$(1.9) \quad \sigma[a, b] = \sup \{f(a) - f(b) : f \downarrow \text{ on } (0, 1), I_p(f) \leq 1\}.$$

Clearly, we may restrict ourselves to those f which are constant on $[0, a]$ and $[b, 1]$. Here is a typical “growth result” of this paper: *Let $p(x) = x$. Then $\sigma[x, \frac{1}{2}]$ is like $\log^{1/2}(1/x)$ for $0 < x < 1/4$.* From Theorem 1.3 it follows that

$$(1.10) \quad I(f) \leq 1 \Rightarrow f^*(x) - f^*(\tfrac{1}{2}) \leq \text{const } \log^{1/2}(1/x), \quad 0 < x < 1/4.$$

Furthermore, this result is best possible. On the other hand, Theorem 1.2 plus Theorem 1.3 gives only

$$(1.11) \quad I(f) \leq 1 \Rightarrow f^*(x) - f^*(\tfrac{1}{2}) \leq 4 - 4 \log 2 + 4 \log(1/x).$$

In this connection we should mention the well-known theorem of John and

Nirenberg [5] on functions of “bounded mean oscillation.” A function $f \in L^1(0, 1)$ is said to be of B.M.O. if there is a constant M such that

$$\frac{1}{|I|} \int_I |f - f_I| \leq M$$

for all subintervals I of $[0, 1]$. Here, $|I|$ means the length of I , and

$$f_I = \frac{1}{|I|} \int_I f.$$

The John–Nirenberg Theorem gives that if f is of B.M.O. then

$$f^*(x) - \int_0^1 f \leq \text{const} + \text{const} \log(1/x),$$

which is essentially (1.11). It is easy to see, however, that

$$\frac{1}{|I|} \int_I |f - f_I| \leq \int_0^1 \int_0^1 \left[\frac{f(x) - f(y)}{x - y} \right]^2 dx dy$$

for all $I \subset [0, 1]$. In other words, $I(f) < \infty$ is stronger than the B.M.O. condition, and (1.10) is an appropriately stronger result.

For $p(x) = x^{1-\epsilon}$, $0 < \epsilon < \frac{1}{2}$, the growth result from Theorems 1.2 and 1.3 agrees with the best-possible result our method attains, namely, $\sigma[x, \frac{1}{2}]$ is like $x^{-\epsilon}$. For $p(x) = x^{1+\epsilon}$, both methods predict that any f with $I(f) < \infty$ is essentially continuous, with modulus of continuity $\leq \text{const } x^\epsilon$. For $p(x) = x$, we saw that the older result from Theorem 1.2 is too big by a factor $\log^{1/2}(1/x)$. If we then look at the “boundary cases” $p(x) = x \log^\beta(1/x)$, we find a range of β such that $I(f) < \infty$ implies f essentially continuous, yet the integral in (1.6) is infinite. Hence the older results do not predict continuity, but only a certain rate of growth. If $\beta = -1$, for example, our modulus-of-continuity estimate is like $\log^{-1/2}(1/x)$, whereas Theorem 1.2 gives a rate of growth $\leq \text{const} \log \log(1/x)$.

2. Application of potential theory. Our method uses the theory of capacity. The set of f satisfying (1.1) is a Dirichlet space in the sense of Beurling and Deny. But the capacities we use are not with respect to this space. These f do not play the role of potentials. Indeed, we will transform $I(f)$ in such a way that df takes the role of a charge distribution whose energy is $I(f)$.

Define the positive symmetric kernel K by

$$(2.1) \quad K(x, y) = K(y, x) = 2 \int_0^x ds \int_y^1 dt \frac{1}{p^2(t-s)}, \quad 0 \leq x \leq y \leq 1.$$

Obviously $K(x, 0) = K(0, x) = 0$, and $K(1-x, 1-y) = K(x, y)$.

Theorem 2.1. *For any monotonic function f on $(0, 1)$,*

$$(2.2) \quad I(f) = \int_0^1 \int_0^1 K(x, y) df(x) df(y).$$

Proof. We first write

$$I(f) = 2 \int_0^1 dx \int_x^1 dy \left[\frac{f(y) - f(x)}{p(y - x)} \right]^2.$$

We may assume that f is right-continuous, and replace $[f(y) - f(x)]^2$ by

$$\int_{(x, y]} df(s) \int_{(x, y]} df(t), \quad 0 < x < y < 1.$$

Then interchanging orders of integration gives (2.2). Observe that jumps of f at the endpoints affect neither side of (2.2).

If μ and ν are positive Borel measures on $(0, 1)$ we write

$$E(\mu, \nu) = \int_0^1 \int_0^1 K(x, y) d\mu(x) d\nu(y),$$

$$E(\mu) = E(\mu, \mu),$$

and also use this notation for signed measures μ and ν , provided $E(|\mu|)$ and $E(|\nu|)$ are finite.

Corollary 2.1. i) If f is of locally bounded variation on $(0, 1)$ with total variation function v and $I(v) < \infty$, then $I(f) < \infty$ and (2.2) holds.

ii) The kernel K is positive definite in the sense that for any signed measure μ ,

$$\mu \neq 0, E(|\mu|) < \infty \Rightarrow E(\mu) > 0.$$

Proof. (i) follows from $|f(x) - f(y)| \leq |v(x) - v(y)|$ and the Jordan decomposition.

To prove (ii), let $\mu = df$. If $E(|\mu|) < \infty$ and $E(\mu) = 0$, then $I(f) = 0$, by (i). Hence $f(x) - f(y) = 0$ for a.e. (x, y) . Hence $\mu = 0$.

We now interpret a measure μ on $(0, 1)$ as a charge distribution, and $K(x, \cdot)$ as the potential of a unit charge at x between two grounded points 0 and 1. The function u_μ defined by

$$u_\mu(x) = \int_0^1 K(x, y) d\mu(y)$$

is the *potential* of μ ; the *energy* of μ is

$$E(\mu) = \int_0^1 u_\mu d\mu;$$

the *mutual energy* of μ and ν is

$$E(\mu, \nu) = \int_0^1 u_\mu d\nu = \int_0^1 u_\nu d\mu.$$

Let S be a subset of $(0, 1)$ and set

$$\sigma(S) = \sup \mu(0, 1),$$

the sup being over all positive measures μ with closed support in S such that $E(\mu) \leq 1$. In other words, $\sigma(S)$ is the most charge you can put on S while keeping the energy ≤ 1 . For $S = [a, b]$ this is the same as $\sigma[a, b]$ defined by (1.9).

The set function σ is directly related to another set function C . The *capacity* $C(S)$ of any $S \subset (0, 1)$ is defined as

$$C(S) = \sup \mu(0, 1),$$

the sup being over positive measures μ with closed support in S such that $u_\mu \leq 1$ on S . Thus $C(S)$ is the most charge you can put in S while keeping the potential ≤ 1 . The theory of capacity can be found in L. Carleson's book [1] and C. Preston's thesis [6]. The essential fact is that, for compact F , the variational problem $\sigma(F)$ has a unique solution μ which is the solution of the integral equation

$$u_\mu(x) = \int_F K(x, y) d\mu(y) = \text{const}$$

for all $x \in F$ minus a set of capacity 0.

Since Carleson and Preston present the theory for translation kernels $K(x - y)$, we will run through the sequence of theorems needed for our work. The proofs that can be found in [1] or [6] will be omitted but we will point out the properties of our K that make the proofs go through.

Theorem 2.2. *Let (μ_n) be a sequence of positive finite measures on $[0, 1]$ which converges weakly to μ . Then*

$$\liminf_{n \rightarrow \infty} E(\mu_n) \geq E(\mu).$$

For Preston's proof we merely need to have a sequence of continuous functions K_m that increases to K . For example, we may set

$$p_m(x) = \max(1/m, p(x)),$$

and define K_m by (2.1), with p replaced by p_m .

Lemma 2.1. *For each $x \in (0, 1)$ the function $K(x, \cdot)$ is convex on the intervals $[0, x]$ and $(x, 1]$ (on $[0, x]$ and $[x, 1]$ if $K(x, \cdot)$ is continuous).*

Proof.

$$\begin{aligned} \frac{\partial^2}{\partial y^2} K(x, y) &= \frac{2}{p^2(y-x)} - \frac{2}{p^2(y)}, \quad y > x \\ &= \frac{2}{p^2(x-y)} - \frac{2}{p^2(1-y)}, \quad y < x. \end{aligned}$$

Since p is increasing on $[0, 1]$, the second derivative of $K(x, \cdot)$ is nonnegative.

Theorem 2.3. (The Maximum Principle) *If μ is a positive measure on $(0, 1)$ and $u_\mu \leq 1$ on the closed support S_μ of μ , then $u_\mu \leq 1$ everywhere.*

Proof. Our situation resembles Carleson's, rather than Preston's. As we are working in one dimension, the proof is easy, and we will give it.

Let I be a component interval of $(0, 1) \setminus S_\mu$. Then $u_\mu \leq 1$ at the endpoints of I . Hence we will be done if we show u_μ is convex on $\bar{I}$. But this is an easy consequence of the convexity of K , as follows: If $x, y \in \bar{I}$, $0 \leq \lambda \leq 1$, $z = (1 - \lambda)x + \lambda y$, $t \notin I$, then by Lemma 2.1,

$$K(z, t) \leq (1 - \lambda)K(x, t) + \lambda K(y, t).$$

(One or both sides may be ∞). Now integrate this with respect to $d\mu(t)$ over I to obtain

$$u_\mu(z) \leq (1 - \lambda)u_\mu(x) + \lambda u_\mu(y).$$

Theorem 2.4. *The set function C is increasing and countably subadditive.*

Theorem 2.5. *Let $S \subset (0, 1)$ be a Borel set and μ a positive measure. If u_μ is bounded on S and $C(S) = 0$ then $\mu(S) = 0$.*

The proof of the main theorem to come involves the set function

$$\tau(S) = \inf E(\mu)^{1/2},$$

the inf being over measures μ supported in S such that $\mu(S) \geq 1$. The functions τ and σ are obviously reciprocal:

Lemma 2.2. $\sigma(S) = 1/\tau(S)$ (where $1/\infty = 0$, $1/0 = \infty$).

Proof. Assume $0 < \tau(S) < \infty$, $0 < \sigma(S) < \infty$. Suppose μ is supported in S , $E(\mu) \leq 1$. Then

$$E(\theta\tau(S)\mu) < \tau^2(S)$$

for all $0 < \theta < 1$. By the definition of $\tau(S)$, $\theta\tau(S)\mu(S) < 1$ and hence $\mu(S) < 1/\theta\tau(S)$ for all $0 < \theta < 1$. This shows $\sigma(S) \leq 1/\tau(S)$. By going the other way, we prove $\sigma(S) \geq 1/\tau(S)$.

The cases $\tau(S)$ or $\sigma(S) = 0, \infty$ are just as easily handled.

If something happens except on a set of capacity 0, we say it happens *C-a.e.* Here is the main result.

Theorem 2.6. *Let F be a compact subset of $(0, 1)$. Then $C(F) < \infty$, and $C(F) = \sigma^2(F)$. There is a unique measure μ_0 supported in F , called the equilibrium measure for F , such that*

- i) $u_{\mu_0} \leq 1$ everywhere, $u_{\mu_0} = 1$ *C-a.e.* on F
- ii) $\mu_0(F) = C(F)$.

Furthermore, the $\sigma(F)$ and $\tau(F)$ variational problems are both solved uniquely by multiples of μ_0 , namely $\mu_0/\sigma(F)$ and $\mu_0/C(F)$.

In fact, the solution μ_1 of the $\tau(F)$ problem is first obtained as a weak limit of approximate solutions. It is proved that $u_{\mu_1} = \text{const } C\text{-a.e. on } F$. Then μ_0 is defined as a multiple of μ_1 .

We are finally at the point of application to our own problem. Recall that we wanted to evaluate

$$\begin{aligned}\sigma[a, b] &= \sup \{f(a) - f(b) : f \downarrow, \quad I(f) \leq 1\} \\ &= \sup \{\mu[a, b] : \mu \text{ positive, } E(\mu) \leq 1\}.\end{aligned}$$

Letting $[a, b] = F$ in Theorem 2.6, we see that there is a maximizing f for this problem, unique to within an additive constant and values at jumps. To obtain $\sigma[a, b]$ we may solve the integral equation

$$(2.3) \quad \begin{aligned}\int_{[a, b]} K(x, y) d\mu_0(y) &\leq 1 \quad \text{on } F \\ &= 1 \quad C\text{-a.e. on } F\end{aligned}$$

for the positive measure μ_0 . Then $C[a, b] = \mu_0[a, b]$ and $\sigma[a, b] = C[a, b]^{1/2}$.

We do not, however, demand to calculate $\sigma[a, b]$ exactly. We will be content here to estimate $\sigma[a, b]$ to within multiplicative constants for fixed b and all small a . To do this we do not have to solve (2.3) exactly; an approximate solution will do. More precisely,

Theorem 2.7. *Suppose the measure μ is supported in the compact set F , and there are positive constants m, M such that*

$$m \leq u_\mu \leq M \text{ on } F.$$

Then

$$\mu(F)/M \leq C(F) \leq \mu(F)/m.$$

Proof. Let μ_0 be the equilibrium measure for F . Then

$$E(\mu, \mu_0) = \int_F u_{\mu_0} d\mu = \int_F u_\mu d\mu_0.$$

We know $u_{\mu_0} = 1$ on F except for a set of capacity 0. Since μ has a bounded potential, this set has μ -measure 0, by Theorem 2.5. Therefore

$$\int_F u_{\mu_0} d\mu = \mu(F).$$

On the other hand, $\mu_0(F) = C(F)$ and

$$m\mu_0(F) \leq \int_F u_\mu d\mu_0 \leq M\mu_0(F).$$

3. The approximate equilibrium measures. Given p satisfying (1.2), we are trying to calculate a function $\varphi(x)$ such that $\sigma[x, b]$ (for some fixed b) is like $\varphi(x)$ for all small x . According to Theorem 2.7, we can do this by finding, for

each x , a measure μ_x which sits on $[x, b]$ and whose potential function is like 1 on $[x, b]$, uniformly in x . Then $\sigma^2[x, b]$ is like $\mu_x[x, b]$. We have been able to do this for special cases of p by the following heuristic argument. Suppose that $K(x, y)$ is like a δ -function: $K(x, y) = k(x)\delta(y - x)$, and we look for a measure $\rho(y) dy$ whose potential is 1. Thus

$$\int K(x, y)\rho(y) dy = k(x)\rho(x) = 1,$$

$$\rho(x) = 1/k(x).$$

Now we might identify $k(x)$ with $\int_0^1 K(x, y)dy$. If x is near 0, however, the long tail of $K(x, \cdot)$ extending toward 1 contributes too much to this integral. Therefore we use $\int_0^x K(x, y)dy$ for $k(x)$, and set

$$(3.1) \quad \rho(x) = \left[\int_0^x K(x, y) dy \right]^{-1}.$$

We must then look at the potential of the measure $\rho(y)dy$, restricted to $[x, b]$.

It will be useful to have an asymptotic approximation for $K(x, y)$, x and y small. Define

$$(3.2) \quad P(x) = \int_x^1 \frac{du}{p^2(u)}, \quad 0 < x \leq 1.$$

If $P \in L^1(0, 1)$ let

$$q(x) = 2 \int_0^x P(t) dt, \quad 0 \leq x \leq 1.$$

If $P \notin L^1(0, 1)$ let

$$q(x) = 2 \int_x^1 P(t) dt, \quad 0 < x \leq 1.$$

Theorem 3.1.

$$(3.3) \quad K(x, y) \sim |q(y) - q(y - x)|, \quad x < y, \quad x \rightarrow 0,$$

uniformly in y for $0 < y \leq \frac{1}{2}$.

Proof. For $x < y$,

$$\begin{aligned} K(x, y) &= 2 \int_0^x ds \int_{y-s}^{1-s} \frac{du}{p^2(u)} \\ &= 2 \int_0^x [P(y - s) - P(1 - s)] ds. \end{aligned}$$

If $0 \leq s \leq x < y \leq \frac{1}{2}$, $x \leq \delta$, then

$$\frac{P(1 - s)}{P(y - s)} \leq \frac{P(1 - \delta)}{P(\frac{1}{2})},$$

which tends to 0 with δ . Therefore as $x \rightarrow 0$,

$$(3.4) \quad K(x, y) \sim 2 \int_0^x P(y - s) ds = 2 \int_{y-x}^y P(t) dt,$$

and the theorem follows.

We will actually use (3.4) rather than (3.3).

Corollary 3.1.

$$(3.5) \quad \int_0^x K(x, y) dy \sim 2 \int_0^x P(t)t dt \quad \text{as } x \rightarrow 0.$$

Proof. We just plug in the right side of (3.4) for $K(x, y)$ (with x and y interchanged), and interchange orders of integration.

It is possible that the integral

$$(3.6) \quad \int_0^1 P(t)t dt = \int_0^1 \frac{t^2 dt}{p^2(t)}$$

is infinite. Later, we will show that in this case the only functions f with $I(f) < \infty$ are the constants.

4. Capacities for $p(x) = x^\alpha$, $\frac{1}{2} < \alpha < \frac{3}{2}$. Growth results when $\frac{1}{2} < \alpha \leq 1$. For these p we get, from (3.1), (3.2), and (3.5), that

$$\begin{aligned} P(x) &\sim \frac{1}{(2\alpha - 1)^{2\alpha - 1}}, \quad \text{as } x \rightarrow 0. \\ \rho(x) &\sim \frac{(2\alpha - 1)(3 - 2\alpha)}{2} \frac{1}{x^{3 - 2\alpha}} \end{aligned}$$

We will reset

$$\rho(x) = \frac{1}{x^{3 - 2\alpha}}$$

and test the potential u of the measure $\rho(y)dy$ on intervals $[a, b]$ for likeness to 1 on $[a, b]$, uniformly in a and b .

First define

$$v(s, t) = \int_s^t K(t, y)\rho(y) dy,$$

$$w(s, t) = \int_s^t K(s, y)\rho(y) dy.$$

Then

$$(4.1) \quad u(x) = v(a, x) + w(x, b), \quad a \leq x \leq b.$$

Lemma 4.1. *If $0 < 2s \leq t$ and t is small enough, then $v(s, t)$ and $w(s, t)$ are both like 1.*

Proof. If we keep t small enough, then whenever K occurs we may use (3.4).

Take v first. Write

$$v(s, t) = \int_{t/2}^t K(t, y) \rho(y) dy + \int_0^{t/2} K(t, y) \rho(y) dy = v_1 + v_2.$$

For $t/2 \leq y \leq t$, $\rho(y)$ is like $\rho(t)$, for all the ρ under consideration. Accordingly, v_1 is like

$$\rho(t) \int_{t/2}^t dy \int_{t-y}^t du P(u) = \rho(t) \left[t/2 \int_{t/2}^t P(u) du + \int_0^{t/2} P(u) u du \right].$$

This is like 1, as can immediately be verified for the various α .

The quantity v_2 is like

$$\int_0^{t/2} dy \rho(y) \int_{t-y}^t du P(u).$$

The inner integral is like $yP(t)$ since $P(t/2)$ is like $P(t)$. Hence

$$v_2 \leq \text{const } P(t) \int_0^{t/2} \rho(y) y dy \leq \text{const}.$$

Therefore $v = v_1 + v_2$ is like 1.

We similarly write

$$w(s, t) = \int_s^{2s} K(s, y) \rho(y) dy + \int_{2s}^t K(s, y) \rho(y) dy = w_1 + w_2.$$

First, w_1 is like

$$\rho(s) \int_s^{2s} dy \int_{y-s}^y du P(u) = \rho(s) \int_s^{2s} dy \int_s^y du P(u) + \rho(s) \int_s^{2s} dy \int_{y-s}^s du P(u).$$

The first term of the right side is like

$$\rho(s) P(s) \int_s^{2s} (y - s) dy,$$

and the second term is precisely

$$\rho(s) \int_0^s P(u) u du.$$

Both of these are like 1.

The term w_2 is like

$$\int_{2s}^t dy \rho(y) \int_{y-s}^y du P(u).$$

The inner integral is like $sP(y)$. Therefore

$$w_2 \leq \text{const } s \int_s^t \rho(y)P(y) dy \leq \text{const } s \int_s^\infty y^{-2} dy = \text{const}.$$

This proves that w is like 1.

Theorem 4.1. *Let $p(x) = x^\alpha$, $\frac{1}{2} < \alpha < \frac{3}{2}$. For $0 < a \leq b/4$, and b sufficiently small, the potential u of the measure $y^{2\alpha-3}dy$ on $[a, b]$ is like 1 on $[a, b]$, uniformly in a and b (but not necessarily in α).*

Proof. Observe that $v(s, t)$ is a decreasing function of s , and $w(s, t)$ is an increasing function of t .

Suppose first that $a \leq x \leq 2a$. Then $v(a, x) \leq v(x/2, x)$, and so by (4.1),

$$w(x, b) \leq u(x) \leq v(x/2, x) + w(x, b).$$

By Lemma 4.1, the left and right sides of this are both like 1 if b is small enough, since $2x \leq b$.

Suppose next that $2a \leq x \leq b/2$. Then both $v(a, x)$ and $w(x, b)$ are like 1.

Finally, if $b/2 \leq x \leq b$, then $w(x, b) \leq w(x, 2x)$, and so

$$v(a, x) \leq u(x) \leq v(a, x) + w(x, 2x).$$

The left and right sides are like 1, since $2a \leq x$.

According to Theorem 2.7, $C[a, b]$ is now like

$$\int_a^b y^{2\alpha-3} dy,$$

and by Theorem 2.6, $\sigma[a, b] = C[a, b]^{1/2}$. Therefore we get:

Theorem 4.2. *Let $p(x) = x^\alpha$, $\frac{1}{2} < \alpha < \frac{3}{2}$. For $0 < a \leq b/4$ and b sufficiently small, $\sigma[a, b]$ is like*

$$\begin{aligned} a^{\alpha-1} & \text{ if } \frac{1}{2} < \alpha < 1 \\ \log^{1/2}(b/a) & \text{ if } \alpha = 1 \\ b^{\alpha-1} & \text{ if } 1 < \alpha < \frac{3}{2} \end{aligned}$$

(nonuniformly in α).

From this, we obtain growth results in the first two cases, where $\sigma[a, b] \rightarrow \infty$ as $a \rightarrow 0$.

Theorem 4.3. *Let $p(x) = x^\alpha$, $\frac{1}{2} < \alpha \leq 1$. For sufficiently small x , $\sigma[x, \frac{1}{2}]$ is like*

$$\begin{aligned} x^{\alpha-1} & \text{ if } \frac{1}{2} < \alpha < 1 \\ \log^{1/2}(1/x) & \text{ if } \alpha = 1. \end{aligned}$$

Proof. The set function σ is obviously monotone and subadditive. If $0 < x < b \leq \frac{1}{2}$ then

$$\sigma[x, b] \leq \sigma[x, \tfrac{1}{2}] \leq \sigma[x, b] + \sigma[b, \tfrac{1}{2}].$$

We fix b small enough for Theorem 4.2. Then $\sigma[x, \frac{1}{2}] \sim \sigma[x, b]$ as $x \rightarrow 0$. Theorem 4.2 completes the proof.

For $\frac{1}{2} < \alpha < 1$, this growth result is like the Theorem 1.2 estimate, but for $\alpha = 1$, Theorem 1.2 gives only $\log(1/x)$.

5. Continuity results for $p(x) = x^\alpha$, $1 < \alpha < \frac{3}{2}$. For this case, Theorem 4.2 shows that for a fixed b , $\sigma[a, b]$ is bounded as $a \rightarrow 0$. This will quickly give that $I(f) < \infty$ implies f bounded. Define

$$\sigma(0, b] = \sup \{f(0+) - f(b) : f \downarrow, I(f) \leq 1\}.$$

Lemma 5.1. *For each b in $(0, 1)$,*

$$\sigma(0, b] = \lim_{x \rightarrow 0} \sigma[x, b].$$

Proof. The limit exists since $\sigma[x, b] \uparrow$ as $x \downarrow$.

It is obvious that “ $\geq$ ” holds, for if $I(f) \leq 1$, $f \downarrow$, then

$$f(x) - f(b) \leq f(0+) - f(b) \leq \sigma(0, b].$$

Let L be the above limit, and suppose $\sigma(0, b] > L$. Then there is an $f \downarrow$ with $I(f) \leq 1$ and $f(0+) - f(b) > L$. But then for small x , $f(x) - f(b) > L$. Hence $\sigma[x, b] > L$, which is impossible.

Theorem 5.1. *Let $p(x) = x^\alpha$, $1 < \alpha < \frac{3}{2}$.*

- i) *For small x , $\sigma(0, x]$ is like $x^{\alpha-1}$.*
- ii) *If $I(f) < \infty$, then f is essentially bounded.*

Proof. (i) follows from Lemma 5.1 and Theorem 4.2. For (ii) we observe

$$\sigma(0, \tfrac{1}{2}] \leq \sigma(0, x] + \sigma[x, \tfrac{1}{2}], \quad 0 < x < \tfrac{1}{2}.$$

Therefore $\sigma(0, \frac{1}{2}]$ is finite. By symmetry, so is the corresponding quantity $\sigma[\frac{1}{2}, 1)$. Therefore we have (ii) for decreasing f . But then Theorem 1.3 gives it for general f , since $f \in L^\infty$ if and only if f^* is bounded.

As we pointed out in the introduction, Theorem 1.1 predicts continuity for f if $I(f) < \infty$ and $p(x) = x^\alpha$, $\alpha > 1$. Indeed, (1.7) gives

$$|f(x) - f(y)| \leq \frac{16\alpha}{\alpha - 1} |x - y|^{\alpha-1}$$

if $I(f) \leq 1$. Theorem 5.1 (i) gives a like modulus of continuity for monotone functions at the endpoints. We now set out to show that we can make our method predict not only boundedness, but continuity, of any f with $I(f) < \infty$.

We use the following change of scale argument, which was also used in one

proof of Theorem 1.1. Return to a general p satisfying (1.2). For $0 < r \leq 1$ let $I_r(f)$ denote the integral (1.1), except that $p(x)$ is replaced by the function $p_r(x) = p(rx)$.

Suppose $I(f) \leq 1$. Then for $0 \leq x < y \leq 1$,

$$(5.1) \quad \int_x^y \int_x^y \left[\frac{f(s) - f(t)}{p(s-t)} \right]^2 ds dt \leq 1.$$

Let $r = y - x$. If we make the change of variables $s = x + ru$, $t = y + rv$, then (5.1) becomes

$$(5.2) \quad I_r(g) \leq 1/r^2,$$

where $g(u) = f(x + ru)$, $0 \leq u \leq 1$.

Define $\sigma_r(S)$ using the integral I_r , just as $\sigma(S)$ is defined using I . In particular let

$$\sigma_r(0, 1) = \sup \{g(0+) - g(1-): g \downarrow, I_r(g) \leq 1\}.$$

Then (5.2) and Theorem 1.3 imply

$$I_r(rg^*) \leq 1,$$

and hence

$$g^*(0+) - g^*(1-) \leq r^{-1}\sigma_r(0, 1).$$

Translating back to f we have

Lemma 5.2. *Let $I(f) \leq 1$ (general p). If $0 \leq x < y \leq 1$, and $r = y - x$, then for a.e. s and a.e. t in $[x, y]$,*

$$(5.3) \quad |f(s) - f(t)| \leq r^{-1}\sigma_r(0, 1).$$

To investigate $\sigma_r(0, 1)$, we look at the kernel K_r associated with the integral I_r . For $x < y$,

$$K_r(x, y) = 2 \int_0^x ds \int_y^1 dt \frac{1}{p^2(r(t-s))}.$$

Recall the definition of P in (3.2).

Lemma 5.3. *Assume that $P(2x)/P(x)$ is bounded away from 1 for $0 < x \leq \frac{1}{2}$. Then $K_r(x, y)$ is like*

$$r^{-2}K(rx, ry)$$

for $0 \leq x, y \leq \frac{1}{2}$, $0 < r \leq 1$.

Proof. For $x < y$,

$$(5.4) \quad K_r(x, y) = \frac{2}{r^2} \int_0^{rx} ds \int_{ry}^1 dt \frac{1}{p^2(t-s)}$$

$$= \frac{2}{r^2} \int_0^{rx} [P(ry - s) - P(r - s)] ds.$$

On the other hand, from the proof of Theorem 3.1,

$$(5.5) \quad \frac{1}{r^2} K(rx, ry) = \frac{2}{r^2} \int_0^{rx} [P(ry - s) - P(1 - s)] ds.$$

Consider (5.4). If $0 \leq s \leq rx < ry \leq r/2$, then

$$(r - s)/(ry - s) \geq r/(ry) \geq 2.$$

Therefore, by our assumption, $P(r - s)/P(ry - s)$ is bounded away from 1 (from below). This is also true of the ratio $P(1 - s)/P(ry - s)$ from (5.5), since $r - s \leq 1 - s$. It follows that both $K_r(x, y)$ and $r^{-2}K(rx, ry)$ are like

$$\frac{2}{r^2} \int_0^{rx} P(ry - s) ds,$$

uniformly in r .

By relating K and K_r , we can also relate the corresponding capacities C and C_r .

Lemma 5.4. *For any set $S \subset (0, \frac{1}{2}]$, $C_r(S)$ is like $r^2 C(rS)$ for $0 < r \leq 1$, where $rS = \{rx : x \in S\}$.*

Proof. From the last lemma, there is an $m > 1$ such that

$$\frac{1}{m} \frac{K(rx, ry)}{r^2} < K_r(x, y) < m \frac{K(rx, ry)}{r^2}$$

for $0 \leq x, y \leq \frac{1}{2}$, $0 < r \leq 1$.

By the definition of $C_r(S)$, there is a measure df supported in S such that

$$\int_S K_r(x, y) df(y) \leq 1, \quad x \in S,$$

$$(5.6) \quad \int_S df \geq \frac{1}{2} C_r(S).$$

Therefore

$$\int_S r^{-2} K(rx, ry) df(y) \leq m,$$

$$\frac{1}{mr^2} \int_{rS} K(rx, s) df(s/r) \leq 1, \quad rx \in rS.$$

Since $df(s/r)$ is supported in rS , this gives that

$$\frac{1}{mr^2} \int_{rS} df(s/r) \leq C(rS),$$

$$\frac{1}{mr^2} \int_S df \leq C(rS).$$

Combining this with (5.6), we get

$$C_r(S) \leq 2mr^2 C(rS).$$

Working the argument the other way gives an opposite inequality.

It is now easy to prove a continuity result.

Theorem 5.2. *Let the assumption on P of Lemma 5.3 hold, and assume that $\sigma(0, x) \rightarrow 0$ as $x \rightarrow 0$. If $I(f) \leq 1$, then we may make f continuous on $[0, 1]$ by adjusting it on a set of measure 0. For the adjusted f ,*

$$(5.7) \quad |f(x) - f(y)| \leq \text{const } \sigma(0, \tfrac{1}{2} |x - y|), \quad 0 \leq x, y \leq 1,$$

where the constant depends only on the function p .

Proof. From Lemma 5.4 and Theorem 2.6 we get that $\sigma_r[a, \frac{1}{2}]$ is like $r\sigma[ra, r/2]$ for $0 < a < \frac{1}{2}$, $0 < r \leq 1$. Letting $a \rightarrow 0$ we see that $\sigma_r(0, \frac{1}{2})$ is like $r\sigma(0, r/2)$. Since $\sigma_r(0, \frac{1}{2}) \leq \sigma_r(0, 1) \leq 2\sigma_r(0, \frac{1}{2})$, the right side of (5.3) may be replaced by

$$\text{const } \sigma(0, r/2) = \text{const } \sigma(0, \tfrac{1}{2} |x - y|).$$

A little real-variable theory then gives the essential continuity of f , plus the estimate (5.7).

Returning to the case $p(x) = x^\alpha$, $1 < \alpha < 3/2$, we see from Theorems 5.1 and 5.2 that the functions f with $I(f) \leq 1$ are essentially continuous, and

$$|f(x) - f(y)| \leq \text{const } |x - y|^{\alpha-1}.$$

6. The boundary cases $p(x) = x \log^\beta(1/x)$. We look at these p in order to compare our results more closely with Theorems 1.1 and 1.2. From (3.1), (3.2), and (3.5),

$$\begin{aligned} P(x) &\sim (1/x) \log^{-2\beta}(1/x) \quad \text{as } x \rightarrow 0. \\ \rho(x) &\sim (1/2x) \log^{2\beta}(1/x) \end{aligned}$$

The reader may easily verify that Lemma 4.1 goes through as before. We then get the following growth and continuity results:

$$(6.1) \quad \text{If } \beta > -\tfrac{1}{2}, \text{ then } \sigma[x, \tfrac{1}{2}] \text{ is like } \left(\log \frac{1}{x}\right)^{\beta+\frac{1}{2}}.$$

$$(6.2) \quad \text{If } \beta = -\tfrac{1}{2}, \text{ then } \sigma[x, \tfrac{1}{2}] \text{ is like } \left(\log \log \frac{1}{x}\right)^{\frac{1}{2}}$$

$$(6.3) \quad \text{If } \beta < -\tfrac{1}{2}, \text{ then } \sigma(0, x) \text{ is like } \left(\log \frac{1}{x}\right)^{\beta+\frac{1}{2}}.$$

We compare these with Theorems 1.1, 1.2, and 1.3, which give, for $I(f) \leq 1$:

$$(6.4) \quad \text{If } \beta > -1, \text{ then } f^*(x) - f^*\left(\frac{1}{2}\right) \leq \text{const} \left(\log \frac{1}{x} \right)^{\beta+1}.$$

$$(6.5) \quad \text{If } \beta = -1, \text{ then } f^*(x) - f^*\left(\frac{1}{2}\right) \leq \text{const} \log \log \frac{1}{x}.$$

$$(6.6) \quad \text{If } \beta < -1, \text{ then } |f(x) - f(y)| \leq \text{const} \left(\log \frac{1}{|x - y|} \right)^{\beta+1}.$$

In general, these earlier estimates are too large by a factor $\log^{1/2}(1/x)$. Moreover, if $-1 \leq \beta < \frac{1}{2}$, then (6.3) and Theorem 5.2 predict continuity of f if $I(f) < \infty$, whereas (6.4) and (6.5) only predict a certain rate of growth.

7. The trivial case $p(x) = x^\alpha$, $\alpha \geq \frac{3}{2}$. We will show that in this case the only functions f with $I(f) < \infty$ are the constants. In fact, this happens if and only if the integral

$$(7.1) \quad \int_0^1 \frac{t^2 dt}{p^2(t)}$$

is infinite.

Lemma 7.1. *If (7.1) is finite, then $K(x, \cdot) \in L^1(0, 1)$ for all $x \in (0, 1)$. If (7.1) is infinite, then $K(x, \cdot) \notin L^1$ for no $x \in (0, 1)$.*

Proof. First observe

$$(7.2) \quad \begin{aligned} \int_0^1 K(x, y) dy &= \int_0^x K(x, y) dy + \int_x^1 K(x, y) dy \\ &= \int_0^x K(x, y) dy + \int_0^{1-x} K(1-x, y) dy. \end{aligned}$$

Then

$$\begin{aligned} \int_0^x K(x, y) dy &= 2 \int_0^x dy \int_0^y ds \int_x^1 dt \frac{1}{p^2(t-s)} \\ &= 2 \int_0^x dz z \int_z^{1+z-x} \frac{du}{p^2(u)}. \end{aligned}$$

By inspecting the region of integration, we see that the last integral is equal to

$$\int_0^m \frac{u^2 du}{p^2(u)}$$

(where $m = \min(x, 1-x)$), plus an integral which is always finite. This plus (7.2) proves the lemma.

Lemma 7.2. *If S is a Borel set of capacity zero then any measure μ supported in S has infinite energy.*

Proof. Let $E_n = \{u_\mu \leq n\} \cap S$. Then $C(E_n) = 0$ and u_μ is bounded on E_n . By Theorem 2.5, $\mu(E_n) = 0$. Hence $\mu(\{u_\mu < \infty\} \cap S) = 0$. Therefore

$$E(\mu) = \int_S u_\mu d\mu = \infty.$$

Theorem 7.1. *If (7.1) is finite, then any f on $(0, 1)$ satisfying a Lipschitz condition has finite $I(f)$. If (7.1) is infinite, then $I(f) < \infty$ only when $f = \text{const a.e.}$*

Proof. The first part is immediate.

Suppose (7.1) is infinite. We will show that $E(\mu) = \infty$ for any nonzero μ on $(0, 1)$. This will prove the theorem. For if f is not constant a.e. and $I(f) < \infty$, then $E(df^*) = I(f^*) \leq I(f)$ by Theorems 1.3 and 2.1, and df^* is a nonzero measure.

Let ν be any measure on $(0, 1)$. Then

$$E(\nu, dx) = \int_0^1 u_\nu(x) dx = \int_0^1 d\nu(y) \int_0^1 dx K(x, y) = \infty,$$

by Lemma 7.1. Therefore u_ν is unbounded. Since every nonzero measure has an unbounded potential, the capacity of every set must be zero. Lemma 7.2 completes the proof.

A version of this theorem for general Ψ and p has been given by Preston [7]. His theorem is for the circle group rather than $(0, 1)$.

8. The cases $p(x) = x^\alpha$, $0 < \alpha \leq \frac{1}{2}$. We consider first $0 < \alpha < \frac{1}{2}$. We abandon the heuristic argument of section 3, for the situation is quite different from the preceding cases. In fact,

$$P(x) \sim 1/(1 - 2\alpha) \quad \text{as } x \rightarrow 0,$$

(8.1)

$$K(x, y) \sim 2 \int_{y-x}^y P \sim 2x/(1 - 2\alpha), \quad x < y, \quad y \rightarrow 0.$$

This K is like the Newtonian potential of a plane charge between grounded plates. Let δ_a be the δ -function concentrated at a . Its potential function is $K(a, x)$. Now (8.1) shows that $K(a, x)$ is close to the constant $2a/(1 - 2\alpha)$ for $a \leq x \leq b$, if b is small. Thus δ_a serves as an approximate equilibrium measure. By Theorem 2.7,

$$C[a, b] \sim (1 - 2\alpha)/2a \quad \text{as } b \rightarrow 0, \quad a < b.$$

Then the proof of Theorem 4.3 gives

$$(8.2) \quad \sigma^2[x, \frac{1}{2}] \sim (1 - 2\alpha)/2x \quad \text{as } x \rightarrow 0$$

But this is no stronger than what we would get if we set $I(f) = \int_0^1 f^2$!

By comparison, Theorem 1.2 gives

$$\sigma[x, \tfrac{1}{2}] \leq \text{const } x^{\alpha-1},$$

which is too large by a factor $x^{\alpha-\frac{1}{2}}$.

The boundary case $p(x) = x^{1/2}$ is curious in that we have not found approximate equilibrium measures for intervals $[a, b]$. Nevertheless, the $\sigma[a, b]$ problem is solved approximately by a multiple of δ_a , even though the potential $K(a, x)$ is hardly like 1. Since

$$K(a, b) \leq K(a, x) \leq K(a, a), \quad a \leq x \leq b,$$

Theorem 2.7 gives

$$(8.3) \quad 1/K(a, a) \leq C[a, b] \leq 1/K(a, b).$$

By (3.2) and (3.4),

$$\begin{aligned} P(x) &\sim \log(1/x), \\ K(x, x) &\sim 2 \int_0^x P(t) dt \sim 2x \log(1/x) \text{ as } x \rightarrow 0. \end{aligned}$$

Hence

(8.4)

$$C[x, \tfrac{1}{2}] \geq \text{const} \left(x \log \frac{1}{x} \right)^{-1}.$$

To reverse this inequality, we will use the subadditivity of capacity. Suppose $2^{-n} \leq x < 2^{-n+1}$, $n = 2, 3, \dots$. Then

$$C[x, \tfrac{1}{2}] \leq \sum_{k=1}^{n-1} C[2^{-k-1}, 2^{-k}].$$

By (3.4) and (8.3),

$$K(t, 2t) \sim 2 \int_t^{2t} \log(1/u) du \sim 2t \log(1/t) \text{ as } t \rightarrow 0,$$

$$C[t, 2t] \leq \frac{\text{const}}{t \log(1/t)}, \quad 0 < t \leq 1/4.$$

Hence

$$C[x, \tfrac{1}{2}] \leq \text{const} \sum_{k=1}^{n-1} \frac{2^{k+1}}{k+1} \leq \text{const } 2^n/n \leq \text{const} \left(x \log \frac{1}{x} \right)^{-1}.$$

It follows that $\sigma[x, \frac{1}{2}]$ is like

$$\left(x \log \frac{1}{x} \right)^{-1/2}.$$

The result of Theorem 1.2,

$$\sigma[x, \tfrac{1}{2}] \leq \text{const } x^{-1/2},$$

is again too large by a factor $\log^{1/2}(1/x)$.

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Date communicated: APRIL 5, 1971