Unbounded General Wiener-Hopf Operators

VICTOR J. PELLEGRINI

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1. Introduction. Let H be a separable Hilbert space and P a self adjoint projection on H . If A is a linear operator with domain $D(A)$ and range $R(A)$ contained in H , we define a general Wiener-Hopf operator to be an operator of the form

$$T(A) = PA \mid R(P) \cap D(A).$$

When A is an invertible operator on H , A. Devinatz and M. Shinbrot have proven in [2] that the following conditions are equivalent:

- (a) $T(A)$ is invertible.
- (b) There exists an invertible operator W on H such that $R(WQ) = R(Q)$ and $\operatorname{Re} (WAx, x) \geq \delta \|x\|^2$ for each $x \in H$ (where $Q = I - P$).
- (c) There exists invertible operators A_- and A_+ on H such that $A = A_-A_+$, $R(A_+P) = R(P)$ and $R(A_-^{-1}Q) = R(Q)$.

The purpose of this paper is to prove a version of this theorem which is valid for a certain class of unbounded operators on H . The techniques employed are similar to those in [5]. Our main result is 2.7. Section 3 contains an example.

2. Unbounded Wiener-Hopf operators. Let A be a closed, 1-1 linear operator on H with domain $D(A)$ and range $R(A)$ dense in H . For each $u \in D(A)$ let

$$\|u\|_+ = \|Au\|$$

where $\|\cdot\|$ denotes the norm in H . The completion of the pre-Hilbert space $\{D(A), \|\cdot\|_+\}$ is denoted by H_+ . The following is clear.

2.1. Lemma. *There exists an isometry A of H_+ onto H such that $A \mid H_+ \cap H = A$.*

Let P be a self adjoint projection on H and $Q = I - P$. Let P_+ be the orthogonal projection in H_+ onto the closure of $R(P) \cap D(A)$ in H_+ . Let $Q_+ = I_+ - P_+$ where I_+ is the identity on H_+ .

2.2. Lemma. *Let $R(P) \cap D(A)$ be dense in $R(P)$. Then $\dim R(P) = \dim R(P_+)$.*

Proof. Since $R(P) \cap D(A)$ is dense in the separable Hilbert space $R(P)$ we have $\dim R(P) = \dim R(P) \cap D(A)$. Similarly $\dim R(P) \cap D(A) = \dim R(P_+)$. Thus 2.2 follows.

By our assumption on A we are assured that A has an adjoint A^* and that A^* is 1-1 on its domain.

2.3. Lemma. *Let $D(A^{*-1}) \cap R(Q)$ be dense in $R(Q)$. Then $\dim R(Q) \leq \dim R(Q_+)$.*

Proof. Consider the 1-1 map $\Phi: D(A^{*-1}) \rightarrow H_+$ defined by $\Phi(v) = A^{-1}A^{*-1}v$. If $p \in R(P) \cap D(A)$ and $v \in D(A^{*-1}) \cap R(Q)$ then $(p, \Phi(v))_+ = (Ap, A^{*-1}v) = 0$. Since $R(P) \cap D(A)$ is dense in $R(P_+)$, we conclude that $\Phi[R(Q) \cap D(A^{*-1})] \subset R(Q_+)$ and thus that $\dim R(Q) \cap D(A^{*-1}) \leq \dim R(Q_+)$. By hypothesis $\dim R(Q) \cap D(A^{*-1}) = \dim R(Q)$ and thus the proof of 2.3 is complete.

2.4. Lemma. *Let $R(Q) \cap D(A^{*-1})$ be dense in $R(Q)$. Then either (i) $\dim R(Q) = \dim H$, or (ii) A bounded implies that $\dim R(Q) = \dim R(Q_+)$.*

Proof. If (i) holds then by 2.3 we have that

$$\dim H = \dim R(Q) \leq \dim R(Q_+) \leq \dim H_+.$$

and hence we have equality by 2.1.

Suppose that (ii) holds. Then $D(A) = H \subset H_+$ and $\|u\|_+ \leq \|A\| \cdot \|u\|$, $u \in H$. Thus for each $v \in H_+$ the map $h \rightarrow (h, v)_+$ is a bounded linear functional on H . Let $\Phi(v)$ be the unique element in H such that $(h, \Phi(v)) = (h, v)_+$ for each $h \in H$. Since H is densely contained in H_+ , the map $\Phi: H_+ \rightarrow H$ is 1-1. For each $v \in R(Q_+)$ and $p \in R(P)$ we have $(p, \Phi(v)) = (p, v)_+ = 0$ and thus $\Phi[R(Q_+)] \subset R(Q)$. Hence $\dim R(Q_+) \leq \dim R(Q)$ and by 2.3 we have our result.

For invertible operators, the following lemma was first proved by Rabinathan in the case where $H = L^2(T)$ and T is the circle group [4]. Later it was generalized by Devinatz and Shinbrot to the case where H is an arbitrary separable Hilbert space [2].

2.5. Lemma. *Let $R(P) \cap D(A)$ be dense in $R(P)$ and $R(Q) \cap D(A^{*-1})$ be dense in $R(Q)$. If $\dim R(Q) < \dim H$ we assume that A is bounded. Then there exists a unitary operator V on H , and an invertible operator $C_+: H_+ \rightarrow H$ satisfying $R(C_+P_+) = R(P)$, such that*

$$A = VC_+.$$

Proof. Since A is an isometry $A R(P_+)$ is closed in H . Let $h \in H$ be perpendicular to $A R(P_+)$. Then $0 = (Ap, h) = (p, A^*h)_+$ for each $p \in R(P_+)$ where $A^*: H \rightarrow H_+$ is the adjoint of A . Hence $h \in A^{*-1}R(Q_+)$. Conversely, it is seen that $A R(P_+) \perp A^{*-1}R(Q_+)$. Hence we have shown that

$$H = A R(P_+) \oplus A^{*-1}R(Q_+).$$

Since we also have the decomposition $H = R(P) \oplus R(Q)$, 2.2 and 2.4 imply that there exist a unitary operator V on H such that $V^*A R(P_+) = R(P)$ and $V^*A^{*-1}R(Q_+) = R(Q)$. If we let $C_+ = V^*A$ we have our result.

2.6. Definition. Let $F: H_+ \rightarrow H$ be a bounded linear operator. Let

$$\mathbf{T}(F) = PF \mid R(P_+).$$

Lemma 2.5 enables us to write

$$\mathbf{T}(\mathbf{A}) = T(V)\mathbf{T}(C_+)$$

where $\mathbf{T}(C_+)$ is invertible. Thus $\mathbf{T}(\mathbf{A})$ is invertible if and only if $T(V)$ is invertible. Since V is a unitary operator on H , we have by [2; Corollary 1] that $T(V)$ is invertible if and only if $QV \mid R(Q)$ is invertible. It is also shown in [2; Theorem 2] that, when $T(V)$ is invertible, the operator

$$G = PVP + QVQ$$

satisfies the condition

$$\|V - G\| < 1.$$

We now are able to state and prove the main result of this paper.

2.7. Theorem. Let A be a closed 1-1 operator on a Hilbert space H . Let P be a self adjoint projection on H such that $R(P) \cap D(A)$ is dense in $R(P)$ and $R(Q) \cap D(A^{*-1})$ is dense in $R(Q)$. If $\dim R(Q) < \dim H$ we assume that A is bounded. Then the following are equivalent:

- (i) $\mathbf{T}(\mathbf{A})$ is invertible.
- (ii) There exists an invertible map $L: H \rightarrow H_+$ such that $\operatorname{Re} LA \geq \delta > 0$, $R(LQ) = R(Q_+)$ and $R(L^*P_+) = R(P)$.
- (iii) The operator $\mathbf{A}$ may be factored as $\mathbf{A} = \mathbf{A}_-\mathbf{A}_+$ where $\mathbf{A}_+: H_+ \rightarrow H$ and $\mathbf{A}_-: H \rightarrow H$ are invertible operators such that $R(\mathbf{A}_+P_+) = R(P)$ and $R(\mathbf{A}_-^{-1}Q) = R(Q)$.

Proof. Suppose that $\mathbf{T}(\mathbf{A})$ is invertible. Then by the above discussion there exists a δ , $0 < \delta < 1$, such that $\|V - G\| < 1 - \delta$. Thus for each $h \in H$ we have

$$\|Vh\|^2 - 2 \operatorname{Re} (Vh, Gh) + \|Gh\|^2 \leq (1 - \delta)^2 \|h\|^2.$$

Since V is an isometry this implies that

$$-2 \operatorname{Re} (Vh, Gh) \leq \delta(\delta - 2) \|h\|^2.$$

The fact that $C_+: H_+ \rightarrow H$ is invertible implies that

$$\begin{aligned} \operatorname{Re} (VC_+u, GC_+u) &\geq \delta' \|C_+u\|^2 \\ &\geq \delta'' \|u\|_+^2 \end{aligned}$$

for each $u \in H_+$ where δ' and δ'' are constants > 0 . Hence if we let $L = (C_+)^*G^*$ we have

$$\begin{aligned} \operatorname{Re} (LAu, u)_+ &= \operatorname{Re} ((C_+)^*G^*Au, u)_+ \\ &= \operatorname{Re} (GVC_+u, C_+u) \\ &\geq \delta'' \|u\|_+^2. \end{aligned}$$

Since G is invertible and $R(GP) = R(P)$ it follows that $R(G^*Q) = R(Q)$. Thus $R(LQ) = R((C_+)^*Q) = R(Q_+)$. From this it follows that $R(L^*P_+) = R(P)$.

Now suppose that (ii) holds. Then $LA: H_+ \rightarrow H_+$ is an invertible operator with a strictly positive real part. Hence by [2, Theorem 5] there exists invertible operators B_+ and B_- on H_+ such that $R(B_+P_+) = R(P_+)$ and $R(B_-^{-1}Q_+) = R(Q_+)$. Since our hypotheses insure that $\dim R(P_+) = \dim R(P)$ and $\dim R(Q_+) = \dim R(Q)$ (see 2.2 and 2.4), we conclude that there exists an invertible operator $T: H_+ \rightarrow H$ such that $R(TP_+) = R(P)$ and $R(TQ_+) = R(Q)$. If we let $D_+ = TB_+$ and $D_- = B_-T^{-1}$, we have $LA = D_-D_+$ where $R(D_+P_+) = R(P)$ and $R(D_-^{-1}Q_+) = R(Q)$. Thus setting $A_+ = D_+$ and $A_- = L^{-1}D_-$ we have (iii).

Now suppose we are given (iii). Let $v \in R(P)$ and consider $u \in R(P_+)$ given by $u = A_+^{-1}PA_-^{-1}v$. Then

$$\begin{aligned} T(A)u &= PA[A_+^{-1}(I - Q)A_-^{-1}v] \\ &= Pv - A_-QA_-^{-1}v \\ &= v \end{aligned}$$

since $A_-QA_-^{-1}v \in R(Q)$. Hence $T(A)$ is onto. If $T(A)u = 0$, then $Au = q$ where $q \in R(Q)$. Hence $A_+u = A_-^{-1}q \in R(P) \cap R(Q)$ and thus $u = 0$. This completes the proof of 2.7.

For completeness we state a theorem which was verified in the proof of 2.7.

2.8. Corollary. *If $T(A)$ is invertible then*

$$T(A)^{-1} = A_+^{-1}PA_-^{-1}.$$

We conclude this section by considering the special cases where A is bounded below (*i.e.* there exists a $\delta > 0$ such that $\|Au\| \geq \delta \|u\|$ for each $u \in D(A)$) and where A is invertible.

If A is bounded below then $\|u\| \leq K \cdot \|u\|_+$ for some constant $K > 0$. This implies $H_+ = D(A)$ and $R(P_+) = R(P) \cap D(A)$ as point sets (the topologies are different). The operator $T(A)$ becomes $PA \upharpoonright R(P) \cap P(A)$. If $PA \upharpoonright R(P) \cap D(A): R(P) \cap D(A) \rightarrow R(P)$ is 1-1 and onto it may be inverted by $A_+^{-1}PA_-^{-1}$ which is now a bounded operator from H into H (since A_+^{-1} is bounded on H).

If A is invertible then 2.7 reduces to the theorem of Devinatz and Shinbrot mentioned in the introduction. More precisely we will show that when A is invertible (i), (ii), and (iii) of 2.7 are equivalent to the conditions (a), (b), and (c) stated in section 1.

If A is invertible then there exists $\delta_1, \delta_2 > 0$ such that $\|u\| \delta_1 \leq \|u\|_+ \leq \delta_2 \|u\|$ for each $u \in H$. Hence the map $\Psi: H \rightarrow H_+$ given by $h \rightarrow h$ is invertible. We have

$$A\Psi = A.$$

Since $R(P_+) = R(\Psi P)$ the relation $T(A)x = T(A)\Psi x$ for each $x \in R(P)$ implies the equivalence of (i) and (a).

Suppose (ii) holds. Then since Ψ is invertible there exists a $\delta > 0$ such that $\operatorname{Re} (LA\Psi x, \Psi x)_+ \geq \delta \cdot \|x\|$ for each $x \in H$. Hence using $(LA\Psi x, \Psi x)_+ = (A^*A\Psi^{-1}LAx, x)$ and setting $W = A^*A\Psi^{-1}L$ we have the first assertion of (b). A computation shows $R(Q_+) = R(\Psi A^{-1}A^{*-1}Q)$ and thus $R(WQ) = R(Q)$. Since W is invertible we also have $R(W^*P) = R(P)$. Conversely, if (b) holds we get (ii) by setting $L = \Psi A^{-1}A^{*-1}W$.

Now suppose that $A = A_-A_+$ as in (iii). Then $A = A\Psi = (A_-)(A_+\Psi)$ is the required factorization in (c). Conversely, if $A = A_-A_+$ as in (c), then $A = A\Psi^{-1} = (A_-)(A_+\Psi^{-1})$ is a factorization of A as in (iii).

3. An example. Let X be a compact Hausdorff space and D a Dirichlet algebra on X . Let m be a finite non-zero positive Baire measure on X which is multiplicative with respect to D . By D_0 we will denote the space of functions $f \in D$ such that $\int_X f dm = 0$.

For $1 \leq p \leq \infty$, L^p will denote the usual Lebesgue space with respect to dm and for $1 \leq p < \infty$, H^p will be the closure of D in L^p . H^∞ shall be $H^1 \cap L^\infty$ with the essential supremum norm. By H_0^2 we will mean the closure of D_0 in L^2 and by $\bar{H}_0^2$ the space of all complex conjugates of functions in H_0^2 .

For any $\phi \in L^2$, L_ϕ will denote the closed operator on L^2 given by $f \rightarrow \phi \cdot f$. In this setting the space H_+ of Theorem 2.7 becomes $L^2(|\phi|^2)$ the space of m measurable function f , such that $\int_X |f|^2 \cdot |\phi|^2 dm < \infty$. With P taken to be the orthogonal projection of L^2 onto H^2 the space $R(P_+)$ becomes $H^2(|\phi|^2)$, the closure of D in $L^2(|\phi|^2)$.

We now make the further assumption that $\ln |\phi| \in L^1$. This implies that there exists a $\Psi \in H^2$ which is an outer function such that $|\phi| = |\Psi|$ [3]. Hence, if we set $g = \phi/\Psi$ we have

$$g \cdot \Psi = \phi$$

where $|g| = 1$ a.e. This factorization, as we shall see in proof of 3.1, is the analog of the factorization in 2.5. We point out that the density conditions in 2.7 are consequences of the conditions $\phi \in L^2$ and $\ln |\phi| \in L^1$. Clearly, $\phi \in L^2$ implies that $D(L_\phi) \cap H^2$ is dense in H^2 . If $h \in D_0$, then $\bar{\Psi}h \in \bar{H}_0^2$ and thus

$$\int_X |\bar{\phi}^{-1}\bar{\Psi}h|^2 dm = \int_X |h|^2 dm < \infty$$

Since Beurling's theorem [3] implies that the set of all functions of the form $\bar{\Psi}h$ with $h \in D_0$ is dense in $\bar{H}_0^2$ we have the second density condition. Devinatz has shown me privately that when X is the circle group that the condition $\ln |\phi| \in L^1$ is in fact equivalent to the two density conditions stated above.

3.1. Lemma. $\mathbf{T}(L_\phi)$ is invertible if and only if $T(g)$ is invertible.

Proof. Since $|\phi| = |\Psi|$ we see that $L_\Psi: L^2(|\phi|^2) \rightarrow L^2$ is invertible and $L_\phi = L_g L_\Psi$. The map L_g is a unitary operator on L^2 .

We will show that $L_\Psi[H^2(|\phi|^2)] = H^2$ and thus $\mathbf{T}(L_\phi) = T(g)\mathbf{T}(L_\Psi)$ from which

3.1 follows. Let $f \in H^2(|\phi|^2)$ and $\{f_n\} \subset D$ such that $\int_X |f_n - f|^2 \cdot |\phi|^2 \, dm \rightarrow 0$. This implies $\int_X |\Psi f_n - \Psi f|^2 \, dm \rightarrow 0$, and since $\Psi \cdot f_n \in H^2$, that $\Psi f \in H^2$. Now let $h \in H^2$. Since Ψ is outer Beurling's theorem implies that there exists $\{h_n\} \subset D$ such that $\int_X |\Psi h_n - h|^2 \, dm \rightarrow 0$. Hence $\int_X |h_n - h/\Psi|^2 \cdot |\phi|^2 \, dm \rightarrow 0$. Thus $h/\Psi \in H^2(|\phi|^2)$ and the proof is complete.

We now show that the function L of 2.7 has a special form in the case we are considering.

3.2. Theorem. $T(L)$ is invertible if and only if there exists a function $\eta \in (1/\Psi)\tilde{H}^\infty$ such that

- (i) $\eta \cdot \tilde{H}_0^2 = H^2(|\phi|^2)^\perp$,
- (ii) $\operatorname{Re} (\eta \cdot \phi x, x)_+ \geq \delta \cdot \|x\|_+^2$ for all $x \in H_+$.

Proof. Suppose that $T(L_\phi)$ is invertible. Then by 3.1 $T(g)^* = T(\bar{g})$ is invertible. Hence by [1, page 313] there exists an $h \in H^\infty$ such that $1/h \in H^\infty$ and $|\arg h\bar{g}| \leq \pi/2 - \epsilon$ for some $\epsilon > 0$. Let $\eta = 1/\Psi \cdot \bar{h}$. Then $f \rightarrow \eta \cdot f$ is an invertible map of $L^2 \rightarrow L^2(|\phi|^2)$.

Since $|\arg \eta \cdot \phi| = |\arg g \cdot \bar{h}| \leq \pi/2 - \epsilon$ and $\operatorname{ess. inf} |\eta\phi| = \operatorname{ess. inf} |\bar{h}g| > 0$, we conclude that there exists a $\delta > 0$ such that $\operatorname{Re} \eta \cdot \phi \geq \delta$. Thus (ii) holds.

Let $k \in H^2(|\phi|^2)^\perp$. Then for each $f \in D$ we have $\int_X f \bar{k} \cdot |\phi|^2 \, dm = \int_X (\Psi f)(\bar{\Psi} \bar{k}) \, dm = 0$. By Beurling's theorem this implies that $\int_X f(k\Psi) \, dm = 0$ for each $f \in H^2$. Hence $k \in 1/\Psi \cdot \tilde{H}_0^2$. Reversing the above argument gives us $H^2(|\phi|^2)^\perp = 1/\Psi \tilde{H}_0^2$. From this we see that $\eta \cdot \tilde{H}_0^2 = 1/\Psi \cdot \bar{h} \cdot \tilde{H}_0^2 = 1/\Psi \cdot \tilde{H}_0^2$. This completes the proof of the necessity of (i) and (ii).

If (i) and (ii) hold for an $\eta \in (1/\Psi)\tilde{H}^\infty$ then the map $L: L^2 \rightarrow L^2(|\phi|^2)$ by $f \rightarrow \eta \cdot f$ is invertible and the sufficiency of (i) and (ii) follows from 2.7.

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University of Cincinnati

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