Singular Sets of Inner Functions

P. R. AHERN

Communicated by P. R. HALMOS

We give some results about zero sets and singular sets of inner functions in a polydisc. By singular set we mean the set of points on the distinguished boundary of the polydisc over which the function has no analytic continuation.

We derive the exact form that an inner function must have in order that it have an analytic continuation over a lower dimensional torus. The method of proof yields a not entirely obvious way to construct inner functions.

In one variable a rational inner function cannot have any singularities on the circle. However in more than one variable rational inner functions may have singularities on the torus. We show that there is a sense in which the singular set of a rational inner function must be much smaller than the singular set of a non-rational inner function. Using this criterion it is easy to construct closed sets F such that F is not the singular set of any inner function, just because F is too "large" to be the singular set of a rational inner function and too "small" to be the singular set of a non-rational inner function.

We use the standard notations, $\mathbf{C}$ denotes the complex numbers, U the open unit disc in $\mathbf{C}$ and T the boundary of U . We consider functions bounded and holomorphic in U^N , N a positive integer. Such a function f has finite radial limits almost everywhere on the torus T^N ; the limit function is again denoted by f . If $|f| = 1$ almost everywhere on T^N we say that f is an inner function. In the notes [2], Rudin uses the "edge of the wedge" theorem to prove the following result about the boundary behavior of inner functions.

Theorem A. *If f is inner in U^N and $z_0 \in T^N$, then either there exists a sequence $z_n \in U^N$ such that $f(z_n) \rightarrow 0$ and $z_n \rightarrow z_0$ or f has an analytic continuation across the point z_0 .*

Suppose k, l are positive integers with $k + l = N$, then we write $\mathbf{C}^N = \mathbf{C}^k \times \mathbf{C}^l$, $U^N = U^k \times U^l$ and $T^N = T^k \times T^l$. Let π denote the coordinate projection of $\mathbf{C}^N$ onto $\mathbf{C}^k$. For an inner function f define

$$\sigma(f) = \{z \in T^N: \exists z_n \in U^N, z_n \rightarrow z, f(z_n) \rightarrow 0\}$$

$$Z(f) = \{z \in U^N: f(z) = 0\}$$

$$\sum (f) = \overline{Z(f)} \cup \sigma(f).$$

Theorem 1. Suppose $\xi \in T^k$ and $\xi \notin \pi(\sum (f))$, then f is independent of $z_{k+1}, \dots, z_N$.

Proof. Since $\xi \notin \pi(\overline{Z(f)})$, $\exists$ a neighborhood Δ (in $\mathbf{C}^k$) of ξ such that f does not vanish on $(\Delta \cap U^k) \times U^l$. Since $\xi \notin \pi(\sigma(f))$, $\exists$ an open set $A = \{(w^l, \dots, w_l) : r < |w_i| < 1\}$, for some $0 < r < 1$, such that f is bounded away from zero on $(\Delta \cap U^k) \times U^l$. By Theorem A, f has an analytic continuation across $\Gamma \times T^l$ where $\Gamma = \Delta \cap T^k$. We can continue f analytically across $\Gamma \times \bar{U}^l$ just by Cauchy's integral formula. Consider the function $g = 1/f$ defined in $(\Delta \cap U^k) \times U^l$; g is holomorphic there because f is not zero there, also g has an analytic continuation across $\Gamma \times T^l$ because $|f| = 1$ on $\Gamma \times T^l$, f being inner. Again by Cauchy's integral formula g has an analytic continuation across $\Gamma \times \bar{U}^l$. In other words f is not zero on $\Gamma \times U^l$. Now take $z \notin \Gamma$, then $f(z, \cdot)$ is an inner function in U^l which is analytic on $\bar{U}^l$ and never zero, hence $f(z, \cdot)$ is constant in w for each $z \in \Gamma$. So, for $1 \leq j \leq l$, $(\partial f / \partial w_j)(z, w) \equiv 0$ on $\Gamma \times U^l$. It is easy to see that this implies that $(\partial f / \partial w_j) \equiv 0$ in U^N for $1 \leq j \leq l$.

Remark. If f is an inner function in U^k then $F(z, w) = f(z)$ is an inner function in U^N and $\pi(\sum (F)) = \sum (f) \subseteq T^k$ and one sees that $\sum (f)$ need not be all of T^k .

Corollary. Suppose f is inner and not independent of $z_{k+1}, \dots, z_N$, suppose $\Delta \subseteq U^k$ is open, $\xi \in T^k \cap \bar{\Delta}$ and $\Delta \cap \pi(\sum (f)) = \emptyset$, then f cannot have an analytic continuation over $\{\xi\} \times T^l$.

Proof. Suppose f had an analytic continuation over $\{\xi\} \times T^l$, then it would have a continuation over $\{\xi\} \times \bar{U}^l$ as before. Since f is not zero in $\Delta \times U^l$, the same argument used in the proof of Theorem 1 shows that the inner function $f(\xi, \cdot)$ is constant, a constant of modulus 1. However by Theorem 1 $\exists z_n \in U^k$, $z_n \rightarrow \xi$, and $w_n \in U^l$, $w_n \rightarrow w_0 \in \bar{U}^l$ such that $f(z_n, w_n) \rightarrow 0$. However since f is holomorphic in a neighborhood of $\{\xi\} \times \bar{U}^l$ we must have $f(\xi, w_0) = 0$, a contradiction.

In particular if f is not a function of $z_1, \dots, z_k$ alone and f doesn't vanish in U^N , then we may take $\Delta = U^k$ in the last corollary. This says that non-vanishing inner functions that depend on all the variables must have a lot of singularities.

In general one can't expect to learn much about f by looking at $\pi(\sum (f))$ since if f is any inner function and we multiply it by the monomial z_N we get an inner function g such that $\pi(\sum (g)) = \bar{U}^k$. So now we turn to $\sigma(f)$ which only takes into account those zeros of f near T^N . From Theorem A we see that $\xi \notin \pi(\sigma(f))$ if and only if f has an analytic continuation across $\{\xi\} \times T^l$ and hence across $\{\xi\} \times \bar{U}^l$. Before proceeding we need some facts about rational inner functions. The following can be found in [3], see also [1]. If p and q are relatively prime polynomials in N variables such that p/q is an inner function,

then q has no zero in U^N and if $q(z) = \sum a_\alpha z^\alpha$ (we use the standard multi-index notation) then $p(z)$ must have the form $p(z) = c \sum \bar{a}_\alpha z^{\beta-\alpha}$ where $|c| = 1$ and the multi-index β has the property that $\beta_i - \alpha_i \geq 0$ for all α such that $a_\alpha \neq 0$.

Now we return to our original setting, $\xi \in T^k$, $\xi \notin \pi(\sigma(f))$. As we have seen, this just means that f has an analytic continuation across $\{\xi\} \times \bar{U}^l$, say into a set $\Delta \times U^{*l}$ where Δ is a neighborhood of ξ (in $\mathbf{C}^k$) and U^* is a neighborhood of $\bar{U}$. Now let $\Gamma = \Delta \cap T^k$ and take $z \in \Gamma$, then $f(z, \cdot)$ is an inner function in U^l that is holomorphic in U^{*l} , hence $f(z, \cdot)$ is a rational inner function, [3] and [1]. So we must have $f(z, w) = c \sum \bar{a}_\alpha w^{\beta-\alpha} / \sum a_\alpha w^\alpha$, as in the previous paragraph. If we make the assumption that the denominator q and the numerator p are relatively prime, and that $q(0) = 1$, then the a_α 's, β , and the c are unique. Since the polynomials $\sum c \bar{a}_\alpha w^{\beta-\alpha} = p(w)$ and $\sum a_\alpha w^\alpha = q(w)$ are relatively prime and their quotient is holomorphic in U^{*l} it follows that $q(w)$ has no zero in U^{*l} , see [3] for a proof. Since the quotient is inner it follows that p has no zeros in $V^l = \{w: |w_i| > 1\}$. Now take $1 \leq j \leq l$ and let $w_i = 1$ for $i \neq j$. We get polynomials $p_j(w_j)$, $q_j(w_j)$ which are still relatively prime because if they weren't they would have a common zero but q_j has no zero in U^* and p_j has no zero in V . So p_j/q_j is a finite Blaschke product with exactly β_j zeros. So if we let $f_j(z, w_i)$ be f with all the w_i for $i \neq j$ set equal to 1, then

$$\beta_j = \frac{1}{2\pi i} \int_{|w_j|=1} \left\{ \frac{\partial}{\partial w_j} f_j(z, w_i) / f_j(z, w_i) \right\} dw_j.$$

This integral depends continuously on z because $f(z, w)$ is bounded away from zero on $\Gamma \times T^l$. Since β_j is an integer we see that β doesn't depend on z if we assume that Γ is a connected neighborhood of ξ on T^k . In general, of course, the a_α 's and c will depend on z , so we write

$$(1) \quad f(z, w) = c(z) \sum \overline{a_\alpha(z)} w^{\beta-\alpha} / \sum a_\alpha(z) w^\alpha, \quad z \in \Gamma, \quad w \in U^{*l}.$$

First we want to extend the a_α 's to be meromorphic in U^k . We use the argument due to Rudin and Stout [3]. Now there exist functions f_α , holomorphic in $U^k \cup \Delta$ such that

$$(2) \quad f(z, w) = \sum_\alpha f_\alpha(z) w^\alpha, \quad \alpha_i \geq 0.$$

The sum converges absolutely and uniformly on compact subsets of $(U^k \times U^l) \cup (\Delta \times U^{*l})$. If we substitute (2) into the left hand-side of (1) multiply through by the denominator of the right-hand-side of (1) and equate coefficients of powers of w we get

$$(3) \quad \sum_{\alpha+\delta=\gamma} a_\alpha(z) f_\delta(z) = \overline{a_{\beta-\gamma}(z) c(z)},$$

for all γ , where a_δ is defined to be 0 if $\delta_i < 0$ for any i . In particular for those γ such that $\gamma_i > \beta_i$ for some i we have $\sum_{\alpha+\delta=\gamma} a_\alpha(z) f_\delta(z) = 0$, or since we have $a_0 = 1$, we can write this as

$$(4) \quad \sum_{\alpha+\delta=\gamma, \alpha \neq 0} a_\alpha(z) f_\delta(z) = -f_\gamma(z) \quad \text{if } \gamma_i > \beta_i \quad \text{for some } i.$$

Now consider the infinite system of linear equations (4) with $z = \xi$. Since we chose p and q to be relatively prime and $q(0) = 1$, this system has the unique solution $\{a_\alpha\}$. So some square array of the coefficients must have a non-zero determinant $D(\xi)$. $D(\xi)$ is just a polynomial in the coefficients f_α , and so this determinant is non-zero in a neighborhood of ξ . So we can use these equations to solve for the a_α 's as rational functions of the f_α 's in some neighborhood (again called Γ) of ξ on T . This now yields a meromorphic continuation of the functions a_α into $U^k \cup \Delta$. Now take 3) with $\gamma = \beta$ to get a meromorphic continuation of c into $U^k \cup \Delta$ and then the remaining equations will give meromorphic continuations of the functions $\bar{a}_\alpha$ into $U^k \cup \Delta$; we denote these functions by a_α^* . Note, in general $a_\alpha^*(z) \neq \overline{a_\alpha(z)}$ if $z \notin \Gamma$.

Now we want to show that the functions a_α and the functions $b_\alpha = ca_\alpha^*$ are actually bounded and holomorphic in $U^k \cup \Delta$. First we need a lemma about the coefficients of certain kinds of polynomials.

Lemma 1. *Suppose n, l are positive integers, then there is a constant C such that if $q(w) = \sum a_\alpha w^\alpha$ is a polynomial in l variables of degree at most n (i.e. $|\alpha| \leq n$ for $a_\alpha \neq 0$) such that $q(0) = 1$ and q doesn't vanish in U^l , then $|a_\alpha| \leq C$.*

Proof. If no such constant existed, then there would exist polynomials $q_i(z)$ of the type described in the lemma such that $\sum_\alpha |a_\alpha^{(i)}|$ is unbounded as $j \rightarrow \infty$ where $q_i(z) = \sum a_\alpha^{(i)} w^\alpha$. Now we can take a subsequence, again denoted by q_i , and find a sequence of positive numbers d_i such that $d_i \rightarrow 0$ and $d_i q_i$ converges to a non zero polynomial q . Since $q_i(0) = 1$, $q(0) = 0$. On the other hand q can't vanish in U^l because q is not identically zero and none of q_i 's vanish in U^l . This contradiction shows that such a constant C must exist.

Next we need a few facts about resultants of polynomials. Let F be a field and let n, m be positive integers. Then there is a polynomial R in $(n+1) + (m+1)$ variables such that if p, q are polynomials in $F[x]$ with degrees at most n, m respectively such that if we evaluate R at the coefficients of p and q we get zero if and only if either i) p and q have a non-trivial common factor or ii) degree $p < n$ and degree $q < m$. See [5] for details, page 83. Now suppose $n = (n_1, \dots, n_l)$ and $m = (m_1, \dots, m_l)$, the n_i, m_i are non-negative integers. If p, q are polynomials in l variables over F let p_i, q_i be p and q looked at as polynomials in w_i with coefficients polynomials in the other variables. Suppose degree $p_i \leq n_i$ and degree $q_i \leq m_i$. Let R_i be the resultant of p_i and q_i as discussed above. Then R_i is a polynomial in $l-1$ variables (which we call $w_1, \dots, w_l$) with coefficients that are polynomials in the coefficients of p and q . Now let $R = R_1 \cdots R_l$. R is a polynomial in $l-1$ variables with coefficients that are polynomials in the coefficients of p and q . Now $R \equiv 0$ if and only if $R_i \equiv 0$ for some $j, 1 \leq j \leq l$. This happens if and only if i) p_i and q_i are not relatively prime or ii) degree $p_i < n_i$ and degree $q_i < m_i$. Recalling that p and q are relatively prime if and only if p_i and q_i are relatively prime

for every j we see that the “resultant polynomial” R of p and q is identically zero if and only if p, q have a common factor of degree $p_i < n_i$ and degree $q_i < m_i$ for some j .

Now we can prove the following theorem.

Theorem 2. *Suppose $\Omega \subseteq \mathbf{C}^k$ is open, and we have functions a_α, d_α meromorphic in Ω (α ranges over a finite set of multi-indices) such that the functions $p(z, w) = \sum d_\alpha(z)w^\alpha$ and $q(z, w) = \sum a_\alpha(z)w^\alpha$ regarded as polynomials in the l variables $w_1, \dots, w_l$ with coefficients in the field of meromorphic functions in Ω are relatively prime. Suppose further that $q(z, 0) = 1$ for every $z \in \Omega$ and that $f = p/q$ is holomorphic in $\Omega \times U^l$. Then the d_α 's are holomorphic in Ω , the a_α 's are bounded and holomorphic in Ω and the function $q(z, w)$ does not vanish in $\Omega \times U^l$. Moreover if f is bounded in $\Omega \times U^l$, then the d_α 's are bounded in Ω .*

Proof. Let $n = (n_1, \dots, n_l), m = (m_1, \dots, m_l)$ be the exact degree of p, q respectively. Form the resultant polynomial $R(z, w)$ as outlined above. $R(z, w)$ is a polynomial in $l - 1$ variables whose coefficients are polynomials in the coefficients of p and q , and $R \not\equiv 0$. Take a point $z_0 \in \Omega$ where all the coefficients of p and q are holomorphic, then the coefficients of R are holomorphic there also. It is clear from the way that R was constructed that the resultant of the polynomials $p(z_0, w)$ and $q(z_0, w)$ is $R(z_0, w)$. If $R(z_0, w) \not\equiv 0$, then $p(z_0, w)$ and $q(z_0, w)$ are relatively prime and since $p(z_0, w)/q(z_0, w)$ is holomorphic in U^l , it follows that $q(z_0, w)$ does not vanish in U^l ; see [3] for a proof. In general we can argue as follows, suppose $q(z_0, w_0) = 0$ for some $w_0 \in U^l$, since $q(z_0, 0) = 1$, $q(z_0, w) \not\equiv 0$, and hence for all z_1 close to z_0 there is a w_1 in U^l such that $q(z_1, w_1) = 0$. But we can choose such z_1 close to z_0 so that $R(z_1, w)$ is not identically zero because R is not identically zero. But this cannot occur, by the first part of the argument. So we have shown the following: if $z_0 \in \Omega$ and all the a_α 's and d_α 's are holomorphic at z_0 , then the polynomial $q(z_0, w) = \sum a_\alpha(z_0)w^\alpha$ has no zero in U^l . Since $q(z_0, 0) = 1$, it follows that $|a_\alpha(z_0)| \leq C$, where C is the constant given in Lemma 1. So the a_α 's are bounded and holomorphic in the complement of an analytic subvariety of Ω , hence they extend uniquely to be bounded and holomorphic in Ω . Now it follows easily that the d_α must be holomorphic in Ω and then the first part of the argument shows that $q(z, w)$ cannot vanish in $\Omega \times U^l$. The fact that the boundedness of f implies the boundedness of the d_α 's follows by expressing both sides of the equation $p(z, w) = f(z, w)q(z, w)$ as power series in w with coefficients holomorphic in z . This completes the proof.

Now return to the original line of argument, we have our inner function f and our functions a_α, a_α^* meromorphic in $\Omega = U^k \cup \Delta$ and

$$f(z, w) = \sum c(z)a_\alpha^*(z)w^{\beta-\alpha} / \sum a_\alpha(z)w^\alpha, \quad z \in \Gamma, \quad w \in U^l.$$

It follows that this identity must hold in $\Omega \times U^l$. The polynomials $p(z, w) = \sum c(z)a_\alpha^*(z)w^{\beta-\alpha}$ and $q(z, w) = \sum a_\alpha(z)w^\alpha$ must be relatively prime as poly-

nomials in w whose coefficients are meromorphic in Ω because for each $z_0 \in \Gamma$ they are relatively prime and for $z_0 \in \Gamma$ the degree of $p(z_0, w)$ in w_i is β_i which is the degree of $p(z, w)$ in w_i . So it follows from Theorem 2 that the a_α 's and the ca_α^* 's are all bounded and that $q(z, w)$ doesn't vanish in $\Omega \times U^l$. In particular $c = ca_0^*$ is bounded. Now it follows that the functions c, a_α, a_α^* all have finite radial limits almost everywhere on T^k . Since f is an inner function, it follows that for almost all $z_0 \in T^k$ the rational function $p(z_0, w)/q(z_0, w)$ must be an inner function in U^l . On the other hand the resultant $R(z, w)$ of p and q is not identically zero, and its coefficients are polynomials in the ca_α^* 's and the a_α 's. By taking radial limits we see that the resultant of $p(z_0, w)$ and $q(z_0, w)$ is $R(z_0, w)$ for almost all $z_0 \in T^k$. Since $R(z, w)$ is not identically zero, $R(z_0, w)$ is not identically zero for almost all z_0 , i.e., $p(z_0, w)$ and $q(z_0, w)$ are relatively prime for almost all $z_0 \in T^k$. Since the representation of a rational inner function as a quotient of two relatively prime polynomials is unique, given that the constant term in the denominator is 1, we can conclude that $|c(z_0)| = 1$ almost everywhere on T^k , and that $a_\alpha^* = \bar{a}_\alpha$ almost everywhere on T^k .

We can summarize what we have done as follows.

Theorem 3. *Suppose the inner function f has an analytic continuation across $\{\xi\} \times T^l$ for some $\xi \in T^k$, then there is a neighborhood Δ of ξ and there is an inner function c in $U^k \cup \Delta$ and meromorphic functions a_α^* in $U^k \cup \Delta$, and bounded holomorphic functions a_α in $U^k \cup \Delta$ such that:*

- i) $a_0 = 1$ and $\sum a_\alpha(z)w^\alpha$ has no zero in $(U^k \cup \Delta) \times U^l$,
- ii) $\bar{a}_\alpha = a_\alpha^*$ almost everywhere on T^k ,
- iii) ca_α^* is bounded and holomorphic in $U^k \cup \Delta$, and
- iv) $f(z, w) = c(z) \sum a_\alpha^*(z)w^{\beta-\alpha} / \sum a_\alpha(z)w^\alpha$, for some β such that $\beta_i - \alpha_i \geq 0$ for all α such that $a_\alpha \neq 0$.

This suggests a method for constructing inner functions. Take an inner function c in U^k and a finite number of meromorphic functions a_α^* in U^k and bounded holomorphic functions a_α in U^k such that ca_α^* is bounded and holomorphic and such that $a_\alpha^* = \bar{a}_\alpha$ a.e. on T^k . We choose the a_α 's with the property that $a_0 = 1$ and $\sum a_\alpha(z)w^\alpha$ has no zero in $U^k \times U^l$. Then consider the function

$$f(z, w) = c(z) \sum a_\alpha^*(z)w^{\beta-\alpha} / \sum a_\alpha(z)w^\alpha$$

where β is chosen large enough, as usual. We would like to show that f is an inner function. Now f has radial limits almost everywhere that have modulus one, so it is enough to show that f is bounded. We consider $f_r(z, w) = f(z, rw)$ for $0 < r < 1$. Of course $f_r \rightarrow f$ and again f_r has radial limits of modulus at most one almost everywhere. So it is enough to show that f_r is bounded, and for this it is enough to show that $q(z, w)$ is bounded away from zero on $U^k \times rU^l$. This will follow from the next lemma.

Lemma 2. *Suppose n, l and a compact subset K of U^l are given, then $\exists \epsilon > 0$*

such that if q is a polynomial of degree at most n in l variables such that q has no zero in U^l and $q(0) = 1$, then $|q(w)| \geq \epsilon$ for $w \in K$.

Proof. Let V be the space of polynomials of degree at most n in l variables. Define $\|q\| = \sup_{U^l} |q(w)|$. Now let

$$L = \{q \in V : q(0) = 1, q \text{ has no zero in } U^l\}.$$

L is closed and by Lemma 1 we see that L is bounded, hence compact, since V is finite dimensional. Now consider the map $T: L \times K \rightarrow \mathbf{C}$ given by $T(q, w) = q(w)$. T is continuous and non-vanishing on the compact set $L \times K$, hence T is bounded from 0. Q.E.D.

Notice that the construction works for any inner function c and has nothing to do with analytic continuation over any points of T^N .

Let c be an inner function in U^k , let us consider for a moment the problem of finding functions a_α, a_α^* satisfying the requirements of the above construction. The a_α^* 's should be such that $ca_\alpha^* = b_\alpha$, a bounded holomorphic function. Also since $a_\alpha^* = \bar{a}_\alpha$ almost everywhere on T^k we have $c\bar{a}_\alpha = b_\alpha$ almost everywhere on T^k . Conversely if we can find a_α such that $\sum a_\alpha(z)w^\alpha$ has no zeros in U^N and such that $c\bar{a}_\alpha = b_\alpha$ almost everywhere on T^k for some bounded holomorphic functions b_α , then we can define $a_\alpha^* = b_\alpha/c$ and we will have functions satisfying the appropriate conditions. Now if $k = 1$, then $c\bar{a} = b$ almost everywhere on T for some bounded holomorphic function b if and only if $a \in (zcH^2)^\perp$, the orthogonal complement in H^2 of the invariant subspace zcH^2 . Now the bounded functions are dense in $(zcH^2)^\perp$ and $(zcH^2)^\perp$ is infinite dimensional unless c is a finite Blaschke product. So if $k = 1$ what we are doing is picking a finite number of bounded functions a_α in $(zcH^2)^\perp$ such that $\sum a_\alpha(z)w$ has no zero in $U \times U^l$, and this determines an inner function in U^{1+l} .

Now consider the inner function $f = p/q$ constructed as above in $U^N = U^k \times U^l$ and suppose that f has an analytic continuation over a point $(\xi, \eta) \in T^k \times \bar{U}^l$. We will assume that the polynomials p, q are relatively prime when regarded as polynomials in l variables whose coefficients lie in the integral domain of functions bounded and holomorphic in U^k . Since f has a continuation over (ξ, η) , f has a continuation over $(\xi, 0)$ by Theorem 4.9.1, page 98 of [4]. Say f has a continuation over $\Delta \times (rU)^l$, where Δ is a neighborhood of ξ and $0 < r < 1$. Now there are functions f_α holomorphic in $\Omega = U^k \cup \Delta$ such that $f(z, w) = \sum_\alpha f_\alpha(z)w^\alpha$, the series converging uniformly and absolutely on compact subsets of $U^k \times U^l$ and in $\Delta \times (rU)^l$. Take a point $z_0 \in \Delta \cap U^k$ such that $p(z_0, w)$ and $q(z_0, w)$ are relatively prime. By the argument of Rudin and Stout used earlier, we can solve for the a_α 's and a_α^* 's and c as rational functions of the f_α 's in some neighborhood of z_0 . This gives a meromorphic continuation of all the coefficients of p and q into $U^k \cup \Delta$. Since the meromorphic functions f and p/q agree in $U^k \times U^l$, they also agree in $\Delta \times (rU)^l$, and so p/q is holomorphic in $\Delta \times (rU)^l$. Now we can apply Theorem 2 (with $(rU)^l$ in place of U^l) to conclude that c , the a_α 's, and the ca_α^* 's are holomorphic in Δ .

Theorem 4. *If f is inner and has a analytic continuation across $\{\xi\} \times T^i$ for some $\xi \in T^k$ and across $T^k \times \{\eta\}$ for some $\eta \in T^l$, then f is rational.*

Proof. By what was just said the inner function c will have an analytic continuation across all of T^k , and hence will be rational. So what we have to show is that a_α , a_α^* are rational. We show that if c is a rational inner function in U^k and if $a, b \in H^1(U^k)$ and $c\bar{a} = b$ almost everywhere on T^k then a and b are also rational. It is enough to prove it for a because $c\bar{a} = b$ implies $c\bar{b} = a$ almost everywhere, since c is inner. Consider the function B defined on V^k , where $V^k = \{z \in \mathbb{C}^k : |z_i| > 1, i = 1, \dots, k\}$ by $B(z_1, \dots, z_k) = \sum \bar{b}_\alpha z_1^{-\alpha_1} \dots z_k^{-\alpha_k}$, where $b(z) = \sum b_\alpha z^\alpha$. Then B is holomorphic in V^k and if we define for $R > 1$, $z \in T^k$, $B_R(z) = B(Rz)$, then $B_R \rightarrow \bar{b}$ in $L^1(T^k)$. Now let $c = p/q$, p, q relatively prime polynomials and consider the function $H = pB$ in V^k . Define, for $R > 1$, and $|z_i| < R$,

$$K(z) = \frac{1}{(2\pi i)^k} \int_{|\xi_1|=R} \dots \int_{|\xi_k|=R} H(\xi_1, \dots, \xi_k) (z_1 - \xi_1)^{-1} \dots \cdot (z_k - \xi_k)^{-1} d\xi_1 \dots d\xi_k.$$

The integral on the right hand side is independent of R , by Cauchy's Theorem and hence K is an entire function. If we take $z \in U^k$ and let R approach 1, we get $K(z) = q(z)a(z)$ since $qa = pb$ almost everywhere on T^k , and $B_R \rightarrow \bar{b}$ in $L^1(T^k)$. So we need to show that K is a polynomial. Take a number $r \geq 2$ and let $R = 2r$ in the integral representation for K , then we see that we get an estimation

$$\sup_{|z_i| \leq r} |K(z)| \leq C \sup_{|z_i| \leq 2r} |p(z)|,$$

since $|B(z)|$ is bounded if $|z_i| \geq 2$. Since K grows like a polynomial, it is a polynomial.

We need a slightly stronger result in order to be able to differentiate the singular sets of rational inner functions from those of non-rational inner functions.

Theorem 5. *Let f be inner in U^N , and suppose that for each j , $1 \leq j \leq N$, there exist $\xi_i \in T$, $1 \leq i \leq N$, $i \neq j$, such that f has an analytic continuation across*

$$\{\xi_1\} \times \dots \times \{\xi_{j-1}\} \times T \times \{\xi_{j+1}\} \times \dots \times \{\xi_N\}.$$

Then f is rational.

Proof. Do induction on N . If $N = 1$, everything is trivial. Suppose true for $N - 1$ variables. Since f has a continuation across a circle of the form $\{\xi\} \times T$ for some $\xi \in T^{N-1}$, we can write

$$f(z, w) = c(z) \sum a_n^*(z) w^{M-n} / \sum a_n(z) w^n,$$

as in Theorem 3. Now c is an inner function in $N - 1$ variables, and by the

remarks preceding Theorem 4 we may apply the induction hypothesis to c and now conclude that c is rational. As we saw in the proof of Theorem 4, this implies that the α_α 's and the α_α^* 's are rational, and hence that f is rational.

Finally we wish to show in a certain sense the singular set of a rational inner function cannot be very large. Let p, q be relatively prime polynomials in N variables such that p/q is inner. Since p, q are relatively prime we can find polynomials A, B, c such that $Ap + Bq = c$, c is not identically zero and c is independent of z_N . Now if $q(z_0) = 0$ for some $z_0 \in T^N$ then $p(z_0) = 0$ since p/q is bounded in U^N , hence $c(z_0) = 0$. So if we take $\xi \in T^{N-1}$ such that $c(\xi) \neq 0$, then $q(\xi, z) \neq 0$ for any $z \in T$ hence p/q has an analytic continuation across $\{\xi\} \times T$. Note that since c is a non identically zero polynomial $\{\xi \in T^{N-1} : c(\xi) \neq 0\}$ has measure 1. Of course we can do the same thing for the other variables z_i .

Now it is easy to give examples of closed sets on the torus T^N that are not the singular sets for any inner function. For instance when $N = 2$, the singular set of a rational inner function is finite, while if F is the singular set of a non rational inner function, then for every $(\xi, \eta) \in T^2$, $\sum (F) \cap (\{\xi\} \times T \cup T \times \{\eta\}) \neq \emptyset$. There are many closed sets that are neither finite nor satisfy this second condition.

One final remark. If $k + l = N$, then it is not in general true that a rational inner function f must have an analytic continuation across sets of the form $\{\xi\} \times T^l$ and $T^k \times \{\eta\}$ for some $\{\xi\} \in T^k$, $\{\eta\} \in T$. For instance let c be a rational inner function in U^2 with a singularity at some point $\xi \in T^2$. Consider the inner function in U^3 given by $f(z, w) = (c(z)w + 1/2)/(1 + 1/2c(z)w)$. Now f can't have a continuation across $T^2 \times \{\eta\}$ for any $\eta \in T$ because if it did then as we have seen this would imply that c has an analytic continuation across T^2 .

REFERENCES

1. A. PFISTER, *Über das Koeffizientenproblem der beschränkten Funktionen von zwei Veränderlichen*. Math. Ann. **146** (1962), 249–262.
2. W. RUDIN, *Lectures on the edge of the wedge theorem* (Lecture notes).
3. W. RUDIN & E. L. STOUT, *Boundary properties of functions of several complex variables*, J. Math. Mech. **14** (1965), 991–1005.
4. W. RUDIN, *Function theory in polydiscs*, Benjamin, 1969.
5. B. L. VAN DER WAERDEN, *Modern Algebra*, Vol. 1, Ungar, New York, 1949.

University of Wisconsin

Date communicated: FEBRUARY 3, 1971