Weighing Operator Spectra

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Introduction. Our primary interest in this paper relates to nonseparable spaces and the extension of classical statements involving the ideal of compact operators to more general statements which uniformly describe the same phenomena, as they relate to each of the closed two sided ideals of $\mathcal{L}(\mathcal{H})$. For example, Atkinson's theorem [1, 2] describes which operators are invertible modulo the ideal of compact operators, but does not give the corresponding description for invertibility modulo the other ideals. We shall give a formulation of Atkinson's theorem (Theorem 2.8) which shows no such favoritism towards a particular ideal.

Our main conceptual tool in developing these generalizations is the idea of weighing spectra. Just as the notion of nullity may be used to specify the multiplicities of eigenvalues, we consider the concept of "approximate nullity" which, using cardinals, can be used to "weigh" the elements of the approximate point spectrum of an operator. In this way we obtain, for each cardinal α which is less than or equal to the dimension of the space, a nonempty closed subset of the spectrum of an operator, which we call the spectrum of weight α . For normal operators, the spectrum of weight $\aleph_0$ is precisely the essential spectrum. Weyl's theorem [21] states that, for Hermitian operators, the essential spectrum is undisturbed by a perturbation with a compact Hermitian operator. The more general phenomenon (Theorem 4.6) is that the spectrum of weight α is undisturbed by a perturbation with an operator from the two sided ideal corresponding to α .

Since the closed two-sided ideals of $\mathcal{L}(\mathcal{H})$ are referred to throughout our work on nonseparable spaces, we next summarize the description of these ideals, obtained independently by L. A. Coburn and A. Lebow [4], B. Gramsch [8] and E. Luft [14].

Let h denote the dimension of the Hilbert space $\mathcal{H}$, where $h \geq \aleph_0$. For each cardinal α , $\aleph_0 \leq \alpha \leq h$, let $\mathcal{I}_\alpha$ denote the two sided ideal in $\mathcal{L}(\mathcal{H})$ of all bounded linear operators of rank less than α . Let $\mathcal{J}_\alpha$ denote the norm closure of $\mathcal{I}_\alpha$, in $\mathcal{L}(\mathcal{H})$. Then the $\mathcal{J}_\alpha$ are precisely the proper closed two sided ideals of $\mathcal{L}(\mathcal{H})$. Of course $\mathcal{J}_{\aleph_0}$ is the ideal of compact operators and $\mathcal{J}_h$ is the maximal closed

two sided ideal of $\mathfrak{L}(\mathfrak{H})$. If $\aleph_0 \leq \alpha < \beta \leq h$ then $\mathfrak{I}_\alpha \subset \mathfrak{I}_\beta$ and $\mathfrak{I}_\alpha \neq \mathfrak{I}_\beta$. The ideals $\mathfrak{I}_\alpha$ are self-adjoint. For α finite we shall also have occasion to consider the closed self-adjoint set $\mathfrak{J}_\alpha$ (not an ideal) of operators of rank less than α .

A subset S of $\mathfrak{H}$ is called α -bounded if, for each $\epsilon > 0$, there exists a set of vectors $\{f_m\}$, $m \in M$, such that $f_m \in S$, $\text{card } M < \alpha$ and

$$S \subset \bigcup_{m \in M} \mathcal{O}_\epsilon(f_m).$$

Here $\mathcal{O}_\epsilon(f_m)$ denotes the ball about f_m , with radius ϵ . A linear operator A is called α -compact if the image of the unit ball, under A , is α -bounded. The many characterizations of the ideals $\mathfrak{I}_\alpha$, found in the three papers [4], [8] and [14], may be conveniently summarized as one theorem.

Theorem 0. *Let A be an operator on a Hilbert space $\mathfrak{H}$ and let α be a cardinal, $\aleph_0 \leq \alpha \leq \dim \mathfrak{H}$. Then the following five conditions are equivalent.*

- i) $A \in \mathfrak{I}_\alpha$ (i.e., A is in the norm closure of the set $\mathfrak{J}_\alpha$ of operators of rank less than α).
- ii) A is α -compact.
- iii) Each closed linear subspace $\mathfrak{H}_0$ of $\mathfrak{H}$ for which $\mathfrak{H}_0 \subset A(\mathfrak{H})$, has dimension less than α .
- iv) (Rellich condition) For each $\epsilon > 0$, there exists a closed linear subspace $\mathfrak{H}_\epsilon \subset \mathfrak{H}$ with codimension less than α such that $\|A|_{\mathfrak{H}_\epsilon}\| < \epsilon$.
- v) Every subspace of $\mathfrak{H}$ on which A is bounded below has dimension less than α .

We can now be more explicit concerning the contents of this paper. The first section is preliminary. The second section has two major theorems listing a number of conditions equivalent to left invertibility modulo $\mathfrak{I}_\alpha$ (Theorem 2.6) and also to invertibility modulo $\mathfrak{I}_\alpha$ (Theorem 2.8). In this section we introduce the notions of α -closed subspace and α -Fredholm operator (α is a cardinal). In the third section we develop the general theory of weighing operator spectra. In Section 4 we obtain more explicit information on the weighted spectra of normal operators (Theorem 4.1). The observation that, for normal operators, the spectrum $\Lambda_{\aleph_0}(A)$ of weight $\aleph_0$ is precisely the essential spectrum of A , leads us to generalizations of Weyl's theorem (Theorem 4.5 and Corollary 4.7). In section 5 we formulate a natural generalization (Theorem 5.7) of von Neumann's theorem (a converse of Weyl's theorem) for a Hilbert space of dimension h , and show that this formulation is true for some cardinals h (even for normal operators) and false for other cardinals (even for positive operators). We also obtain a nonseparable analogue (Theorem 5.6) of a recent result of I. D. Berg [3], which asserts that for certain nonseparable Hilbert spaces, every normal operator may be expressed as a sum of a diagonal operator and an h -compact operator. The last Section 6 uses the notion of weighted spectra to obtain a nonseparable version of Radjavi's theorem on self-commutators.

The authors are the three members of a discovery course on Hilbert space, given at the University of California, Santa Barbara in 1969–70, based on the text, *A Hilbert Space Problem Book* by Paul Halmos [10]. All the results of

this paper are “spin-offs” from the prior consideration of problems posed in Halmos’ book. We therefore want to make perfectly clear the indebtedness of this research to that work, both as a source of motivation and as a mathematical resource.

1. Approximate nullity. T. Kato (*cf.* p. 233 of [13]) has considered the notion of approximate nullity. We here modify that notion somewhat by distinguishing between different infinite cardinals. In this section we state two basic lemmas, and leave their proofs as exercises. Lemma 1.1 is obtained first for normal operators by using the spectral theorem and then applying this to the positive part of the polar decomposition of an arbitrary operator.

Lemma 1.1. *Let A be a bounded linear operator on the Hilbert space $\mathcal{H}$ and let $\epsilon > 0$. Then there exists a closed subspace $\mathcal{K}$ of $\mathcal{H}$, containing the kernel of A , such that*

$$\|Af\| < \epsilon \|f\| \quad \text{for all } f \in \mathcal{K}, \quad f \neq 0,$$

and

$$\|Af\| \geq \epsilon \|f\| \quad \text{for all } f \in \mathcal{K}^\perp.$$

Furthermore, if A is a normal operator, there exists such a closed subspace which reduces A .

Lemma 1.2. *Let A be an operator on $\mathcal{H}$ and let $\epsilon > 0$. Suppose $\mathcal{K}$ is a closed subspace of $\mathcal{H}$ such that $\|Af\| < \epsilon \|f\|$ for all $f \in \mathcal{K}$, $f \neq 0$, and suppose $\mathcal{L}$ is a closed subspace such that $\|Af\| \geq \epsilon \|f\|$ for all $f \in \mathcal{L}^\perp$. Then $\dim \mathcal{K} \leq \dim \mathcal{L}$ and $\dim \mathcal{L}^\perp \leq \dim \mathcal{K}^\perp$.*

The subspace $\mathcal{K}$ given in Lemma 1.1 is not unique. However it is an immediate corollary of Lemma 1.2 (with $\mathcal{K} = \mathcal{L}$) that all the subspaces satisfying the conditions of Lemma 1.1 have the same dimension. We are thus justified in defining $\delta_\epsilon(A)$ to be the common dimension of all subspaces $\mathcal{K}$ satisfying the conditions of Lemma 1.1.

Definition 1.3. We define the *approximate nullity* $\delta(A)$ of an operator A to be

$$\delta(A) = \min_{\epsilon > 0} \delta_\epsilon(A).$$

2. Atkinson’s theorem. Atkinson’s theorem [1, 2] (*cf.* problem 142 of [10]) precisely characterizes the elements of $\mathcal{L}(\mathcal{H})$ which are invertible modulo the ideal $\mathcal{K}_\infty$ of compact operators. They are precisely the Fredholm operators, *i.e.*, those operators A with closed range, finite nullity $\nu(A)$ and finite corank $\rho'(A)$. This characterization has motivated a more general study of operators invertible modulo an arbitrary ideal $\mathcal{I}$, the so called $\mathcal{I}$ -Fredholm operators of H. O. Cordes [5]. In this section we shall characterize the operators which are invertible modulo the ideals $\mathcal{I}_\alpha$ and $\mathcal{J}_\alpha$ in a manner analogous to Atkinson’s theorem.

For this purpose it is necessary to introduce the notion of an α -closed subspace. This concept is hinted at by L. A. Coburn and A. Lebow [4], who show that if A is left invertible modulo the ideal $\mathcal{I}_\alpha$, then its range is closed "except for a space of dimension less than α ." (Throughout this paper the term subspace is used for a linear subspace, which need not be closed.)

Definition 2.1. A subspace $\mathcal{K}$ of a Hilbert space $\mathcal{H}$ is called α -closed if there is a closed subspace $\mathcal{L}$ of $\mathcal{H}$ such that $\mathcal{L} \subset \mathcal{K}$ and such that $\dim(\mathcal{K} \cap \mathcal{L}^\perp) < \alpha$.

Note that a subspace $\mathcal{K}$ is 1-closed if and only if it is closed. Further whenever $\mathcal{L}$ is a closed subspace of $\mathcal{H}$ contained in $\mathcal{K}$, then $\dim(\mathcal{K} \cap \mathcal{L}^\perp) = \dim(\mathcal{K}^\perp \cap \mathcal{L}^\perp)$ and hence either expression may be used in Definition 2.1. This is justified by the following lemma, whose proof we leave as an exercise.

Lemma 2.2. Let $\mathcal{K}$ be a linear subspace of a Hilbert space $\mathcal{H}$ and let $\mathcal{L}$ be a closed subspace of $\mathcal{H}$, which is contained in $\mathcal{K}$. Then

$$(\mathcal{K} \cap \mathcal{L}^\perp)^\perp = \mathcal{K}^\perp \cap \mathcal{L}^\perp.$$

Lemma 2.3. Let $\mathcal{K}$ be a subspace of $\mathcal{H}$. Then $\mathcal{K}$ is $\aleph_0$ -closed if and only if it is closed.

Proof. Suppose $\mathcal{K}$ is $\aleph_0$ -closed. Then there exists a closed subspace $\mathcal{L}$, $\mathcal{L} \subset \mathcal{K}$, such that $\mathcal{K} \cap \mathcal{L}^\perp$ is finite dimensional and hence closed. Thus $\mathcal{K} = (\mathcal{K} \cap \mathcal{L}^\perp) \oplus \mathcal{L}$ is closed.

Lemma 2.4. Let $\mathcal{H}$ be a Hilbert space of dimension h and $1 \leq \alpha \leq h$. If $\mathcal{K}$ and $\mathcal{L}$ are subspaces of $\mathcal{H}$ and $\mathcal{K}$ is closed and $\dim \mathcal{L} < \alpha$, then $\mathcal{K} + \mathcal{L}$ is α -closed.

Lemma 2.5. Suppose A is a one-to-one bounded linear operator of a Hilbert space $\mathcal{K}$ into another Hilbert space $\mathcal{L}$. Let $\mathfrak{M}$ be a subspace of $\mathcal{K}$ such that $A(\mathfrak{M})$ is closed. Then

$$\dim \mathfrak{M}^\perp \leq \dim (A(\mathfrak{M}))^\perp.$$

Theorem 2.6. Let $\mathcal{H}$ be a Hilbert space of infinite dimension h . Let $A \in \mathcal{L}(\mathcal{H})$ and let α be a cardinal such that $1 \leq \alpha \leq h$. Then the following five conditions are equivalent.

- i) A is left invertible modulo $\mathcal{I}_\alpha$.
- ii) A is left invertible modulo $\mathcal{J}_\alpha$.
- iii) A is bounded below on some closed subspace of codimension less than α .
- iv) $\delta(A) < \alpha$.
- v) $\nu(A) < \alpha$ and $\text{ran } A$ is α -closed.

Proof. The implication i) implies ii) is trivial. We next show ii) implies iii). Let B be an operator in $\mathcal{L}(\mathcal{H})$ such that $I - BA \in \mathcal{I}_\alpha$. Clearly $B \neq 0$. By Lemma 1.1 there exists a closed subspace $\mathcal{K}$ of $\mathcal{H}$ such that

$$\|BAf\| < \frac{1}{2} \|f\| \quad \text{for } f \in \mathcal{K}, \quad f \neq 0$$

and

$$\|BAf\| \geq \frac{1}{2} \|f\| \quad \text{for } f \in \mathcal{K}^\perp.$$

Thus

$$\|Af\| \geq 2^{-1} \|B\|^{-1} \|f\| \quad \text{for } f \in \mathcal{K}^\perp.$$

Hence it is sufficient to show that $\dim \mathcal{K} < \alpha$. But for $f \in \mathcal{K}$ we have

$$\|(I - BA)f\| \geq \|f\| - \|BAf\| \geq (1 - \frac{1}{2}) \|f\| = \frac{1}{2} \|f\|.$$

Thus $(I - BA)$ is bounded below on $\mathcal{K}$. Since $(I - BA) \in \mathcal{J}_\alpha$, Theorem 0 (characterization v)) implies that $\dim \mathcal{K} < \alpha$.

We next show iii) implies iv). Suppose A is bounded below on a closed subspace $\mathcal{L}$ such that $\dim \mathcal{L}^\perp < \alpha$. Choose $\epsilon > 0$ such that $\|Af\| \geq \epsilon \|f\|$ for all f in $\mathcal{L}$. Using this ϵ , let $\mathcal{K}$ denote a closed subspace satisfying the conditions of Lemma 1.1. By Lemma 1.2, $\dim \mathcal{K} \leq \dim \mathcal{L}^\perp < \alpha$. But $\delta_\epsilon(A) = \dim \mathcal{K}$. Hence $\delta(A) \leq \delta_\epsilon(A) < \alpha$.

We next show iv) implies v). Since $\delta(A) = \min_{\epsilon > 0} \delta_\epsilon(A)$, there exists an $\epsilon_0 > 0$ such that $\delta(A) = \delta_{\epsilon_0}(A)$. By Lemma 1.1 there exists a closed subspace $\mathcal{K}$ such that $\ker A \subset \mathcal{K}$, $\dim \mathcal{K} = \delta_{\epsilon_0}(A) < \alpha$, $\|Af\| < \epsilon_0 \|f\|$ for $f \in \mathcal{K}$, $f \neq 0$ and $\|Af\| \geq \epsilon_0 \|f\|$ for $f \in \mathcal{K}^\perp$. Since A is bounded below on $\mathcal{K}^\perp$, $A(\mathcal{K}^\perp)$ is closed. Clearly $\dim A(\mathcal{K}) \leq \dim \mathcal{K} < \alpha$. Further $\text{ran } A = A(\mathcal{K}^\perp) + A(\mathcal{K})$. Hence Lemma 2.4 implies that $\text{ran } A$ is α -closed. Finally $\nu(A) \leq \delta(A) < \alpha$.

The proof is concluded by showing v) implies i). Since $\text{ran } A$ is α -closed, there exists a closed subspace $\mathfrak{N}$ contained in the range of A such that

$$\dim ((\text{ran } A)^\perp \cap \mathfrak{N}^\perp) < \alpha.$$

(If α is finite, $\text{ran } A$ is closed and we choose $\mathfrak{N} = \text{ran } A$.) Let $\mathfrak{M} = A^{-1}(\mathfrak{N}) \cap (\ker A)^\perp$. Then A is a one-to-one bounded linear map of $(\ker A)^\perp$ into $(\text{ran } A)^\perp$ which maps $\mathfrak{M}$ onto the closed subspace $\mathfrak{N}$. We may therefore apply Lemma 2.5, with $\mathcal{K} = (\ker A)^\perp$ and $\mathcal{L} = (\text{ran } A)^\perp$ to conclude that

$$\dim (\mathfrak{M}^\perp \cap (\ker A)^\perp) \leq \dim (\mathfrak{N}^\perp \cap (\text{ran } A)^\perp).$$

Since A maps $\mathfrak{M}$ one-to-one onto $\mathfrak{N}$, $A|_{\mathfrak{M}}$ admits an inverse map of $\mathfrak{N}$ onto $\mathfrak{M}$. Define B in $\mathcal{L}(\mathfrak{M})$ by requiring that it agree with this inverse on $\mathfrak{N}$, and annihilate $\mathfrak{N}^\perp$. Thus $BA|_{\mathfrak{M}} = I|_{\mathfrak{M}}$ or $(I - BA)|_{\mathfrak{M}} = 0$. Hence $\rho(I - BA) \leq \dim \mathfrak{M}^\perp$. Since $\mathfrak{M} \subset (\ker A)^\perp$ we have $\ker A \subset \mathfrak{M}^\perp$ and thus

$$\dim \mathfrak{M}^\perp = \dim (\ker A) + \dim (\mathfrak{M}^\perp \cap (\ker A)^\perp).$$

Hence

$$\begin{aligned} \rho(I - BA) &\leq \dim (\ker A) + \dim (\mathfrak{M}^\perp \cap (\ker A)^\perp) \\ &\leq \nu(A) + \dim (\mathfrak{N}^\perp \cap (\text{ran } A)^\perp) \\ &< \alpha + \alpha = \alpha \quad \text{if } \alpha \text{ is infinite} \\ &< \alpha + 0 = \alpha \quad \text{if } \alpha \text{ is finite.} \end{aligned}$$

L. A. Coburn and A. Lebow [4] have previously obtained a condition equivalent to left invertibility modulo $\mathcal{G}_\alpha$ which is almost the same as our condition iii).

Definition 2.7. Let $\mathcal{H}$ be a Hilbert space of dimension h and let α be a cardinal such that $1 \leq \alpha \leq h$. An operator A on $\mathcal{H}$ is called an α -Fredholm operator if $\nu(A) < \alpha$, $\rho'(A) < \alpha$ and the range of A is α -closed.

Note that an operator A is a Fredholm operator in the usual sense (i.e., $\nu(A) < \aleph_0$, $\rho'(A) < \aleph_0$ and $\text{ran } A$ is closed) if and only if A is an $\aleph_0$ -Fredholm operator. One need only recall Lemma 2.3 which asserts that a subspace is $\aleph_0$ -closed if and only if it is closed.

The equivalent statements of Theorem 2.6 give us two other characterizations of the α -Fredholm operators. Thus A is an α -Fredholm operator if and only if $\rho'(A) < \alpha$ and $\delta(A) < \alpha$. Alternatively, A is an α -Fredholm operator if and only if $\rho'(A) < \alpha$ and A is bounded below on some subspace of codimension less than α . We can now give our generalization of Atkinson's theorem.

Theorem 2.8. Let $\mathcal{H}$ be a Hilbert space of infinite dimension h and let α be a cardinal, $1 \leq \alpha \leq h$. An operator A on $\mathcal{H}$ is (1) an α -Fredholm operator if and only if it is (2) invertible modulo $\mathcal{G}_\alpha$ (the set of operators of rank less than α), or alternatively, if and only if it is (3) invertible modulo $\mathcal{G}_\alpha$.

Proof. We first prove that condition (1) implies condition (2). Since $\text{ran } A$ is α -closed, there exists a closed subspace $\mathcal{K}$ contained in $\text{ran } A$ such that $\dim((\text{ran } A)^\perp \cap \mathcal{K}^\perp) < \alpha$. (If α is finite, then $\text{ran } A$ is closed (cf. Lemma 2.3), in which case we take $\mathcal{K} = \text{ran } A$.) Let $\mathfrak{N} = A^{-1}(\mathcal{K}) \cap (\ker A)^\perp$. A maps $(\ker A)^\perp$ one-to-one onto $\text{ran } A$, and maps $\mathfrak{N}$ one-to-one onto $\mathcal{K}$. Since $\mathcal{K}$ is closed, A has a bounded linear inverse on $\mathfrak{N}$. Define $B \in \mathcal{L}(\mathcal{H})$ such that $BA \upharpoonright \mathfrak{N} = I_\mathfrak{N}$, $AB \upharpoonright \mathcal{K} = I_\mathcal{K}$ and $B \upharpoonright \mathcal{K}^\perp = 0$. Thus $(I - BA) \upharpoonright \mathfrak{N} = 0$. We may now argue, exactly as we did in the proof that condition v) implies condition i) in Theorem 2.6, to conclude that $\rho(I - BA) < \alpha$ and hence that $(I - BA) \in \mathcal{G}_\alpha$. Similarly $\rho(I - AB) \leq \dim \mathcal{K}^\perp = \dim((\text{ran } A)^\perp \cap \mathcal{K}^\perp) + \rho'(A) < \alpha$.

As the implication (2) implies (3) is obvious, we complete the proof by showing (3) implies (1). Since A is left invertible modulo $\mathcal{G}_\alpha$, we have $\nu(A) < \alpha$ and $\text{ran } A$ is α -closed by Theorem 2.6. Since A is right invertible, modulo $\mathcal{G}_\alpha$, A^* is left invertible modulo $\mathcal{G}_\alpha$ and hence, again by Theorem 2.6, we have $\nu(A^*) = \rho'(A) < \alpha$.

3. Weighted Spectra. The nullity ν may be used to indicate how much an eigenvalue counts in the spectrum. In the same spirit, we shall use the approximate nullity δ and the corank ρ' , to weight elements in the approximate point spectrum and the compression spectrum respectively.

Definition 3.1. Let H be a Hilbert space of dimension h . Let $A \in \mathcal{L}(\mathcal{H})$ and let α be a cardinal, $1 \leq \alpha \leq h$.

The *approximate point spectrum* of A , of weight α , denoted $\Pi_\alpha(A)$, is the set of complex numbers λ such that $\delta(A - \lambda I) \geq \alpha$.

The *compression spectrum* of A , of weight α , denoted $\Gamma_\alpha(A)$, is the set of complex numbers λ such that $\rho'(A - \lambda I) \geq \alpha$.

The *spectrum* of A , of weight α , denoted $\Lambda_\alpha(A)$ is defined by $\Lambda_\alpha(A) = \Pi_\alpha(A) \cup \Gamma_\alpha(A)$.

Note that $\Pi_1(A) = \Pi(A)$, the usual approximate point spectrum of A . Similarly $\Gamma_1(A) = \Gamma(A)$ and $\Lambda_1(A) = \Lambda(A)$. Further if $1 \leq \alpha < \beta \leq h$, then $\Pi_\beta(A) \subset \Pi_\alpha(A)$, $\Gamma_\beta(A) \subset \Gamma_\alpha(A)$ and $\Lambda_\beta(A) \subset \Lambda_\alpha(A)$.

The point spectrum $\Pi_0(A)$ may also be placed in such a hierarchy by defining, for $1 \leq \alpha \leq h$, $\Pi_{0\alpha}(A)$ to be the set of eigenvalues of A , of multiplicity at least α , i.e., $\Pi_{0\alpha}(A) = \{\lambda : \nu(A - \lambda I) \geq \alpha\}$. Thus $\Pi_{01}(A)$ is the usual point spectrum $\Pi_0(A)$.

The weighted spectra at the top of the hierarchy (largest cardinal, but smallest set) play an important enough role in the theory to deserve a special label. Recalling that h denotes $\dim \mathcal{H}$, we call $\Pi_h(A)$, $\Gamma_h(A)$, $\Lambda_h(A)$ and $\Pi_{0h}(A)$ the *heavy approximate point spectrum*, the *heavy compression spectrum*, the *heavy spectrum* and the *heavy point spectrum* respectively.

Our original motivation in developing Theorems 2.6 and 2.8 was to obtain various characterizations of these weighted spectra. For example, Theorem 2.6 (interchanging conditions i) and iv)) may be reformulated as follows.

Theorem 3.2. *Let A be an operator on a Hilbert space of infinite dimension h , and let α be a cardinal, $1 \leq \alpha \leq h$. Then the following conditions are equivalent*

- i) $\lambda \in \Pi_\alpha(A)$.
- ii) $A - \lambda I$ is not left invertible modulo $\mathcal{J}_\alpha$.
- iii) Every subspace of $\mathcal{H}$ on which $A - \lambda I$ is bounded below, has codimension $\geq \alpha$.
- iv) $A - \lambda I$ is not left invertible modulo $\mathcal{S}_\alpha$.
- v) Either λ is an eigenvalue of A of multiplicity at least α , or the range of $A - \lambda I$ is not α -closed.

Corollary 3.3. *If $A \notin \mathcal{J}_\alpha$, then $\Pi_\alpha(A) = \{0\}$.*

Corollary 3.4. *If n is a positive integer, then*

$$\Pi_n(A) = \Pi_{n_0}(A) \cup \Pi_{0n}(A).$$

Just as Theorem 3.2 is a reformulation of Theorem 2.6, Theorem 2.8 gives us some useful characterizations of the spectrum of weight α .

Theorem 3.5. *Let A be an operator on a Hilbert space $\mathcal{H}$ of infinite dimension h . Let α be a cardinal, $1 \leq \alpha \leq h$. Then the following are equivalent.*

- i) $\lambda \in \Lambda_\alpha(A)$.
- ii) $A - \lambda I$ is not invertible modulo $\mathcal{J}_\alpha$.
- iii) $A - \lambda I$ is not invertible modulo $\mathcal{S}_\alpha$.
- iv) Either λ is an eigenvalue of multiplicity at least α , or $\rho'(A - \lambda I) \geq \alpha$, or the range of $A - \lambda I$ is not α -closed.

Corollary 3.6. *If α is an infinite cardinal, then $\Lambda_\alpha(A)$ is precisely the ordinary spectrum of the canonical image of A , in the quotient C^* -algebra $\mathcal{L}(\mathcal{H})/\mathcal{I}_\alpha$. Hence $\Lambda_\alpha(A)$ is non-empty and compact.*

Theorem 3.7. *Let A be an operator on a Hilbert space $\mathcal{H}$ of infinite dimension h . Let α be an infinite cardinal, $\aleph_0 \leq \alpha \leq h$. Then $\Pi_\alpha(A)$ contains the boundary of $\Lambda_\alpha(A)$.*

Proof. Let $\mathfrak{B}$ denote the quotient algebra $\mathcal{L}(\mathcal{H})/\mathcal{I}_\alpha$, which is a C^* -algebra with unit (cf. Proposition 1.8.2 of [6]). Thus $\mathfrak{B}$ has a faithful representation as a C^* -algebra of operators acting on some Hilbert space $\mathcal{H}_0$. Let $\mathfrak{A} = \mathcal{L}(\mathcal{H}_0)$. We make the appropriate identification and consider $\mathfrak{B}$ as a subalgebra of $\mathfrak{A}$, where $\mathfrak{A}$ and $\mathfrak{B}$ have a common unit element I_0 . Let q denote the canonical map of $\mathcal{L}(\mathcal{H})$ onto $\mathfrak{B}$.

According to Theorem 3.2, $\Pi_\alpha(A)$ is precisely the set of complex numbers λ such that $(q(A) - \lambda I_0)$ is not left invertible in $\mathfrak{B}$. Thus $\Pi_\alpha(A)$ clearly contains $\Pi_\alpha(q(A))$, the set of complex numbers λ for which $q(A) - \lambda I_0$ is not left invertible in $\mathfrak{A}$. By the known facts about ordinary spectra and approximate point spectra of operators (cf. problem 63 of [10]) we have that $\Pi_\alpha(q(A))$ contains the boundary of $\Lambda_\alpha(q(A))$, the spectrum of $q(A)$ in the algebra $\mathfrak{A} = \mathcal{L}(\mathcal{H}_0)$. However $\Lambda_\alpha(q(A))$ is equal to $\Lambda_\mathfrak{B}(q(A))$, the spectrum of $q(A)$ in the algebra $\mathfrak{B}$ (cf. Proposition 1.3.10 of [6]), which in turn is equal to $\Lambda_\alpha(A)$ by Corollary 3.6. Putting all this together we have

$$\text{boundary}(\Lambda_\alpha(A)) = \text{boundary}(\Lambda_\alpha(q(A))) \subset \Pi_\alpha(q(A)) \subset \Pi_\alpha(A).$$

Remark 3.8. By now it is quite apparent that most of the well-known facts about spectra have appropriate analogies for weighted spectra. We summarize a few of the results we have obtained, leaving their proofs as exercises. For any α , $\aleph_0 \leq \alpha \leq h$, $\Pi_\alpha(A)$ is a non-empty compact set. The spectral mapping theorem for polynomials is true for Π_α and Λ_α . In fact essentially all of Chapter 8 of [10] is valid for Π_α and Λ_α . We also have:

- (1) $(\Pi_{0\alpha}(A))^* = \Gamma_\alpha(A^*)$.
- (2) $\Lambda_\alpha(A^*) = \Pi_\alpha(A^*) \cup \Pi_\alpha(A)^*$ and hence $\Lambda_\alpha(A^*) = (\Lambda_\alpha(A))^*$.
- (3) If A is normal, then $\Gamma_\alpha(A) = \Pi_{0\alpha}(A)$ and hence $\Lambda_\alpha(A) = \Pi_\alpha(A)$.

As an example we mention that for the unilateral shift U on l_2 we have, for $2 \leq \alpha \leq \aleph_0$, that

$$\Pi_{0\alpha}(U) = \Pi_{0\alpha}(U^*) = \Gamma_\alpha(U) = \Gamma_\alpha(U^*) = \emptyset$$

and

$$\Lambda_\alpha(U) = \Lambda_\alpha(U^*) = \Pi_\alpha(U) = \Pi_\alpha(U^*) = \{\lambda : |\lambda| = 1\}.$$

4. Weyl's theorem. We begin this section with a more explicit description of the weighted spectra for normal operators. For this purpose we shall use a version of the spectral theorem which is valid for nonseparable Hilbert spaces (cf. Section 97 of [10] and Section X.5 of [7]). Every normal operator may be represented as

a multiplication by a bounded measurable function φ on a space $L^2(X, \mu)$ of square integrable complex valued functions on a (possibly not σ -finite) measure space (X, μ) .

Theorem 4.1. *Let A be a normal operator on an infinite dimensional Hilbert space, represented as a multiplication by the function φ on the space $L^2(X, \mu)$. Let $h = \dim L^2(X, \mu)$ and let α be a cardinal number, $1 \leq \alpha \leq h$. For any measurable set Y of complex numbers, let*

$$\mathcal{R}_Y = \{f : f \in L^2(X, \mu) \text{ and } f \text{ vanishes a.e. off } \varphi^{-1}(Y)\}.$$

Then $\lambda \in \Pi_\alpha(A)$ if and only if $\dim \mathcal{R}_U \geq \alpha$ for every measurable neighborhood U of λ .

Proof. Consider first neighborhoods of the form $U_\epsilon = \{z : |z - \lambda| < \epsilon\}$ where $\epsilon > 0$. Then it is an easy computation to show $\|(A - \lambda)f\| < \epsilon \|f\|$ for all $f \in \mathcal{R}_{U_\epsilon}$, $f \neq 0$ and $\|(A - \lambda)f\| \geq \epsilon \|f\|$ for all $f \in \mathcal{R}_{U_\epsilon}^\perp$. Hence $\delta_\epsilon(A - \lambda) = \dim \mathcal{R}_{U_\epsilon}$.

Suppose $\lambda \in \Pi_\alpha$ and U is any measurable neighborhood of λ . Choose $\epsilon > 0$ such that $\lambda \in U_\epsilon \subset U$. Then by the above remark

$$\dim \mathcal{R}_U \geq \dim \mathcal{R}_{U_\epsilon} = \delta_\epsilon(A - \lambda) \geq \delta(A - \lambda) \geq \alpha.$$

Conversely suppose $\dim \mathcal{R}_U \geq \alpha$ for every measurable neighborhood U of λ . Applying this to neighborhoods of the form U_ϵ , we have $\delta_\epsilon(A - \lambda) \geq \alpha$ for every $\epsilon > 0$. Thus $\delta(A - \lambda) \geq \alpha$ and $\lambda \in \Pi_\alpha(A)$.

This theorem may be interpreted in the special case of a diagonal operator, since a diagonal operator is a multiplication where the measure space (X, μ) is discrete. Let D denote a diagonal operator. Then $\lambda \in \Pi_\alpha(D)$ if and only if, for every neighborhood U of λ , U contains at least α diagonal elements of D , where a diagonal element is counted just as many times as it appears. In particular if λ itself appears in the diagonal α times, then λ is an eigenvalue of multiplicity α and hence $\lambda \in \Pi_{0\alpha}(D) \subset \Pi_\alpha(D)$.

Lemma 4.2. *If S is a compact set of complex numbers and $\dim \mathcal{R}_S \geq \alpha \geq \aleph_0$, then $\Pi_\alpha(A) \cap S \neq \emptyset$.*

Proof. In this lemma we are, of course, continuing with the hypotheses of Theorem 4.1, i.e., A is a normal operator represented as a multiplication φ on the space $L^2(X, \mu)$.

If Z is a compact set of complex numbers, we shall say Z is α -large if $\dim \mathcal{R}_Z \geq \alpha$. Notice that if $Z_1, \dots, Z_m$ are compact sets of complex numbers and $Z = \bigcup_{i=1}^m Z_i$ is α -large, then at least one of the Z_i is α -large.

The set S is clearly nonempty since $\dim \mathcal{R}_S \geq \aleph_0$. We define a sequence S_n of nonempty compact sets inductively. Let $S_0 = S$. Suppose S_{n-1} has been defined and is a nonempty compact α -large set of diameter less than or equal to $1/(n-1)$. Write $S_{n-1} = \bigcup_{i=1}^m Z_i$ where the Z_i are compact sets of diameter $\leq 1/n$. At least one of the Z_i is α -large. Let this be S_n . Thus $\{S_n\}$ is a nested collection of

nonempty α -large compact sets with diameters converging to zero. Hence $\bigcap_{n=1}^{\infty} S_n = \{\lambda\}$, for some $\lambda \in S$.

We claim $\lambda \in \Pi_{\alpha}(A)$. Indeed let $\epsilon > 0$ and let n be a positive integer such that $1/n < \epsilon$. Then $S_n \subset \{z : |z - \lambda| < \epsilon\} = U_{\epsilon}$ and hence $\delta_{\epsilon}(A - \lambda) = \dim \mathcal{R}_{U_{\epsilon}} \geq \dim \mathcal{R}_{S_n} \geq \alpha$. Since $\epsilon > 0$ was arbitrary, we have $\delta(A - \lambda) \geq \alpha$ so that $\lambda \in \Pi_{\alpha}(A)$.

Theorem 4.3. *Let A be a normal operator on a Hilbert space of dimension h , and let α be a cardinal, $\aleph_0 \leq \alpha \leq h$. Then $A \in \mathcal{J}_{\alpha}$ if and only if $\Pi_{\alpha}(A) = \{0\}$.*

Proof. We continue to argue in terms of the representation of A as a multiplier φ on $L^2(X, \mu)$. The fact that $\Pi_{\alpha}(A) = \{0\}$ for elements of $\mathcal{J}_{\alpha}$ has already been observed for general operators (Corollary 3.3).

Suppose $\Pi_{\alpha}(A) = \{0\}$. Let $\epsilon > 0$ and let $S = \{z : z \in \mathbb{C}, \epsilon/2 \leq |z| \leq \|A\|\}$. Then S is a compact set having no intersection with $\Pi_{\alpha}(A)$. By Lemma 4.2 we conclude that $\dim \mathcal{R}_S < \alpha$. Further it is easy to see that $\|Af\| < (\epsilon/2) \|f\|$ for all $f \in \mathcal{R}_S^{\perp}$, $f \neq 0$. By the Rellich condition characterizing the ideal $\mathcal{J}_{\alpha}$ (Theorem 0, Part iv)) we conclude that $A \in \mathcal{J}_{\alpha}$.

We remark that the normality condition on the operator A is necessary for the implication: $\Pi_{\alpha}(A) = \{0\}$ implies $A \in \mathcal{J}_{\alpha}$. Let α be any infinite cardinal and let $\mathcal{H}_0$ be a Hilbert space of dimension α . Define A on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0$ by $A = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$. Then A is not normal, $A \notin \mathcal{J}_{\alpha}$ and yet $\Pi_{\alpha}(A) = \{0\}$.

Recall that if A is a normal operator, the *essential spectrum* of A , $\Lambda_e(A)$, is defined to be the set of limit points of the spectrum $\Lambda(A)$, together with the eigenvalues of infinite multiplicity. A straightforward proof based on Theorem 4.1 and the spectral theorem gives us the following observation.

Theorem 4.4. *Let A be a normal operator on an infinite dimensional space. Then $\Lambda_e(A) = \Lambda_{\aleph_0}(A)$.*

When A is normal, we have $\Pi_{\aleph_0}(A) = \Lambda_{\aleph_0}(A) = \Lambda_e(A)$. Thus $\Pi_{\aleph_0}(A)$ and $\Lambda_{\aleph_0}(A)$ are both candidates for a generalized notion of essential spectrum for nonnormal operators. However we are rather late in the game of generalizing the notion of essential spectrum. Indeed K. Gustafson and J. Weidmann [9] have surveyed the literature and found seven concepts of essential spectrum, for nonnormal operators, both bounded and unbounded. Thus it is not surprising that $\Pi_{\aleph_0}(A)$ and $\Lambda_{\aleph_0}(A)$ can be found in their list.

For example, F. Wolf [23] has defined the essential spectrum of A (denoted σ_e^3 by Gustafson and Weidmann) to be the set of complex numbers λ such that $(A - \lambda)$ is not a Fredholm operator. By Atkinson's theorem, this is precisely those complex λ for which $(A - \lambda)$ is not invertible modulo the ideal $\mathcal{J}_{\aleph_0}$ of compact operators. By Theorem 3.5 this is precisely $\Lambda_{\aleph_0}(A)$.

Similarly T. Kato [13] considered the set (denoted $\sigma_e^{\nu}(A)$ by Gustafson and Weidmann) of complex numbers λ such that either $\nu(A - \lambda I) \geq \aleph_0$ or $\text{ran}(A - \lambda I)$ is not closed. By Theorem 3.2, with $\alpha = \aleph_0$ (and recalling that a space is $\aleph_0$ -closed if and only if it is closed) we have $\sigma_e^{\nu}(A) = \Pi_{\aleph_0}(A)$.

T. Kato [13] considered still another notion of essential spectrum for non-

normal operators, denoted $\Sigma_*(A)$ by Kato and $\sigma_*^2(A)$ by Gustafson and Weidmann [9]. This is the set of λ such that either $\nu(A - \lambda I) \geq \aleph_0$ and $\rho'(A - \lambda I) \geq \aleph_0$ or $\text{ran } (A - \lambda I)$ is not closed. It is an easy observation that

$$\sigma_*^2(A) = \Pi_{\aleph_0}(A) \cap \Pi_{\aleph_0}(A^*)^*$$

once one has noted that $\text{ran } (A - \lambda I)$ is closed if and only if $\text{ran } (A^* - \bar{\lambda} I)$ is closed (cf. Theorem 5.13 of [13]).

Weyl's theorem [22] (cf. page 367 of [18]) asserts that if A and $A + C$ are normal and C is compact, then $\Lambda_*(A) = \Lambda_*(A + C)$. Thus the following theorem may be considered a generalization of Weyl's theorem, both in the direction of nonnormality and in the consideration of perturbations by operators in the other closed two sided ideals of $\mathfrak{L}(\mathfrak{H})$.

Theorem 4.5. *Let A be an operator on a Hilbert space of infinite dimension h . Let α be a cardinal, $\aleph_0 \leq \alpha \leq h$. Then*

- i) $\Pi_\alpha(A) = \Pi_\alpha(A + C)$ for any $C \in \mathfrak{J}_\alpha$
- ii) $\Lambda_\alpha(A) = \Lambda_\alpha(A + C)$ for any $C \in \mathfrak{J}_\alpha$.

This theorem is an immediate consequence of the characterization of $\Pi_\alpha(A)$ and $\Lambda_\alpha(A)$ given in Theorems 3.2 and 3.5 respectively.

Theorem 4.5 states that the spectrum of weight α is not disturbed by a perturbation by an element of the ideal $\mathfrak{J}_\alpha$. Can all the other points of the spectrum be annihilated by an appropriate perturbation from $\mathfrak{J}_\alpha$? Our next theorem gives an affirmative answer in the case where the operator is normal.

Theorem 4.6. *Let A be a normal operator on a Hilbert space $\mathfrak{H}$ of infinite dimension h . Let α be a cardinal, $\aleph_0 \leq \alpha \leq h$. Then there exists a normal operator C in $\mathfrak{J}_\alpha$ such that C commutes with A and*

$$\Lambda(A + C) = \Lambda_\alpha(A).$$

Proof. We shall first prove the theorem under the assumption that $0 \in \Lambda_\alpha(A)$. As usual, let A be represented as a multiplication φ on the space $L^2(X, \mu)$, where φ is a bounded measurable function on X .

Note that $\Lambda_\alpha(A)$ is a compact subset of complex numbers (Corollary 3.9) and hence we may define the bounded measurable function ψ on X by $\psi(x) = \varphi(x)$ if $\varphi(x) \notin \Lambda_\alpha(A)$ and $\psi(x) = 0$ if $\varphi(x) \in \Lambda_\alpha(A)$. Let B denote the operator given by multiplication on $L^2(X, \mu)$ by ψ .

Suppose λ is a nonzero element of $\Lambda(B)$. Since $\Lambda(B)$ is the essential range of ψ , we have $\lambda \notin \Lambda_\alpha(A)$. By Theorem 4.1, there exists a measurable neighborhood U of λ such that

$$\dim \{f : f \in L^2(X, \mu) \text{ and } f \text{ vanishes a.e. off } \varphi^{-1}(U)\} < \alpha.$$

Since $\Lambda_\alpha(A)$ is closed and $\lambda \notin \Lambda_\alpha(A)$, there exists an open neighborhood V of λ such that $V \subset U$ and $V \cap \Lambda_\alpha(A) = \emptyset$. Then it is easy to verify that $\psi^{-1}(V) \subset \varphi^{-1}(V) \subset \varphi^{-1}(U)$. Hence

$$\dim \{f : f \in L^2(X, \mu) \text{ and } f \text{ vanishes a.e. off } \psi^{-1}(V)\} < \alpha.$$

By Theorem 4.1 we conclude that $\lambda \notin \Lambda_\alpha(B)$. Hence $\Lambda_\alpha(B) = \{0\}$ and thus, by Theorem 4.3, $B \in \mathcal{J}_\alpha$.

Let $C = -B$. Then C is a normal operator which commutes with A , is contained in $\mathcal{J}_\alpha$, and such that $\Lambda(A + C) = \Lambda_\alpha(A)$.

Throughout this proof we have assumed $0 \in \Lambda_\alpha(A)$. The general result may now be obtained by applying this result to the translated operator $A - \lambda I$, where λ is some element of $\Lambda_\alpha(A)$.

Corollary 4.7. *If A is a normal operator on a Hilbert space of infinite dimension h , and if α is a cardinal, $\aleph_0 \leq \alpha \leq h$, then*

$$\Lambda_\alpha(A) = \bigcap_{C \in \mathcal{J}_\alpha} \Lambda(A + C).$$

Theorem 4.5 states that a perturbation by an α -compact operator (*i.e.*, an operator in the ideal $\mathcal{J}_\alpha$) does not disturb the spectrum of weight α , of any operator. We next show that this property actually characterizes the ideal $\mathcal{J}_\alpha$.

Theorem 4.8. *An operator B is contained in $\mathcal{J}_\alpha$ if and only if*

$$\Lambda_\alpha(A) = \Lambda_\alpha(A + B)$$

for all A in $\mathcal{L}(\mathcal{H})$.

Proof. In one direction this is Theorem 4.5. The other direction is a generalization of Theorem 4.1 of [9] and the proof is essentially the same as that given by Gustafson and Weidmann for the $\alpha = \aleph_0$ case.

5. Von Neumann's theorem. Weyl's theorem states that two normal operators which are unitarily equivalent modulo a compact operator, have the same essential spectrum. Von Neumann's theorem [15], is a converse of this theorem, under the additional assumptions that the space $\mathcal{H}$ is separable and the operators are Hermitian (*cf.* page 367 of [18]).

We shall consider a possible generalization of von Neumann's theorem to a nonseparable Hilbert space of dimension h . Noting that on a separable space, the essential spectrum and the heavy spectrum are the same (Theorem 4.4) we formulate the question as follows. If two operators A and B have the same heavy spectrum, $\Lambda_h(A) = \Lambda_h(B)$, are they then unitarily equivalent modulo an h -compact operator? For self-adjoint operators and $h = \aleph_0$, the yes answer is precisely von Neumann's theorem. For cardinals larger than $\aleph_0$ we were somewhat surprised to discover that the answer is "yes" for some cardinals (even for normal operators), and "no" for other cardinals (even for positive operators). Following Bernhard Gramsch (*cf.* Definition 1.3 of [8]) we define a cardinal α to be $\aleph_0$ -irregular if it is the sum of countably many cardinals strictly smaller than α . A cardinal which is not $\aleph_0$ -irregular is said to be $\aleph_0$ -regular. The answer is "yes" for normal operators A and B (Theorem 5.7 below) if h is an $\aleph_0$ -irregular

cardinal. The answer is “no” for positive operators A and B , if h is an $\aleph_0$ -regular cardinal.

The proof of von Neumann’s theorem in the separable case uses Hilbert-Schmidt operators. We next introduce the notion of α -Hilbert-Schmidt operators which will play an analogous role in the proof of von Neumann’s theorem in the nonseparable case.

Definition 5.1. Let $\mathcal{H}$ be a Hilbert space of infinite dimension h , and let α be a cardinal, $\aleph_0 \leq \alpha \leq h$. An operator $C \in \mathcal{L}(\mathcal{H})$ will be called an α -Hilbert-Schmidt operator if $\mathcal{H}$ admits a decomposition $\mathcal{H} = \sum_{i=1}^{\infty} \oplus \mathcal{H}_i$, as a direct sum of countably many subspaces such that $\rho(C \upharpoonright \mathcal{H}_i) < \alpha$, $i = 1, 2, \dots$, and $\sum_{i=1}^{\infty} \|C \upharpoonright \mathcal{H}_i\|^2 < +\infty$.

Every Hilbert-Schmidt operator is an α -Hilbert-Schmidt, but not conversely. It is also clear that every α -Hilbert-Schmidt operator is an α -compact operator. We next obtain (Lemma 5.3) a rather specialized uncountable analogue of a result of von Neumann [15] on permutations of sequences. (See also [12] and [24].)

Definition 5.2. Let $S = \{a_n : n \in N\}$ be an indexed set in a metric space (X, d) , where cardinality $N = h$. If α is a cardinal, $\aleph_0 \leq \alpha \leq h$, and $x \in X$, we say x is an α -cluster point of the indexed set S if, for every neighborhood U of x

$$\text{card } \{n : n \in N \text{ and } a_n \in U\} \geq \alpha.$$

Note that if $N = \mathbb{Z}^+$ so that S is a sequence, then x is an $\aleph_0$ -cluster point of S if and only if it is a cluster point of the sequence in the usual sense. This concept is relevant to operator theory since the spectrum of weight α of a diagonal operator D is precisely the set of α -cluster points of the indexed set of diagonal elements of D .

Lemma 5.3. Let $\mathcal{A} = \{a_n : n \in N\}$ and $\mathcal{B} = \{b_n : n \in N\}$ denote two indexed sets in a compact metric space (X, d) , both indexed by the same set N . We further suppose $\text{card } N = h$ is $\aleph_0$ -irregular. If $\mathcal{A}$ and $\mathcal{B}$ have the same set S of h -cluster points, and $\mathcal{A} \subset S$ and $\mathcal{B} \subset S$, then there exists a countable partition $\{N_i : i \in \mathbb{Z}^+\}$ of N such that $\text{card } N_i < h$ for all i , and a permutation σ of N such that

$$d(a_n, b_{\sigma(n)}) < i^{-1} \text{ for all } n \in N_i.$$

Proof. Since h is an $\aleph_0$ -irregular cardinal, we may express N as a countable disjoint union of subsets M_i , such that $\text{card } M_i < h$ for $i = 1, 2, \dots$. We order N in such a way that if $i < j$ and $n \in M_i$ and $m \in M_j$, then $n < m$.

We proceed in a manner similar to that used by Halmos [12], by defining, inductively, an injective map ψ of N into N . If n_1 is the first element of N , a_{n_1} is an h -cluster point of $\mathcal{B}$ and hence there is an element $b_m \in \mathcal{B}$ such that $d(a_{n_1}, b_m) < 1$. Define $\psi(n_1) = m$. Suppose $k \in N$ and ψ has been defined for all $n \in N$, $n < k$. Suppose $k \in M_i$. Since a_k is an h -cluster point of $\mathcal{B}$,

$$\text{card } \{m : m \in N, d(a_k, b_m) < j^{-1}\} = h$$

while

$$\text{card } \{n: n \in N \text{ and } n < k\} \leq \sum_{i=1}^i \text{card } M_i < h.$$

Hence there exists $m_k \in N$ such that $d(a_k, b_{m_k}) < j^{-1}$ and $m_k \neq \psi(n)$ for any $n < k$. Define $\psi(k) = m_k$. We thus have an injective map ψ of N into N such that $d(a_n, b_{\psi(n)}) < j^{-1}$ for all $n \in M_i$ and $j = 1, 2, \dots$. By symmetry there exists an injective map φ of N into N such that $d(a_{\varphi(n)}, b_n) < j^{-1}$ for all $n \in M_i$ and $j = 1, 2, \dots$.

We now apply the Schröder-Bernstein theorem, including some information contained in its proof, to conclude that there exists a one-to-one map σ of N onto N such that, for every $n \in N$, either $\sigma(n) = \psi(n)$ (in which case $d(a_n, b_{\sigma(n)}) < j^{-1}$ if $n \in M_i$) or $\sigma(n) = \varphi^{-1}(n)$ (in which case $d(a_n, b_{\sigma(n)}) < j^{-1}$ if $n \in \sigma^{-1}(M_i)$). Then the countable partition of N satisfying the conditions of the lemma is defined, for each positive integer i , by

$$N_i = \left(\bigcup_{i \leq i} (M_i \cap \sigma^{-1}(M_i)) \right) \cup \left(\bigcup_{i \leq i} (M_i \cap \sigma^{-1}(M_i)) \right).$$

Theorem 5.4. *Let $\mathcal{H}$ be a Hilbert space of infinite dimension h , where h is an $\aleph_0$ -irregular cardinal. If A and B are two diagonal operators on $\mathcal{H}$ such that $\Lambda_h(A) = \Lambda_h(B)$, then A and B are unitarily equivalent modulo $\mathcal{J}_h$.*

Proof. We first note that we may assume, without loss of generality, that $\Lambda(A) = \Lambda_h(A)$ and $\Lambda_h(B) = \Lambda(B)$. Indeed if not we may use Theorem 4.8 to obtain operators C and D in $\mathcal{J}_h$ such that $\Lambda(A + C) = \Lambda_h(A + C) = \Lambda_h(A)$ and $\Lambda(B + D) = \Lambda_h(B + D) = \Lambda_h(B)$. An examination of the proof of Theorem 4.6, specialized to diagonal operators, shows that such a C and D can be found such that $A + C$ and $B + D$ are again diagonal operators. Finally it is easy to see that $A + C$ is unitarily equivalent to $B + D$ modulo $\mathcal{J}_h$ if and only if A is unitarily equivalent to B modulo $\mathcal{J}_h$.

Let A be the diagonal operator $\{a_n\}$ with respect to the orthonormal basis $\{e_n\}$ of $\mathcal{H}$. Similarly let B be the diagonal operator $\{b_n\}$ with respect to the orthonormal basis $\{f_n\}$, and let the index set for both bases be denoted N . Then $\mathcal{A} = \{a_n: n \in N\} \subset \Lambda(A) = \Lambda_h(A)$, the set of h -cluster points of $\mathcal{A}$. Similarly $\mathcal{B} = \{b_n: n \in N\} \subset \Lambda(B) = \Lambda_h(B)$, the set of h -cluster points of $\mathcal{B}$, and $\Lambda_h(A) = \Lambda_h(B)$. Thus there exists a countable partition $\{N_i\}$ of N and a permutation σ of N , satisfying the conditions of Lemma 5.3. Define a unitary operator U on $\mathcal{H}$ by $Ue_n = f_{\sigma(n)}$ for all $n \in N$. Hence with respect to the basis $\{e_n\}$, the operator $C = U^*BU - A$ is the diagonal operator $\{b_{\sigma(n)} - a_n: n \in N\}$.

Finally we note that C is an h -Hilbert-Schmidt operator and hence an h -compact operator. Let $\mathcal{H}_i$ denote the closed subspace of $\mathcal{H}$ spanned by $\{e_n: n \in N_i\}$. Then $\mathcal{H} = \sum_{i=1}^{\infty} \oplus \mathcal{H}_i$ and $\rho(C \upharpoonright \mathcal{H}_i) < h$ since $\dim \mathcal{H}_i < h$. Finally

$$\begin{aligned} \sum_{i=1}^{\infty} \|C \upharpoonright \mathcal{H}_i\|^2 &= \sum_{i=1}^{\infty} \left(\sup_{n \in N_i} |a_n - b_{\sigma(n)}| \right)^2 \\ &\leq \sum_{i=1}^{\infty} i^{-2} < +\infty. \end{aligned}$$

As a corollary of this result and a recent result of I. D. Berg [3] we obtain the separable von Neumann theorem for normal operators. This result has also been obtained, independently, by W. Sikonja [19].

Corollary 5.5. *If A and B are normal operators on a separable Hilbert space which have the same essential spectrum, then A and B are unitarily equivalent modulo a compact operator.*

Proof. When the space is separable, the essential spectrum and the heavy spectrum are the same for normal operators (Theorem 4.4) and $\mathcal{J}_\lambda$ is the ideal of compact operators. Thus the corollary follows from Theorem 5.4 and the recent result of I. D. Berg [3] which asserts that every normal operator may be written as a sum of a diagonal operator and a compact operator. (Recall that a perturbation by a compact operator does not change the essential spectrum.)

Our next result is an extension to nonseparable spaces of a theorem of Weyl [22], Berg [3] and Sikonja [19].

Theorem 5.6. *Let $\mathcal{H}$ be a Hilbert space of dimension h , where h is an $\aleph_0$ -irregular cardinal, $h > \aleph_0$. Then every normal operator A on $\mathcal{H}$ may be written $A = D + C$, where D is a diagonal operator and C is an h -Hilbert-Schmidt operator (and hence an h -compact operator).*

Proof. Since $h > \aleph_0$, $\mathcal{H}$ may be decomposed as a direct sum of h separable subspaces, which reduce A . Since h is $\aleph_0$ -irregular we may group these h subspaces to obtain a decomposition $\mathcal{H} = \sum_{i=1}^{\infty} \oplus \mathcal{H}_i$ where $\aleph_0 \leq \dim \mathcal{H}_i < h$ and each $\mathcal{H}_i$ reduces A . Let $A = \sum_{i=1}^{\infty} \oplus A_i$. Then each A_i is a normal operator on $\mathcal{H}_i$. It is an easy exercise, using the spectral theorem for normal operators, to show that the set of diagonal operators on a Hilbert space is norm dense in the set of normal operators. Hence for each i , there is a diagonal operator D_i on $\mathcal{H}_i$ such that $\|A_i - D_i\| < 2^{-i}$, $i = 1, 2, \dots$. Finally, let $D = \sum_{i=1}^{\infty} \oplus D_i$ and $C = \sum_{i=1}^{\infty} \oplus (A_i - D_i)$.

Theorem 5.7. *Let $\mathcal{H}$ be a Hilbert space of dimension h , where h is an $\aleph_0$ -irregular cardinal, $h > \aleph_0$. Then any two normal operators with the same heavy spectrum are unitarily equivalent modulo an h -compact operator.*

Theorem 5.7 is our extension of von Neumann's theorem to nonseparable Hilbert spaces of $\aleph_0$ -irregular dimension. It is an immediate consequence of Theorems 5.4 and 5.6. The example, following Theorem 4.3, of a nonnormal operator A , with heavy spectrum $\Lambda_h(A) = \{0\}$, which is *not* an h -compact operator, can be used to show that the hypothesis of normality in Theorem 5.7 is necessary.

We conclude this section with a counter-example which shows Theorem 5.7 is false for infinite Hilbert spaces of dimension h , when h is $\aleph_0$ -regular. For this purpose we shall first prove a lemma which extends a theorem of E. Luft (cf. Theorem 6.4 of [14]). E. Luft has shown that if α is not a limit cardinal, then $\mathcal{J}_\alpha = \mathcal{J}_\alpha$. We next show that even if α is a limit cardinal, we get $\mathcal{J}_\alpha = \mathcal{J}_\alpha$ whenever α is $\aleph_0$ -regular.

Lemma 5.8. *If α is an $\aleph_0$ -regular cardinal, then $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha$.*

Proof. By the previously mentioned Luft result we may assume, without loss of generality, that α is a limit cardinal. First note that $\mathfrak{g}_\beta \subset \mathfrak{g}_\alpha$ for any $\beta < \alpha$. Indeed if $A \in \mathfrak{g}_\beta$ then $\rho(A) \leq \beta < \alpha$.

However, according to Theorem 2.3 of [8], if α is $\aleph_0$ -regular, and is a limit cardinal, then $\mathfrak{g}_\alpha = \bigcup_{\beta < \alpha} \mathfrak{g}_\beta$. Hence $\mathfrak{g}_\alpha \subset \mathfrak{g}_\alpha$.

We proceed with our counter-example. Let h be an infinite cardinal which is $\aleph_0$ -regular. Let $\mathcal{H}$ be a Hilbert space of dimension h and let $\{e_i\}$, $i \in N$, be an orthonormal basis for $\mathcal{H}$. Let E and F denote disjoint, countable, dense subsets of $[0, 1]$. Let A denote the diagonal operator on $\mathcal{H}$ (relative to the basis $\{e_i\}$) where the diagonal elements are the elements of E , each repeated h times. Similarly let B denote the diagonal operator where the diagonal elements are the elements of F , each repeated h times. Then A and B are positive operators and $\Lambda_h(A) = \Lambda(A) = [0, 1] = \Lambda(B) = \Lambda_h(B)$. Thus A and B are positive operators with the same heavy spectrum and the same essential spectrum. Nevertheless we shall show that A is not unitarily equivalent to B modulo $\mathfrak{g}_h$.

Suppose, to the contrary, that A is unitarily equivalent to B modulo $\mathfrak{g}_h$. Then there exists a unitary operator U and an operator C , $C \in \mathfrak{g}_h$, such that $A + C = UBU^*$. Since h is $\aleph_0$ -regular we have, by Lemma 5.8 that $\mathfrak{g}_h = \mathfrak{g}_h$ and hence $\rho(C) < h$. Let $\mathcal{K} = (\text{ran } C)^\perp \vee (\ker C)^\perp$. Further $\dim \mathcal{K} \leq \dim ((\text{ran } C)^\perp) + \dim ((\ker C)^\perp) \leq \rho(C) + \rho(C) < h$. Thus $\mathcal{K}$ is a proper subspace of $\mathcal{H}$. Let $\mathcal{L}$ denote the closed subspace of $\mathcal{H}$ generated by those basis elements e_i which are not orthogonal to $\mathcal{K}$. Then clearly $\mathcal{K} \subset \mathcal{L}$. Now it is easy to verify that if $\dim \mathcal{K} = \beta$, then at most $\aleph_0\beta$ of the elements of the orthonormal basis $\{e_i\}$, can have nontrivial projection onto $\mathcal{K}$. Thus $\dim \mathcal{L} = \dim \mathcal{K} < h$ and $\mathcal{L}$ is also a proper subspace of $\mathcal{H}$. Consider the matrix representations $A = (A_{ij})$, $C = (C_{ij})$ and $UBU^* = (B_{ij})$, $i, j = 1, 2$, relative to the decomposition $\mathcal{H} = \mathcal{L} \oplus \mathcal{L}^\perp$. Since $\mathcal{L}$ reduces A , $A_{12} = A_{21} = 0$. Since $(\text{ran } C)^\perp \subset \mathcal{K} \subset \mathcal{L}$ we have $C_{21} = C_{22} = 0$. Since $(\ker C)^\perp \subset \mathcal{K} \subset \mathcal{L}$ we have $\mathcal{L}^\perp \subset \ker C$ and hence $C_{12} = C_{22} = 0$. Since $(A_{ij}) + (C_{ij}) = (B_{ij})$ we have $B_{12} = B_{21} = 0$ and $A_{22} = B_{22}$. Since $\mathcal{L}$ is a proper subspace of $\mathcal{H}$, we have $A_{22} \neq 0$. Since A is a diagonal operator with diagonal entries from E , A_{22} has eigenvalues contained in E . Hence B_{22} and thus B has eigenvalues from E . But $\Pi_0(B) = F$, which is disjoint from E . This contradiction shows that A cannot be unitarily equivalent to B modulo $\mathfrak{g}_h$.

6. Radjavi's theorem on self-commutators. In this last section we turn to the problem which was in fact the beginning of our research program; the problem of identifying the self-commutators (operators of the form $A^*A - AA^*$) on a nonseparable Hilbert space. Robert Thompson [21] identified the self-commutators on a finite dimensional space as the Hermitian operators with trace 0. Heydar Radjavi [17] identified the self-commutators on a separable Hilbert space as the Hermitian operators whose essential spectra contain at least one nonnegative point and at least one nonpositive point. Recall (Theorem 4.4)

that on a separable Hilbert space, the essential spectrum and the heavy spectrum of a Hermitian operator are the same. In this section we shall prove Radjavi's theorem is valid for nonseparable spaces, provided only that the term "essential spectrum" is replaced by the term "heavy spectrum".

Proposition 6.1. *Let $\mathcal{H}$ be a Hilbert space of infinite dimension and let S be a self-commutator on $\mathcal{H}$. Then the heavy spectrum $\Lambda_h(S)$ of S contains at least one nonnegative element and at least one nonpositive element.*

Proof. Since S is a self-commutator, it is clearly self-adjoint and thus $\Lambda_h(S)$ is a nonempty compact subset of the real line. Let $\mathfrak{B}$ denote the algebra $\mathcal{L}(\mathcal{H})/\mathcal{J}_h$, which is a C^* -algebra with unit. The algebra $\mathfrak{B}$ thus has a faithful representation as a C^* -algebra of operators acting on some Hilbert space $\mathcal{H}_0$. By means of an identification, we consider $\mathfrak{B}$ as a subalgebra of $\mathcal{L}(\mathcal{H}_0)$, where $\mathcal{L}(\mathcal{H}_0)$ and $\mathfrak{B}$ have a common unit element I_0 . Let q denote the quotient map of $\mathcal{L}(\mathcal{H})$ onto $\mathfrak{B}$. Clearly $q(S)$ is a self-commutator in $\mathfrak{B} \subset \mathcal{L}(\mathcal{H}_0)$. Furthermore, by Corollary 3.6, the heavy spectrum $\Lambda_h(S)$ of S is precisely the same as the ordinary spectrum of $q(S)$ in $\mathfrak{B}$, which in turn (Proposition 1.3.10 of [6]) is the spectrum $\Lambda(q(S))$ of $q(S)$ in $\mathcal{L}(\mathcal{H}_0)$. Suppose $\Lambda_h(S)$ ($= \Lambda(q(S))$) does not have the property stated in the proposition. Then either $\Lambda(q(S))$ consists only of strictly positive numbers or $\Lambda(q(S))$ consists only of strictly negative numbers. In the first case $q(S)$ is a self-commutator which is invertible (since $0 \notin \Lambda(q(S))$) and which is positive. This contradicts the theorem of C. R. Putnam [16], which states that a positive self-commutator cannot be invertible (cf. problem 188 of [10]). If $\Lambda(q(S))$ consists only of strictly negative numbers, then $q(-S)$ contradicts Putnam's theorem.

The proof of the converse of Proposition 6.1 will require a series of preliminary results. The proof of Proposition 6.2 is straight forward and is left to the reader. Lemma 6.3 is similar, both in its statement and proof, to Lemma 2 of [17] and its proof is therefore omitted.

Proposition 6.2. *An operator A is a self-commutator if and only if it is the difference of two unitarily equivalent Hermitian operators.*

Lemma 6.3. *Let A be a normal operator on a Hilbert space $\mathcal{H}$ of infinite dimension h , and let $\lambda \in \Lambda_h(A)$. Then A may be decomposed $A = \sum_{n=0}^{\infty} \oplus A_n$ such that each operator A_n acts on a space of dimension h and $\|A_n - \lambda\| \leq 2^{-n} \|A - \lambda\|$.*

Lemma 6.4. *Suppose A is a Hermitian operator of the form $A = \sum_{i=0}^{\infty} \oplus A_i$, where each A_i is an operator acting on the same infinite dimensional Hilbert space $\mathcal{H}$. Define the operators B_n, C_n on $\mathcal{H}$, for $0 \leq n < \aleph_0$, by $B_n = \sum_{i=0}^n \oplus A_{2i}$ and $C_n = -\sum_{i=0}^n \oplus A_{2i+1}$. If there is a bound r such that $\|B_n\| < r$ and $\|C_n\| < r$ for all n , $0 \leq n < \aleph_0$, then A is a self-commutator.*

Proof. Define P on the space of A by $P = \sum_{i=0}^{\infty} \oplus P_i$ where $P_1 = 0$, $P_{2i} = B_i$, $0 \leq i < \aleph_0$, and $P_{2i+1} = C_{i-1}$, $1 \leq i < \aleph_0$. Similarly define $Q = \sum_{i=0}^{\infty} \oplus Q_i$ where $Q_0 = 0$, $Q_{2i} = B_{i-1}$, $1 \leq i < \aleph_0$, and $Q_{2i+1} = C_i$, $0 \leq i < \aleph_0$. Then

P and Q are bounded Hermitian operators, and $A = P - Q$. Since P and Q are direct sums of the same operators, simply taken in a different order, they are unitarily equivalent. Hence A is a self-commutator by Proposition 6.2.

Proposition 6.5. *If A is a Hermitian operator on a Hilbert space of infinite dimension h , and $0 \notin \Lambda_h(A)$, then A is a self-commutator.*

Proof. Decompose A according to Lemma 6.3 and then apply Lemma 6.4.

Corollary 6.6. *Every Hermitian operator A with large kernel (i.e., the dimension of the kernel of A is equal to the dimension h of the space on which A acts) is a self-commutator.*

Corollary 6.7. *Let $\mathcal{H}$ be a Hilbert space of dimension h . Every Hermitian h -compact operator is a self-commutator.*

Proof. By Theorem 4.3, all such operators have heavy spectrum equal to $\{0\}$.

Proposition 6.8. *If A is a Hermitian operator on a Hilbert space $\mathcal{H}$ of infinite dimension h , and if $\Lambda_h(A)$ has both a positive and a negative element, then A is a self-commutator.*

Proof. Since A is Hermitian we may use the spectral theorem to obtain the decomposition $A = P \oplus N$, where P and $-N$ are positive operators. Let λ_+ and λ_- denote the positive and negative elements respectively, which are contained in $\Lambda_h(A)$ by hypothesis. According to Theorem 4.1, both P and N act on subspaces of dimension h and $\lambda_+ \in \Lambda_h(P)$ and $\lambda_- \in \Lambda_h(N)$.

Since $\lambda_+ \in \Lambda_h(P)$ we may use Lemma 6.3 to conclude that P has a decomposition, $P = \sum_{i=0}^{\infty} \oplus P_i$ such that all the P_i act on a space of dimension h and further $\|P_i - \lambda_+\| \leq 2^{-i} \|P - \lambda_+\|$, $0 \leq i < \aleph_0$. Similarly N admits a decomposition, $N = \sum_{i=0}^{\infty} \oplus N_i$ such that all the N_i act on a space of dimension h and further $\|N_i - \lambda_-\| \leq 2^{-i} \|N - \lambda_-\|$, $0 \leq i < \aleph_0$.

Next choose a sequence $\lambda_0, \lambda_1, \lambda_2, \dots$, such that each term is either λ_+ or λ_- and such that there is a bound $b > 0$ such that

$$\left| \sum_{i=0}^n \lambda_{2i} \right| < b \quad \text{and} \quad \left| \sum_{i=0}^n \lambda_{2i+1} \right| < b$$

for every n , $0 \leq n < \aleph_0$.

We next choose a sequence $\{A_i\}$ of operators from the two lists $(P_0, P_1, P_2, \dots)$ and $(N_0, N_1, N_2, \dots)$ in the following way. Whenever $\lambda_i = \lambda_+$, choose A_i equal to the next unused P_j . Whenever $\lambda_i = \lambda_-$, choose A_i equal to the next unused N_j . Thus $\sum_{i=0}^{\infty} \oplus A_i$ is simply a reordering of the direct sum

$$\left(\sum_{i=0}^{\infty} \oplus P_i \right) \oplus \left(\sum_{i=0}^{\infty} \oplus N_i \right) = P \oplus N = A$$

and hence is clearly unitarily equivalent to A . Further, by an appropriate unitary equivalence, we may consider all the operators A_i as acting on the same

space $\mathcal{K}$ of dimension h . Finally

$$\begin{aligned} \left\| \sum_{i=0}^n A_{2i} \right\| &\leq \sum_{i=0}^n \|A_{2i} - \lambda_{2i}\| + \left\| \sum_{i=0}^n \lambda_{2i} \right\| \\ &\leq \sum_{i=0}^{2n} \|P_i - \lambda_+\| + \sum_{i=0}^{2n} \|N_i - \lambda_-\| + b \\ &\leq 2 \|P - \lambda_+\| + 2 \|N - \lambda_-\| + b. \end{aligned}$$

Similarly

$$\left\| \sum_{i=0}^n A_{2i+1} \right\| \leq 2 \|P - \lambda_+\| + 2 \|N - \lambda_-\| + b.$$

By the criteria of Lemma 6.4, we conclude that A is a self-commutator.

Theorem 6.9. *Let A be an operator on an infinite dimensional Hilbert space. Then A is a self-commutator if and only if its heavy spectrum contains at least one non-negative element and at least one non-positive element.*

Proof. We have simply combined Propositions 6.1, 6.5 and 6.8 into one statement.

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