Polynomial Approximation and Spectral Properties of Composition Operators on H^2

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1. Introduction. If ϕ is a non-constant analytic function defined on and taking values in the open unit disk $\mathbf{D}$, one can define the **composition operator** C_ϕ by

$$(C_\phi f)(z) = (f \circ \phi)(z) = f(\phi(z))$$

for functions f analytic in $\mathbf{D}$. Such operators are bounded linear operators taking the Hardy classes H^p into themselves. Previous work on these operators has been done by Ryff [6], Nordgren [4], Schwartz [7], and Leon [2].

The spectral properties of these operators vary widely. In some cases the operator is compact, and its spectrum is a sequence of eigenvalues converging to zero [7]. If ϕ is an inner function (except for the identity $\phi(z) = z$), and $\phi(0) = 0$, C_ϕ is a unilateral shift of infinite multiplicity with a one-dimensional identity attached, and $\mathbf{D}$ is compression spectrum for C_ϕ . If ϕ is a linear fractional transformation, Nordgren showed it may have eigenvalues at cyclotomic points of the circle, at each point of the unit circle, or at each point of a closed annulus.

We consider the problem of when the range of C_ϕ or C_ϕ^* is dense in H^2 . For C_ϕ , this is equivalent to the question of when the polynomials are dense in $H^2(G)$, where G is a bounded, simply connected plane domain. If ϕ is continuous in the closed disk $\bar{\mathbf{D}}$, the range of C_ϕ is dense if and only if ϕ maps $\mathbf{D}$ univalently to a Jordan domain. A weaker condition is sufficient that $C_\phi H^2$ be dense in H^2 . On the other hand, C_ϕ^* always has a dense range. The proofs are valid, if interpreted properly, for H^p , $1 \leq p < \infty$.

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2. Polynomial Approximation. For a region G , $H^2(G)$ is defined as the set of all analytic functions f for which $|f|^2$ has a harmonic majorant in G . If G is simply connected, $H^2(G)$ consists exactly of the functions $g \circ \phi^{-1}$, where ϕ is any mapping function from $\mathbf{D}$ to G , $g \in H^2$; and if $H^2(G)$ is normed by letting

$\|f\|^2$ be the value of the least harmonic majorant of $|f|^2$ at $\varphi(0)$, this correspondence is an isometric isomorphism [1].

Lemma 1. *If φ is univalent in $\mathbf{D}$ and maps $\mathbf{D}$ to a region G , then the range of C_φ is dense in H^2 if and only if polynomials are dense in $H^2(G)$.*

Proof. Under the isometric isomorphism $f \rightarrow f \circ \varphi$ polynomials in w correspond to polynomials in φ . Since polynomials are dense in H^2 , the lemma follows easily.

A **Carathéodory set** is a bounded open set whose boundary is the boundary of the unbounded component of the complement of its closure. Thus G has no "inner" boundary points. The Carathéodory domains with continuous curves as boundary are the Jordan domains. Carathéodory domains are the domains in which polynomial approximation "ought" to occur; if $\bar{G}$ has holes in it, the polynomials should converge across them. If the approximation is bounded pointwise convergence, for example, this is what happens [5]. In the case of approximation in $H^2(G)$, one can show that approximation is possible in case G is Carathéodory, but it is more difficult to show it is impossible if G is not. We show the last in the case where the boundary of G is a continuous curve.

If G is a bounded domain, it is contained in a larger Carathéodory set G^* , called its **outer envelope**. G^* is simply the complement of the closure of the unbounded component of the complement of the closure of G . It is clear $H^2(G^*) \subseteq H^2(G)$, and if $f \in H^2(G^*)$, then $\|f\|_G \leq \|f\|_{G^*}$, for the least harmonic majorant for $|f|^2$ in G^* is a harmonic majorant for $|f|^2$ in G .

Theorem 1. *If G is a Carathéodory domain, polynomials are dense in $H^2(G)$. Conversely, if polynomials are dense in $H^2(G)$ and if the mapping function $\varphi: \mathbf{D} \rightarrow G$ is continuous in $\bar{\mathbf{D}}$, then G is a Jordan domain.*

Proof. If G is a Carathéodory domain, G is the kernel with respect to $w_0 \in G$ of a decreasing sequence of open, simply connected supersets G_n , $\bar{G}_{n+1} \subset G_n$. If ψ_n is the mapping function from G_n to G , $\psi_n(w_0) = w_0$, $\psi'_n(w_0) > 0$, then ψ_n converges to the identity function uniformly on compact subsets of G . ([3], v. III).

Let φ be any univalent mapping function from $\mathbf{D}$ to G . We first approximate $(\varphi^{-1})^k$ in $H^2(G)$. By Runge's theorem, there is a polynomial p_n approximating $(\varphi^{-1})^k \circ \psi_n$ uniformly within $1/n$ on G , and $\|p_n\|_\infty < \|(\varphi^{-1})^k\|_\infty + 1/n = 1 + 1/n$. Letting $\|f\|$ denote the $H^2(G)$ norm, $\|p_n\| \leq \|p_n\|_\infty < 1 + 1/n$. Choose a weakly convergent subsequence $\{p_{n_i}\}$ from $\{p_n\}$; let g be its weak limit point. Since evaluation at a point of G is a bounded linear functional on $H^2(G)$, weak convergence implies pointwise convergence. Thus $g(w) = (\varphi^{-1}(w))^k$ for $w \in G$, and $g = (\varphi^{-1})^k$. Since $1 = \|(\varphi^{-1})^k\| \leq \liminf \|p_{n_i}\| \leq 1$, $\|p_{n_i}\| \rightarrow \|(\varphi^{-1})^k\|$. In a Hilbert space, weak convergence and convergence of norm is equivalent to norm convergence, so $(\varphi^{-1})^k$ is approximable by polynomials.

Finally, since $(\varphi^{-1})^k$ is the function corresponding to z^k in H^2 , the collection

of these functions forms an orthonormal basis in $H^2(G)$. Thus every $H^2(G)$ function can be approximated by polynomials.

If G has a continuous boundary curve, G^* is a Jordan domain, for the outer boundary of G^* corresponds to an arc whose image is a simple closed curve. If G^* is mapped conformally to $\mathbf{D}$, the image of G is a subdomain G_1 of the disk whose outer envelope is $\mathbf{D}$. This follows from the possibility of extending the homeomorphism from G^* to $\mathbf{D}$ into a neighborhood of the closure of G^* . Moreover, since the mapping function is conformal on G , $H^2(G)$ and $H^2(G^*)$ are mapped isometrically to $H^2(G_1)$ and H^2 . Since by the first part of the proof, polynomials are dense in $H^2(G^*)$, and the inclusion map from $H^2(G^*)$ to $H^2(G)$ is norm-reducing, it suffices to show that H^2 is not dense in $H^2(G)$ for $G \subset \mathbf{D}$, $G^* = \mathbf{D}$, and $G \neq \mathbf{D}$.

We shall show there is a non-zero bounded linear functional on $H^2(G)$ which annihilates H^2 . Let $E = (\alpha, \beta)$ be a component of the open subset of the unit circle where $|\varphi(z)| < 1$. Then $\varphi(\alpha) = \varphi(\beta)$; for if not, the unit circle and $\varphi(E)$ together bound two disjoint domains, only one of which can be contained in G^* .

Let C_1 and C_2 be the images of the radii extending to α and β . Then if $f \in H^2$,

$$\int_{C_1} f dz = \lim_{|\zeta| \rightarrow 1} \int_0^\zeta f dz,$$

$\zeta \in C_1$, exists because the integral of f is a continuous function in $\bar{\mathbf{D}}$. Moreover, for $f \in H^2$, $\int_{C_1} f dz = \int_{C_2} f dz$.

If $w_0 \in \varphi(E)$, then $g(w) = (w - w_0)^{1/2}$ is bounded in G and is therefore in $H^2(G)$, where an appropriate branch of the square root is chosen. Approaching $\varphi(\beta) = \varphi(\alpha)$ from C_1 or C_2 , one obtains different branches of $(\varphi(\beta) - w_0)^{1/2}$, and hence

$$\int_{C_1} (w - w_0)^{1/2} dw = - \int_{C_2} (w - w_0)^{1/2} dw \neq 0.$$

Letting $\lambda(f) = \int_{C_1} f dz - \int_{C_2} f dz$, we obtain a nonzero bounded linear functional on the span of H^2 and g which can be extended to the desired functional on $H^2(G)$.

Note. By Szegő's theorem, the conclusion follows in case $G \subset \mathbf{D}$ and the harmonic measure for G , restricted to the unit circle, is absolutely continuous with respect to Lebesgue measure and its derivative is log-integrable. It is not clear, however, what this corresponds to geometrically, and it does not hold in general.

3. Composition operators.

Theorem 2. *If φ maps $\mathbf{D}$ conformally to a Carathéodory domain, then the range of C_φ is dense in H^2 . Conversely, if φ is analytic in $\mathbf{D}$ and the range of C_φ is dense in H^2 , then φ is univalent and if φ is continuous in $\bar{\mathbf{D}}$, then φ maps $\mathbf{D}$ to a Jordan domain.*

Proof. All that remains to be shown is that if the range of C_φ is dense, φ is univalent. If not, $\varphi(z_1) = \varphi(z_2)$, then let $g = k_{z_1} - k_{z_2}$, where k_z is the kernel function for evaluation at z : $(f, k_z) = f(z)$. Then g is orthogonal to the range of C_φ , for $(f \circ \varphi, g) = f(\varphi(z_1)) - f(\varphi(z_2))$. Thus if range C_φ is dense, φ is univalent.

We remark that range C_φ is H^2 only if C_φ is invertible and hence φ is a conformal mapping of $\mathbf{D}$ to itself.

In contrast, range C_ϕ^* is always dense. It is impossible that $f \circ \varphi$ be zero for non-zero f and non-constant φ , and so no function can be orthogonal to range C_ϕ^* .

Another way of looking at this in the case in which $\varphi(0) = 0$ may shed some light on the operator C_ϕ^* . If $\varphi(0) = 0$, C_ϕ^* has an upper triangular matrix with respect to the standard basis for H^2 ;

$$(C_\phi^* z^k, z^m) = (z^k, \varphi^m) = \frac{1}{k!} \frac{d^k}{dz^k} \overline{\varphi^m(0)} = 0 \quad \text{if } k < m.$$

The diagonal entries are $\{\overline{\phi'(0)^n} : n = 0, 1, 2, \dots\}$ and since the polynomials of degree less than n form an invariant subspace of C_ϕ^* for each n , C_ϕ^* has a polynomial of degree n as eigenvector corresponding to $\overline{\phi'(0)^n}$ for each $n \geq 0$. If $\phi'(0) = 0$, a similar relationship holds with the first non-zero Taylor coefficient. Thus the polynomials are contained in range C_ϕ^* , which is therefore dense.

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