A Non-Commutative Generalization of Topology

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Communicated by I. N. SNEDDON

§1. Introduction. In a recent paper on the foundations of quantum mechanics [6] the concepts of a q -set and a q -topology were introduced, structures which can be regarded as quantum analogues of the concepts of *set* and *topology*. In particular, a non-commutative analogue of the Gelfand representation theorem for commutative C^* -algebras was stated, in which an arbitrary (non-commutative) C^* -algebra A (with an identity) was represented in terms of a “compact Hausdorff q -space” $Q(A)$. Our object in this paper is to place these concepts in a mathematical setting and to give the proofs of the assertions in [6].

In the meantime Akemann [1, 2], motivated by purely mathematical considerations, independently conceived the *hull-kernel structure* [2] of an arbitrary C^* -algebra A (with identity), which essentially coincides with $Q(A)$, and established its properties. There is a considerable overlap between this work and our discussion of $Q(A)$ in Section 5 of the present paper. Moreover, as explained in [6], a number of our proofs are derived from [1], although the setting is somewhat different—we work with the reduced atomic representation of A .

The earlier sections build up a framework in which this theory finds a natural place: Section 2 describes a non-commutative analogue of a *set*, and Sections 3 and 4 discuss the analogue of a general completely regular topological space.

We should like to express our indebtedness to Karl H. Hofmann and David Topping for discussions which considerably clarified the significance of the following constructions.

After the present paper had been prepared for publication a preprint arrived from F. Combes [10] which discusses some closely related questions.

§2. Q -spaces. We may associate with any set X the algebra $B(X)$ of all complex-valued bounded functions on X . With the sup norm $B(X)$ is a commutative W^* -algebra. (By a W^* -algebra we mean a C^* -algebra which is, as a Banach space, the dual of a Banach space.) The complete lattice of all projections in $B(X)$ may be identified, in an obvious way, with the lattice of all subsets

of X . This lattice is *atomic*, the atoms (minimal non-zero elements) being the points of X , and we shall accordingly call B an *atomic* W^* -algebra. Conversely, the lattice of projections in any commutative W^* -algebra B is a complete atomic distributive lattice and is thus [3] isomorphic to the lattice of all subsets of some set X . Consequently, B is isomorphic to $B(X)$.

Now consider the non-commutative analogue of this situation. Let us define a q -space to be a pair (Q, B) where B is an atomic W^* -algebra and Q is the set of all projections in B ; we shall often refer briefly to *the* q -space Q . The elements of Q will be called q -sets and the minimal non-zero q -sets q -points. The letters e, f (with or without embellishments) will always be used to denote q -sets, and x, y to denote q -points. *Central* q -sets, *i.e.* projections in the centre of B , will be denoted by z, w . Among other things a q -space is a lattice. Union and intersection of q -sets are to be taken in this lattice. However, two q -sets e and f will be called *disjoint* only if $ef = 0$. If $e \leq f$ we call e a q -subset of f and may say e is in f . Any q -space (Q, B) contains a least q -set, the zero projection 0 , and a greatest q -set, the identity 1 of B , which we shall also refer to as *the identity in* Q . A q -set e is *finite* iff it is a finite union of q -points; e is *cofinite* iff $1 - e$ is finite.

The categorical significance of the concept of a q -space was pointed out to us by K. H. Hofmann. Let $\mathfrak{S}$ be the category of sets and let $\mathfrak{W}_a^*$ be the category of commutative atomic W^* -algebras, in which the morphisms are the σ -continuous $*$ -homomorphisms which preserve the identity. (Following Sakai [9], we denote the ultraweak topology in a W^* -algebra by σ .) Then the map Φ , which assigns to each set X the commutative atomic W^* -algebra $\Phi(X) = B(X)$ of all bounded functions on X , and to each morphism $f: X \rightarrow Y$ the morphism $\Phi(f): \Phi(Y) \rightarrow \Phi(X)$ given, for every b in $\Phi(Y)$, by $[\Phi(f)](b) = a$, where $a(x) = b(f(x))$, is a contravariant functor. It is easy to show that Φ is full, faithful, and representative [8] and thus is an *equivalence* between $\mathfrak{S}$ and the dual of the category $\mathfrak{W}_a^*$. Now $\mathfrak{W}_a^*$ is, obviously, a full subcategory of the category $\mathfrak{W}_*^*$ of all atomic W^* -algebras. Thus we can regard the dual category of $\mathfrak{W}_*^*$ as an extension of the category of sets; this may be interpreted as the category of q -spaces.

The structure of an arbitrary q -space (Q, B) is very simple. Indeed, the centre Z of B is (Corollary 2.4 below) a commutative atomic W^* -algebra. Consequently, any central q -set z decomposes into a (disjoint) union of minimal central q -sets. We call this the *central decomposition* of z . In particular, the central decomposition of the identity 1 of Q is $1 = \vee \{z_i : i \in I\}$ where $\{z_i : i \in I\}$ is the set of all minimal central q -sets. Since z_i is minimal central $z_i B$ is a *factor* which, being atomic, is of type I and so isomorphic to $L(H_i)$ for some Hilbert space H_i . Thus B is isomorphic to $\prod \{L(H_i) : i \in I\}$ for some set of Hilbert spaces $\{H_i : i \in I\}$; moreover, it is clear that any W^* -algebra of this form is atomic. We shall call this the *canonical representation* of the q -space Q . It is unique up to a spatial isomorphism (unitary equivalence). In the commutative case, where $B = B(X)$, each H_i is 1-dimensional and the set I may be identified with the set X .

We introduce now some definitions and lemmas which will be needed later.

Let (Q, B) be a q -space, Z the centre of B , e and f q -sets, x a q -point, z a central q -set. Let $a \in B$ and $K \subset B$.

Definition 2.1. The *support* of a is $e(a) = \bigwedge \{g \in Q: ag = a\}$ (cf. [5], p. 323). The *support* of K is $e(K) = \bigvee \{e(a): a \in K\}$. The *central support* of a is $z(a) = \bigwedge \{w \in Z \cap Q: aw = a\}$. The *central support* of K is $z(K) = \bigvee \{z(b): b \in K\}$.

Lemma 2.2.

- (a) $e(a) \leq e \Leftrightarrow ae = a, \quad e(a)f = 0 \Leftrightarrow af = 0.$
- (b) $z(a) \leq z \Leftrightarrow az = a, \quad z(a)z = 0 \Leftrightarrow az = 0.$
- (c) $e \leq f \Leftrightarrow z(e) \leq z(f).$
- (d) $z(e)f = 0 \Leftrightarrow z(e)z(f) = 0 \Leftrightarrow ez(f) = 0.$

If also $\{e_i: i \in J\}$ is any set of q -sets, then:

- (e) $z(\bigvee \{e_i: i \in J\}) = \bigvee \{z(e_i): i \in J\}$, and
- (f) $z(\bigvee \{e_i: i \in J\}) = \bigvee \{ze_i: i \in J\}.$

The proof is straightforward.

Lemma 2.3. $z(x)$ is a minimal central q -set.

Proof. Suppose $z \leq z(x)$. Then $zx \leq x$ so either $zx = x$ in which case $z = z(x)$, or $zx = 0$ in which case $z = zz(x) = 0$.

Corollary 2.4. The centre Z of B is atomic.

Proof. Let z be a central q -set. Choose a q -point $x \leq z$. Then $z(x) \leq z$. Thus every central q -set contains a minimal central q -set.

Definition 2.5. Let e be a q -set and let $Q_e = \{f: f \in Q, f \leq e\}$. Then (Q_e, eBe) is a q -space which we call the q -subspace of (Q, B) determined by e .

In the commutative case $B = B(X)$ for some set X and the hermitian elements of B are real-valued functions on X . This suggests the following definition:

Definition 2.6. Let $a = a^* \in B$ and $\lambda \in \mathbb{R}$. Then we say a takes the value λ on e iff $(a - \lambda 1)e = 0$ (i.e. iff e is the projection on an eigenspace of a of eigenvalue λ). If $\lambda = 0$ we say a vanishes on e .

The following lemma is an immediate consequence of the transitivity theorem ([4], p. 44):

Lemma 2.7. Let (Q, B) be a q -space and A be a σ -dense C^* -subalgebra of B . Let $x_1, \dots, x_n$ be a finite set of mutually disjoint q -points and $\lambda_1, \dots, \lambda_n$ be real numbers. Then there exists a hermitian operator $a \in A$ which takes the value λ_r at x_r for $1 \leq r \leq n$.

§3. Q -topologies. Let Q be a q -space. We are going to define on Q a structure analogous to a topology by distinguishing a certain class of q -sets as *open*. A characterisation resembling the usual definition of a topology is not yet

available. However, for an important class of topological spaces X —the completely regular spaces—the topology is determined by the C^* -algebra $C(X)$ of continuous complex-valued bounded functions on the space: indeed the open sets are exactly the cozero sets of subsets of $C(X)$. Moreover, if we assume also that the topology is Hausdorff then $C(X)$ is σ -dense in $B(X)$. Passing to the non-commutative case, we arrive at the following structure.

Let (Q, B) be a q -space with identity 1 . Let A be a σ -dense C^* -subalgebra of B , not necessarily containing 1 . This notation will be held until the end of Section 4.

Definition 3.1.

(a) The q -topology τ_A determined by A on Q is the class of all open q -sets, where, if e is any q -set,

(b) e is open iff $e = e(K)$ for some $K \subset A$. (Since $e(a^*a) = e(a)$ it is sufficient to consider subsets K of the positive cone A^+ in A .)

(c) e is closed iff $1 - e$ is open.

(d) The interior of e is $e^0 = \bigvee \{f: f \text{ is open, } f \leq e\}$.

(e) The closure of e is $\bar{e} = \bigwedge \{f: f \text{ is closed, } f \geq e\}$.

A q -topological q -space is one with a distinguished q -topology

We note at once:

Lemma 3.2. Let $a = a^* \in A$. Let $I \subset \mathbb{R}$ be an open set. Let $a = \int \lambda \, dE(\lambda)$ be the spectral representation of a and let $e = \int_I dE(\lambda)$. Then e is open.

Proof. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with $f(\lambda) = 0$ if $\lambda \notin I$, $f(\lambda) > 0$ if $\lambda \in I$. Let $b = f(a)$. Clearly, $e = e(b)$ (but if A has no unit b may not belong to A). Let a_α be an increasing directed set of positive elements in A which is also an approximate unit for A ([4], p. 15). Let $b_\alpha = b^{1/2} a_\alpha b^{1/2}$. Then $b_\alpha \in A$ so that our result will follow if we can prove that $\bigvee_\alpha e(b_\alpha) = e$. Clearly, for all α , $e(b_\alpha) \leq e$ (for, if f is any q -set, $bf = 0 \Rightarrow (b^{1/2}f)^*(b^{1/2}f) = 0 \Rightarrow b^{1/2}f = 0 \Rightarrow b_\alpha f = 0$) so that $\bigvee_\alpha e(b_\alpha) \leq e$. On the other hand, by [9], Chapter I, Lemma 3.3, $a_\alpha \rightarrow 1$ and $b_\alpha \rightarrow b$ in the σ -topology. Now, if f is any q -set such that $[\bigvee_\alpha e(b_\alpha)]f = 0$, then $b_\alpha f = 0$ for all α so that $bf = 0$ (since multiplication is separately σ -continuous) which means $e(b)f = 0$. This shows that $e(b) \leq \bigvee_\alpha e(b_\alpha)$. Thus $e = \bigvee_\alpha e(b_\alpha)$.

Theorem 3.3.

(a) 0 and 1 are open and closed.

(b) Any union of open q -sets is open; any intersection of closed q -sets is closed.

(c) For any q -set e , e^0 is the largest open q -set in e and $\bar{e}$ is the smallest closed q -set containing e .

However, the intersection of two open q -sets e and f is not necessarily open. A counterexample has been given by Akemann [1] who has shown also that

this situation can arise only if e and f “make a zero angle”: i.e. if 0 is a non-isolated point of the spectrum of $e + f$.

We now determine the structure of A .

Lemma 3.4. *Let B be given in its canonical representation on a Hilbert space H . Then the identity representation of A is a direct sum of inequivalent irreducible representations.*

Proof. Let $\mathbf{1} = \bigvee \{z_i : i \in I\}$ be the central decomposition of the identity in B . For each i let $\pi_i(a) = z_i a$. Then π_i is a representation of A on $H_i = z_i H$ and the identity representation of A is the direct sum $\bigoplus \{\pi_i : i \in I\}$. Clearly, π_i is nondegenerate. If π_i is not irreducible there exists a q -set f with $z_i > f > 0$ and $af = fa$ for all a in A . But then f belongs to the commutant A' of A . However, $A' = B'$ since A is σ -dense, so f is central, a contradiction. If $\pi_i \sim \pi_j$ then there is a partial isometry t_{ij} from z_j to z_i such that, for a in A , $t_{ij}(z_j a) = (z_i a)t_{ij}$; i.e. $t_{ij}a = at_{ij}$. This means $t_{ij} \in A' = B'$. But then $z_i = t_{ij}z_j t_{ij}^* = z_j(t_{ij}z_j t_{ij}^*) = z_j z_i$ so that $z_i \leq z_j$ whence, by symmetry, $z_i = z_j$.

In view of the relation between pure states and irreducible representations ([4], §2.5) we have:

Corollary 3.5. *To each q -point x assign the linear function φ_x on A defined by $\varphi_x(a) = \text{trace}(xa)$. Then the map $x \rightarrow \varphi_x$ is an injection of the q -points of Q into the pure states of A .*

Theorem 3.6. *The q -topology τ_A is Hausdorff: i.e. given disjoint q -points x and y there exist disjoint open q -sets e and f with $x \leq e$ and $y \leq f$.*

Proof. By Lemma 2.7 there is a hermitian a in A which takes the value 0 at x and 1 at y . Let $a = \int \lambda dE(\lambda)$ be its spectral resolution. Let I and J be disjoint open subsets of $\mathbb{R}$ with $0 \in I$, $1 \in J$. Then, by Lemma 3.2, $e = \int_I dE(\lambda)$, $f = \int_J dE(\lambda)$ have the required properties.

In the same way we can prove the more general:

Theorem 3.7. *Given any disjoint finite q -sets e and f there exist disjoint open q -sets g and h such that $e \leq g$ and $f \leq h$.*

Corollary 3.8. *Any finite q -set e is closed.*

Proof. Let $e' = \mathbf{1} - e$. For each q -point $x \leq e'$ choose an open q -set $f_x \leq e'$ with $x \leq f_x$. Then $e' = \bigvee \{f_x : x \leq e'\}$ which is open; so e is closed.

In particular:

Corollary 3.9. *The q -topology τ_A is T_1 : i.e. every q -point is closed.*

We now introduce in Q an equivalence relation $\sim_A$ which has some of the properties of equivalence of projections in a W^* -algebra.

Definition 3.10. Let $a \in B$, $b \in B$. We write $a \sim_A b$ iff $uau^* = b$ for some unitary u in the C^* -algebra $\tilde{A}$ generated by A and the identity in B .

Lemma 3.11. *If e and f are q -sets then*

- (a) $e \sim_A f \Rightarrow z(e) = z(f)$.
- (b) *If also e and f are finite and of equal rank (in particular, if they are q -points) then $z(e) = z(f) \Rightarrow e \sim_A f$.*

Proof. (a) Assume $e = ufu^*$, $u \in \tilde{A}$, u unitary. If, for a central q -set z , $zz(e) = 0$ then $0 = ze = zufu^* = uzfu^*$ so $zf = 0$ whence $zz(f) = 0$. Thus $z(e) \leq z(f)$ whence, by symmetry, $z(e) = z(f)$. (b) There exists a unitary u in B with $e = ufu^*$. Now apply the transitivity theorem, Dixmier [4], p. 45.

Lemma 3.12. *For any q -set e , $z(e) = \bigvee \{f: f \sim_A e\}$.*

Proof. Let $\bigvee \{f: f \sim_A e\} = g$. Clearly, $ugu^* = g$ for every unitary u in $\tilde{A}$. So $g \in A' = B'$: i.e. g is central. But then

$$g = z(g) = \bigvee \{z(f): f \sim_A e\} = z(e),$$

using Lemma 2.2.

Theorem 3.13.

- (a) *If $e \sim_A f$ then e is open iff f is open.*
- (b) *If e is open then $z(e)$ is open.*

Proof. (a) Let $e = e(K)$ for $K \subset A$. Then if $f = ueu^*$, with u unitary and in $\tilde{A}$, $f = e(uKu^*)$ and $uKu^* \subset A$.

- (b) By Lemma 3.12 and (a), $z(e)$ is a union of open q -sets.

§4. Complete regularity. In this section we make use of the ideal structure of A to show that the q -topology τ_A is completely regular. We retain the notation of Section 3. For any subset $K \subset B$, let $\bar{K}$ denote the ultraweak closure ($= \sigma$ -closure) of K in B .

Definition 4.1. For any q -set e let:

- (i) $L(e) = \{a \in A: ae = a\}$.
- (ii) $L^*(e) = (L(e))^* = \{a \in A: ea = a\}$.
- (iii) $A(e) = L(e) \cap L^*(e)$.
- (iv) $A^+(e) = \{a \in A(e): a \geq 0\}$.

Lemma 4.2.

- (a) $L(e)$ is a norm-closed left ideal in A .
- (b) $L(e) = \{a \in A: e(a) \leq e\} = \{a \in A: e(a^*a) \leq e\}$.
- (c) $L(e) = L(e^0)$.
- (d) $e(L(e)) = e^0$.
- (e) $A(e)$ is a C^* -subalgebra of A .
- (f) $A^+(e) = L(e) \cap A^+$.
- (g) $\overline{L(e)} = AA(e) = AA^+(e)$.
- (h) $e^0 \in \overline{A(e)}$.

Proof. (a) is immediate, and (b) follows from Lemma 2.2.

(c) If $a \in A$ then $e(a) \leq e \Rightarrow e(a) \leq e^0$. Now use (b).

(d) By (b), $e(L(e^0)) \leq e^0$. On the other hand, being open, $e^0 = e(K)$ for some $K \subset A$, which (by the definition of $e(K)$) means $K \subset L(e^0)$. Thus $e^0 \leq e(L(e^0))$.

(e) and (f) are obvious, and (g) follows from the fact ([4] p. 47) that any closed left ideal in A is generated by its positive elements. Lastly, (h) follows from Dixmier [5] p. 41, since $e^0 = e(A(e))$.

Corollary 4.3. *The map $e^0 \mapsto L(e^0)$ is an order-preserving injection of the open q -sets into the norm-closed left ideals in A .*

Lemma 4.4. *Let e^0 be any open q -set and x be any q -point. Then the following conditions are equivalent:*

- (a) $xe^0 = 0$.
- (b) $\text{trace}(xa) = 0$ for every a in $L(e^0)$.
- (c) $\text{trace}(xa) = 0$ for every a in $A^+(e^0)$.
- (d) $xA^+(e^0) = 0$.

Proof. Let ξ be a unit vector on the ray xH . Assume (a), and $a \in L(e^0)$. Then $\text{trace}(xa) = \langle \xi, xa\xi \rangle = \langle \xi, xae^0\xi \rangle = 0$, whence (b). Clearly (b) implies (c). Assume (c). By 4.2 (h) and the Kaplansky density theorem ([4] p. 43) $e^0 \in \overline{A^+(e^0)}$. But the map $a \mapsto \text{trace}(xa)$ is σ -continuous, whence (a). Clearly (a) implies (d) and (d) implies (c).

Lemma 4.5. *Let e be any q -set. Let A_1 , B_1 , and $A_1(e)$ be the unit balls in A , B , and $A(e)$ respectively and let $L_1(e) = L(e) \cap A_1$. Then (a) $\overline{A_1(e)} = e^0 B_1 e^0$, (b) $\overline{A(e)} = e^0 B e^0$, (c) $\overline{L_1(e)} = B_1 e^0$, (d) $\overline{L(e)} = B e^0$.*

Proof. Clearly,

$$A_1(e)A_1A_1(e) \subset A_1(e) \subset e^0 B_1 e^0.$$

Since multiplication is jointly ultrastrongly continuous on bounded sets this gives (since the ultrastrong and ultraweak closures coincide for convex sets)

$$\overline{A_1(e)A_1A_1(e)} \subset \overline{A_1(e)} \subset e^0 B_1 e^0.$$

But $\overline{A_1} = B_1$, by the Kaplansky density theorem, so the last relation yields, by Lemma 4.2 (h) and the proof of Lemma 4.4,

$$e^0 B_1 e^0 \subset \overline{A_1(e)} \subset e^0 B_1 e^0.$$

(b) follows at once from (a). Using the relation

$$A_1 L_1(e) \subset L_1(e) \subset B_1 e^0,$$

(c) and (d) can be proved in the same way.

Definition 4.6. Let e be an open q -set. By Lemma 4.5 and Definition 3.1 the C^* -algebra $A(e)$ determines a q -topology $\tau_{A(e)}$ on the q -subspace (Q_*, eBe)

determined by e (see Definition 2.5). One might call this the *subspace q -topology* on the open q -set e . Since $A(e) \subset A$ and $e(A(e)) = e$ it consists of exactly those open q -sets which are contained in e .

Note that, since the intersection of open q -sets is not necessarily open, $\tau_{A(e)}$ does not in general coincide with the intersections with e of τ_A -open q -sets. For a general q -set e even an appropriate definition of a subspace q -topology is not evident. However, if e is central, there is no apparent difficulty in extending the usual theory (eA is σ -dense in eB and so defines a q -topology).

Theorem 4.7. *The q -topology τ_A on Q is completely regular: i.e. if x is a q -point disjoint from a closed q -set e then A contains a hermitian element a satisfying $0 \leq a \leq 1$ and taking the value 0 on e and 1 at x .*

Proof. Let $e' = 1 - e$. Then e' is open and $x \leq e'$. By Lemma 4.5 (b), $\overline{A(e')} = e'Be'$ so that, by Lemma 2.7, $A(e')$ contains a hermitian element b which takes the value 1 at x . Of course, b takes the value 0 on e . Now put $a = \varphi(b)$ where $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function with $\varphi(\lambda) = 0$ for $\lambda \leq 0$ and $\varphi(\lambda) = 1$ for $\lambda \geq 1$.

Corollary 4.8. *τ_A is regular: i.e. if x is a q -point disjoint from a closed q -set e then there are disjoint open q -sets f and g with $f \geq x$, $g \geq e$.*

Proof. Choose $a \in A$ as above. Let f and g be the spectral projections of a corresponding to disjoint open sets in $\mathbb{R}$ containing 1 and 0 respectively.

§5. The pure state q -space of a C^* -algebra.

Definition 5.1. Let A be a C^* -algebra. Let $\{\pi_i : i \in I\}$ be a maximal set of inequivalent (non-null) irreducible representations of A . Let π_i act in a Hilbert space H_i and let $\pi = \bigoplus \{\pi_i : i \in I\}$ be the direct sum ([4] p. 27) of the representations π_i . π acts in the Hilbert space $H = \bigoplus \{H_i : i \in I\}$ and is called the *reduced atomic representation* [7] of A . It is unique up to equivalence and is faithful. When no confusion will arise we shall identify A with $\pi(A)$.

Definition 5.2. Let $B = \prod \{L(H_i) : i \in I\}$, which is an atomic W^* -algebra in its canonical representation. Let $Q(A)$ be the set of all projections in B . Then $(Q(A), B)$ is a q -space. By [4] §2.8.3, A is σ -dense in B . We call the q -space $Q(A)$, equipped with the q -topology determined by A , the *pure state q -space* (or *maximal left ideal q -space*) of A . Let Z denote the centre of B .

Clearly, $Q(A)$ is determined up to isomorphism by A . In Theorem 5.13 we shall show that if A has an identity then A is determined up to isomorphism by $Q(A)$.

The reason for the name given to $Q(A)$ is the existence of a natural bijection between the q -points and the pure states, or equivalently ([4] p. 48) the maximal left ideals, of A :

Lemma 5.3. *The mapping which assigns to each q -point x the functional φ_x on A defined by $\varphi_x(a) = \text{trace}(xa)$ is a bijection of the q -points of $Q(A)$ onto the pure states of A .*

Proof. By 3.5 it is only necessary to show that this map is onto. However, ([4] §2.5) any pure state φ determines an irreducible representation, which we may identify with one of the π_i , and a vector ξ in the corresponding Hilbert space H_i . If we define x to be the projection on the ray spanned by ξ we have $\varphi = \varphi_x$.

Observing that φ_x and φ_y determine equivalent representations iff $z(x) = z(y)$ we have:

Corollary 5.4. *The map $x \mapsto \varphi_x$ determines a bijection between the minimal central q -sets and the equivalence classes of irreducible representations of A : i.e. between the atoms of the centre $Z \cap Q(A)$ of $Q(A)$ and the points of the spectrum $\hat{A}$ of the C^* -algebra A ([4] p. 29); hence also a bijection between the central q -sets and the subsets of $\hat{A}$.*

We now show that this bijection carries the q -topology of $Q(A)$ (which, restricted to $Z \cap Q(A)$, is a true topology) into the Jacobson topology ([4] p. 59) on $\hat{A}$.

Lemma 5.5. *The map $e \mapsto L(e)$ determines a bijection of the open q -sets in $Q(A)$ onto the norm-closed left ideals in A .*

Proof. By Corollary 4.3 it is sufficient to show that every norm-closed left ideal L may be written in the form $L = L(e(L))$. Clearly, $L(e(L)) \supset L$. If $L(e(L)) \neq L$ then there is ([4] p. 48) a maximal left ideal containing L and not containing $L(e(L))$ and thus a q -point x with $Lx = 0$ but $L(e(L))x \neq 0$. However, $Lx = 0$ implies $e(L)x = 0$ (by Lemma 2.2) which implies $L(e(L))x = 0$.

Restricting this bijection to the central open q -sets we get:

Corollary 5.6. *The map $e \mapsto L(e)$ determines a bijection of the central open q -sets onto the norm-closed 2-sided ideals in A .*

In view of Dixmier [4] p. 62 this establishes the relation mentioned above between the q -topology of $Q(A)$ and the Jacobson topology of $\hat{A}$.

Corresponding to the fact that $\hat{A}$ is quasicompact when A has an identity ([4] p. 61) we have:

Theorem 5.7. *The pure state q -space $Q(A)$ of a C^* -algebra with an identity is compact: i.e. the following equivalent conditions hold:*

- (a) *Any covering of 1 by open q -sets has a finite subcovering. (Here 1 is the identity in B_0 and a covering of 1 is a collection $\{e_i\}$ of q -sets such that $\bigvee \{e_i\} = 1$.)*
- (b) *Any collection of closed q -sets with the finite intersection property has a non-empty intersection.*

Proof. That (a) and (b) are equivalent is proved in the usual way—De Morgan's laws, which are all that is needed, hold equally in the non-commutative case. We shall establish (a). The proof follows the pattern used by Dixmier [4] p. 61 for $\hat{A}$.

Suppose that $\{e_i : i \in I\}$ is a family of open q -sets with $1 = \bigvee \{e_i : i \in I\}$. Consider the (not necessarily closed) left ideal in A , $L = \sum_{i \in I} L(e_i)$. If $L \neq A$ then, since A has an identity, L is contained in a maximal proper left ideal. To this ideal there corresponds ([4] p. 48) a pure state of A and so (Lemma 5.3) a q -point x . $\text{Trace}(xa) = 0$ for every a in L and hence for every a in $L(e_i)$ whence (by Lemma 4.4) $xe_i = 0$, and this for every i . This means $x1 = 0$, which is a contradiction. Hence $L = A$. Thus $1 \in L$, so that $1 \in \sum_{i \in J} L(e_i)$ for some finite subset J of I . But then $1e_J = 1$ where $e_J = \bigvee_{i \in J} e_i$, whence $e_J = 1$.

The analogy with the commutative case persists when A does not have an identity:

Theorem 5.8. *Let A be a C^* -algebra. Let Q , with unit 1 , be the pure state q -space of A . Let $\tilde{Q}$, $\tilde{1}$ be the corresponding quantities for the algebra $\tilde{A}$ obtained by adjoining an identity (which we identify with $\tilde{1}$) to A . Then $\tilde{1} = x_\infty + 1$, where x_∞ is a central q -point. Moreover, iff A has no identity, 1 is dense in $\tilde{1}$ (i.e. $\tilde{1} = \bar{1}$) (so that $\tilde{Q}$ might be described as "the one-point compactification of the q -space Q ").*

Proof. We write the elements of $\tilde{A}$ as formal sums of the form $a + \lambda\tilde{1}$ and regard A as embedded in $\tilde{A}$ in the obvious way. Every pure state of A has ([4] p. 51) a unique extension to a pure state of $\tilde{A}$. The only other pure state of $\tilde{A}$ is that which vanishes on A ; to this there corresponds the one-dimensional representation $a + \lambda\tilde{1} \mapsto \lambda$. If A has an identity 1 then $e(\tilde{1} - 1) = \tilde{1} - 1 = e_\infty$, so that e_∞ is open and $\tilde{1} = 1$. Conversely, suppose $\tilde{1} \neq 1$. Then $\tilde{1} = 1$ so e_∞ is open. But then there exists a set $K = \{a_i + \lambda_i\tilde{1} : i \in I\} \subset \tilde{A}$ such that $e(K) = e_\infty$. This means $0 = (a_i + \lambda_i\tilde{1})1$ so that $a_i = \lambda_i 1$, for all i in I . Thus, since $\lambda_i \neq 0$ for some i , $1 \in A$: i.e. A has an identity.

Our next object is to establish that the q -topology τ_A on $Q(A)$ is normal. First we have:

Lemma 5.9. *For any open q -set e in Q , (Q_e, eBe) is the pure state q -space of $A(e)$.*

Proof. The q -topology on Q_e is given by the dense C^* -subalgebra $A(e)$ of eBe (see Definition 4.6). Assume B to be given in its canonical representation. Then, by Lemma 3.4, the identity representation of $A(e)$ is a direct sum of inequivalent irreducible representations. It remains only to show that every irreducible representation of $A(e)$ occurs in this sum, or equivalently that, given any pure state φ of $A(e)$, there is a q -point $x \leq e$ with $\varphi(a) = \text{trace}(xa)$ for all a in $A(e)$. But φ determines ([4] p. 32) an irreducible representation π_φ of $A(e)$ on some Hilbert space K_φ and a vector $\xi_\varphi \in K_\varphi$ such that $\varphi(a) =$

$\langle \xi_\varphi, \pi_\varphi(a)\xi_\varphi \rangle$. The representation π_φ can be extended ([4] p. 51) to an irreducible representation of A on a Hilbert space $K \supset K_\varphi$. If we identify the representation so obtained with a direct summand of the identity representation of A , the projection on ξ_φ becomes identified with a q -point x with the desired property. Moreover, since (by Lemma 4.2 (h)) $e \in \overline{A(e)}$, the fact that $\|\varphi\| = 1$ gives $1 = \text{trace}(xe)$ so that $x \leq e$.

The next theorem generalises Urysohn's lemma:

Theorem 5.10. *If A is a C^* -algebra with an identity 1 and e and f are disjoint closed q -sets in the pure state q -space Q of A then there is a hermitian $a \in A$ satisfying $0 \leq a \leq 1$ and taking the value 1 on e and 0 on f .*

Proof. Let $e' = 1 - e$; e' is open and $f \leq e'$. Let $\tilde{A}(e')$ be the C^* -algebra generated by 1 and $A(e')$. By Theorem 5.8 (neglecting the trivial case in which $e = e'$) the identity of $Q(\tilde{A}(e'))$ is $\tilde{1} = e_\infty + e'$ where e_∞ is a central q -point. It can be shown (see [2]) that f is closed in $Q(\tilde{A}(e'))$. (In a preprint of this paper this fact was tacitly assumed. We are indebted to Dr. Akemann for pointing out the gap in our proof.) Since $Q(\tilde{A}(e'))$ is completely regular $\tilde{A}(e')$ contains a hermitian element a satisfying $0 \leq a \leq \tilde{1}$ and taking the value 1 at e_∞ and 0 on f , and so being of the form $a = 1 + b$ where $b \in A(e')$. But then a , regarded as an element of A , takes the value 1 on e and 0 on f and satisfies $0 \leq a \leq 1$.

Theorem 5.11. *If A is a C^* -algebra with identity then the q -space $Q(A)$ is normal: i.e. given disjoint closed q -sets e and f there are disjoint open q -sets g and h with $g \geq e$ and $h \geq f$.*

Proof. Choose a as in Theorem 5.10. Then let g and h be the spectral projections of a corresponding to disjoint open sets in $\mathbb{R}$ containing the points 1 and 0 respectively.

We next generalise the notion of a continuous function:

Definition 5.12. Let (Q, B) be a q -space with a q -topology τ_A determined by a σ -dense C^* -subalgebra A of B . We say that a hermitian element a of B is q -continuous iff the spectral projection of a corresponding to every open interval in $\mathbb{R}$ is an open q -set.

By Lemma 3.2 every hermitian element of A is q -continuous. We now prove a converse which can be regarded as a non-commutative generalisation of the Gelfand representation theorem for commutative C^* -algebras.

Theorem 5.13. *If (Q, B) is the pure state q -space of a C^* -algebra A with identity then every q -continuous hermitian element of B belongs to A .*

Proof. Let $b \in B$ be q -continuous and hermitian. Without loss of generality we may assume $b \geq 0$. Let $b = \int \lambda dE(\lambda)$ be the spectral representation of b and, for each interval $I \subset \mathbb{R}$ let $e(I) = \int_I dE(\lambda)$. Given $\epsilon > 0$ choose λ_i , $i = 0, 1, 2, \dots, N$ with $0 = \lambda_0$, $\lambda_N > \|b\|$, and $\lambda_{i-1} < \lambda_i < \lambda_{i-1} + \epsilon$, for $i =$

$1, \dots, N$. For $i = 1, \dots, N$ we can (by Theorem 5.10) choose a hermitian element a_i of A with $0 \leq a_i \leq 1$ and $a_i e([0, \lambda_{i-1}]) = 0 = (1 - a_i) e([\lambda_i, \lambda_N])$. (1 denotes the identity in B .) Let $a = \sum_{i=1}^N (\lambda_i - \lambda_{i-1}) a_i$. Then a is an approximation to b . Indeed, if $I_i = [\lambda_{i-1}, \lambda_i]$,

$$\begin{aligned} a + \epsilon 1 - b &= \sum_{i=1}^N \int_{I_i} (a + \epsilon - \lambda) dE(\lambda) \\ &= \sum_{i=1}^N \int_{I_i} \left[\sum_{j=1}^N (\lambda_j - \lambda_{j-1}) a_j + \epsilon - \lambda \right] dE(\lambda) \\ &= \sum_{i=1}^N \int_{I_i} \left[\sum_{j=1}^{i-1} (\lambda_j - \lambda_{j-1}) + (\lambda_i - \lambda_{i-1}) a_i + \epsilon - \lambda \right] dE(\lambda) \\ &= \sum_{i=1}^N \int_{I_i} [(\lambda_{i-1} + \epsilon - \lambda) + (\lambda_i - \lambda_{i-1}) a_i] dE(\lambda) \geq 0 \end{aligned}$$

since, in each term, the integrand is non-negative. Similarly, replacing ϵ by $-\epsilon$ in the last equation, we obtain

$$a - \epsilon 1 - b = \sum_{i=1}^N \int_{I_i} [(\lambda_{i-1} - \lambda) + (\lambda_i - \lambda_{i-1} - \epsilon) a_i - \epsilon(1 - a_i)] dE(\lambda) \leq 0.$$

Thus $\|a - b\| \leq \epsilon$. But $a \in A$ and ϵ is arbitrary, so $b \in A$.

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This work was supported by a grant from the National Research Council of Canada.

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Date communicated: FEBRUARY 23, 1971