Morse Theory of Double Normals of Immersions

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Introduction. Let $f:M^m \rightarrow E^n$ be a C^k ($k \geq 2$) immersion of a connected compact differentiable manifold into Euclidean n -space. A double normal is a line segment between two distinct points $f(x)$ and $f(y)$ of the immersion such that it is perpendicular to the tangent planes at $f(x)$ and $f(y)$. In this paper we give a lower bound for the number of double normals of immersions which are generic in a certain sense. We show that this bound depends on the betti-numbers of the homology groups of $(M \times M, \Delta \cup \Delta_f)$ where Δ is the diagonal of $M \times M$ and Δ_f is the double locus of f taken as a submanifold of $M \times M$. The general non-generic problem for convex bodies in E^n was studied by N. H. Kuiper [3] who showed that the lower bound is n . The non-generic special C^2 case for convex bodies was known by some mathematicians, *e.g.* T. Ganea, to be contained essentially in the work of L. Lusternik, L. Schnirelman and M. Morse.

The problem of double normals of imbeddings of one Riemannian manifold in another Riemannian manifold has been dealt with by M. Morse [5] and A. Riede [6]. Their approach from the point of view of symmetric squares and path spaces makes explicit computation of lower bounds difficult. Here, we give a stronger approach for the special case of generic immersions in E^n which enables easy computation of these bounds. The first section defines the generic class of immersions, the second gives the main theorem, and the third gives many corollaries and examples.

1. We define an immersion $f:M^m \rightarrow E^n$ to be *D-generic* (*distance-generic*) if the function

$$D_f(x, x') = \|f(x) - f(x')\|^2$$

has the following properties:

- 1) $\Delta = \{(x, x') \in M \times M \mid x = x'\}$ and

$$\Delta_f = \{(x, x') \in M \times M \mid f(x) = f(x'), x \neq x'\}$$

are non-degenerate critical manifolds of D_f .

2) All critical points of D_f on $M \times M - \Delta \cup \Delta_f$ are non-degenerate.

We recall that a manifold $V \subset N^n$ is a non-degenerate critical manifold of a function g on N if there are local coordinates $(x_1, \dots, x_v, x_{v+1}, \dots, x_n)$ such that $V = \{x_{v+1} = \dots = x_n = 0\}$ and $g = \text{const} + x_{v+1}^2 + \dots + x_{v+s}^2 - x_{v+s+1}^2 - \dots - x_n^2$. (In our case here the minus signs will not occur.)

Remark 1. Clearly, if f is an immersion Δ is a non-degenerate critical manifold of D_f . Moreover, if f is a self-transversal immersion, Δ_f is also a non-degenerate critical manifold of D_f .

Remark 2. The critical points of D_f outside $\Delta \cup \Delta_f$ are just the double normals.

Proposition. *The set of D -generic immersions is open and dense in the space of all C^k ($k \geq 2$) immersions.*

Proof. Self-transversal immersions are known to be open and dense. Hence, we have only to deal with property 2, and, in fact, with property 2 outside an ϵ -neighborhood of $\Delta \cup \Delta_f$ in $M \times M$, ϵ sufficiently small. We define the following spaces and maps.

- 1) $N_f(M) = \{(x, v) \in M \times E^n \mid f(x) - v \perp f_*(T_x(M)), |f(x) - v| \geq \epsilon\}$,
- 2) $N(f): N_f(M) \rightarrow E^n$ is the map defined by $N(f)(x, v) = v$,
- 3) $G_m(E^n)$ is the bundle over E^n whose fibre over a $p \in E^n$ consists of all non-oriented m -planes in $T_p(E^n)$ and whose projection map shall be denoted π_m ,
- 4) $G_m(N_f(M)) = \{(x, v, W) \in M \times E^n \times G_m(E^n) \mid (x, v) \in N_f(M), \pi_m(W) = v, f(x) - v \perp W\}$,
- 5) $G_m(f): G_m(N_f(M)) \rightarrow G_m(E^n)$ is the natural map which sends (x, v, W) into W , and finally,
- 6) $\tilde{d}f: M \rightarrow G_m(E^n)$ is the map which assigns to each point $x \in M$ the tangent plane $f_*(T_x(M))$ to $f(M)$ at $f(x)$.

We observe that if $\tilde{d}f(x) = G_m(f)(x', v, W)$, then (x, x') is a critical point of D_f , i.e., it is a double normal.

We wish to show that under a small perturbation $\tilde{d}f$ and $G_m(f)$ may be made transversal. To this end choose open sets U_1, U'_1 such that $U_1 \cap U'_1 = \emptyset$ and choose closed subsets $V_1 \subset U_1, V'_1 \subset U'_1$. We want to perturb f slightly in U_1 to obtain an f' with the property that $\tilde{d}f'|_{V_1}$ is transversal with respect to $G_m(f')|_{G_m(N_{f'}(V_1))}$. By Abraham and Robbin [1] this can be done by an arbitrary C^k ($k \geq 2$) small perturbation, and, furthermore, once they are transversal, they remain transversal under a sufficiently small C^2 perturbation.

Now, consider a finite covering of $M \times M - \epsilon$ -neighborhood of $\Delta \cup \Delta_f$ of the form (U_i, U'_i) with $U_i \cap U'_i = \emptyset$ and with closed $V_i \subset U_i, V'_i \subset U'_i$ such that (V_i, V'_i) already cover $M \times M - \epsilon$ -neighborhood of $\Delta \cup \Delta_f$. We

obtain the required transversality by applying the above procedure to all pairs $(V_i \subset U_i, V'_i \subset U'_i)$.

Since $\dim G_m(E^n) = m + \dim G_m(N_f(M))$ and $\dim M = m$, the set

$$\{(x, (x', v, W)) \in M \times G_m(N_f(M)) \mid \tilde{d}f(x) = G_m(f)(x', v, W)\}$$

is a closed manifold of dimension zero, *i.e.*, a finite set of points, because $M \times M$ is compact. Thus the double normals are finite in number and hence isolated. Finally, we make clear why they are non-degenerate.

Suppose for an immersion f the above transversality holds. Then, if f' is C^2 sufficiently close to f , the number of critical points of D_f and $D_{f'}$ outside their respective non-degenerate critical manifolds $\Delta \cup \Delta_f, \Delta \cup \Delta_{f'}$ are the same. But if an (x, x') were a degenerate critical point of D_f , then by an arbitrary C^2 small perturbation of f near x and x' , we would produce a different number of critical points of $D_{f'}$ near (x, x') . This is a contradiction. Hence, (x, x') was a non-degenerate critical point. This concludes the proof of the proposition.

2. Let f be a D -generic C^k immersion of a connected compact differentiable manifold M^m into Euclidean n -space. To each double normal, connecting, say, x and x' , there correspond two critical points of D_f , namely (x, x') and (x', x) . The indices of these two critical points are the same because D_f is invariant under the involution which interchanges the factors of $M \times M$. The *index of the double normal* is defined to be equal to the index of its corresponding critical points. We observe that if c_i is the number of double normals of index i , then D_f has $2c_i$ critical points of index i .

Theorem. Let b_i be the i^{th} betti number of $(M \times M, \Delta \cup \Delta_f)$ and let c_i be the number of double normals of index i . Then

$$2 \sum_{i=0}^i (-1)^i c_{i-i} \geq \sum_{i=0}^i (-1)^i b_{i-i}, \quad i = 0, \dots, 2m.$$

For $i = 2m$, equality holds.

Corollary 1. The minimum number of double normals is given by

$$2 \sum_{i=0}^{2m} \left[\sum_{j=0}^i (-1)^j b_{i-j}/2 \right] - \left[\sum_{j=0}^{2m} (-1)^j b_{2m-j}/2 \right],$$

where $[n/2]$ is the smallest integer $\geq n/2$, n some integer.

Proof of the Theorem. If ϵ is chosen suitably small, then $D_f^{-1}[0, \epsilon]$ is a tubular neighborhood V of $\Delta \cup \Delta_f$ and furthermore has a smooth boundary, ∂V . Hence, by excision

$$H_*(M \times M - \text{int } V, \partial V) \simeq H_*(M \times M, V),$$

where $\text{int } V$ denotes the interior of V . But V can be contracted to $\Delta \cup \Delta_f$. Hence,

$$H_*(M \times M, V) \simeq H_*(M \times M, \Delta \cup \Delta_f).$$

Now D_f is a Morse function on $M \times M - \text{int } V$; it is constant and has no critical points on ∂V . An application of Morse Theory [4] generalized to manifolds with boundary, on which the function is constant and has no critical points, gives the statement of the theorem. The factor of two appears on the left-hand side of the inequality since each double normal is counted twice, once for (x, x') and once for (x', x) .

Proof of the Corollary. Clearly, the minimum number of critical points of index zero occurs when $2c_0 = b_0$, b_0 even, or when $2c_0 = b_0 + 1$, b_0 odd. Hence,

$$(c_0)_{\min} = \left\lceil \frac{b_0}{2} \right\rceil.$$

The minimum number of critical points (or double normals) of index one is given as follows. First, c_0 must be $(c_0)_{\min}$. Hence, from the inequality of the theorem, for c_1 to be a minimum, $2c_1$ must be as close as possible to $2(c_0)_{\min} + b_1 - b_0$. Therefore,

$$(c_1)_{\min} = (c_0)_{\min} + \left\lceil \frac{b_1 - b_0}{2} \right\rceil.$$

A similar analysis yields

$$(c_2)_{\min} = \left\lceil \frac{b_1 - b_0}{2} \right\rceil + \left\lceil \frac{b_2 - b_1 + b_0}{2} \right\rceil,$$

and, in general,

$$(c_i)_{\min} = \left\lceil \sum_{j=0}^{i-1} (-1)^j b_{i-1-j} / 2 \right\rceil + \left\lceil \sum_{j=0}^i (-1)^j b_{i-j} / 2 \right\rceil.$$

It is easily seen that if we have a sequence $c_0, \dots, c_{2m}$ satisfying the inequality of the theorem for which $\sum_i c_i$ is minimal, then $c_i = (c_i)_{\min}$. The minimum number of double normals is thus given by $\sum_{i=0}^{2m} (c_i)_{\min}$ and so the corollary is proved.

3. Examples. From this point on, all imbeddings or immersions shall be C^k ($k \geq 2$) and D -generic.

Corollary 2. Let $f: M^m \rightarrow E^n$ be a D -generic imbedding. Then, if d_i are the betti numbers of M^m , the minimum number of double normals of f is given by

$$2 \sum_{i=0}^{2m} \left[\sum_{j=0}^i (-1)^j \left(\sum_{k=1}^{i-j} d_k \cdot d_{i-j-k} \right) / 2 \right] - \left[\sum_{j=0}^{2m} (-1)^j \left(\sum_{k=1}^{2m-j} d_k \cdot d_{2m-j-k} \right) / 2 \right].$$

This corollary follows from the fact that for an imbedding $\Delta \simeq M$, $\Delta_f = \phi$ and hence

$$H_*(M \times M, \Delta \cup \Delta_f) \simeq H_*(M \times M, M),$$

and from the fact that the betti numbers of $(M \times M, M)$ are just the betti numbers of $M \times M$ minus the betti numbers of M , i.e.

$$\beta_i(M \times M, M) = \beta_i(M \times M) - \beta_i(M) = \sum_{k=1}^i d_k \cdot d_{i-k},$$

where the symbol β_i is used to denote the respective i^{th} betti numbers of $(M \times M, M)$, $M \times M$, and M .

We have also used the fact that M is connected and hence $d_0 = 1$.

From this point on we use the symbol $(DN)_{\min}$ to denote the minimum number of double normals.

Corollary 3a. *Let $f: M^2 \rightarrow E^n$ be a D -generic imbedding of an orientable connected compact surface. Then*

$$(DN)_{\min} = 2g^2 + 3g + 3,$$

where g is the genus of M^2 .

The proof follows from the fact that, for M , $d_0 = d_2 = 0$ and $d_1 = 2g$. A simple computation gives the result. As examples, we see that any C^k imbedding of the two-sphere must have at least three double normals and any C^k imbedding of the torus must have at least eight. The ellipsoid with three unequal axes is an example of an imbedding of a sphere with exactly three double normals.

Corollary 3b. *Let $f: P^2 \rightarrow E^n$ be a D -generic imbedding of the projective plane into E^n for n large enough. Then*

$$(DN)_{\min} = 6.$$

This follows from the fact that the Z_2 -betti numbers of $(P^2 \times P^2, P^2)$ are $b_0 = 0$, $b_1 = 1$, $b_2 = 2$, $b_3 = 2$, $b_4 = 1$ and Corollary 1. Hence, $(c_0)_{\min} = 0$, $(c_1)_{\min} = 1$, $(c_2)_{\min} = 2$, $(c_3)_{\min} = 2$, $(c_4)_{\min} = 1$, and $\sum_i (c_i)_{\min} = 6$.

Corollary 4. *Let $f: M^3 \rightarrow E^n$ be a D -generic imbedding of an orientable connected compact 3-manifold. Then*

$$(DN)_{\min} = 2d_1^2 + 3d_1 + 4 \quad \text{if } d_1 \text{ is even};$$

$$(DN)_{\min} = 2d_1^2 + 3d_1 + 5 \quad \text{if } d_1 \text{ is odd}.$$

The proof follows from the fact that for such a manifold the first and second betti numbers are equal. A computation gives the result. Thus, the three-sphere has at least 4 double normals and the three-torus has at least 32.

Corollary 5. *Let $f: S^{n-1} \rightarrow E^n$ be a D -generic imbedding of the $(n - 1)$ -sphere. Then*

$$(DN)_{\min} = n.$$

Corollary 6. *Let $f: T^{n-1} \rightarrow E^n$ be a D -generic imbedding of the $(n - 1)$ -torus.*

Then

$$(DN)_{\min} = 2^{2n-3}.$$

These two corollaries follow from easy computations. We observe that Corollary 5 recovers the result of Kuiper [3] in the generic case.

Next, we consider the case where $f: M^m \rightarrow E^{2m}$ is a D -generic immersion of M^m into E^{2m} . Here, the intersection points are isolated. Suppose there are r points of self-intersection. Then, Δ_f consists of $2r$ points in $M \times M$. In this case we have, denoting the betti numbers of $(M \times M, M)$ by β_i ,

$$b_0(M \times M, \Delta \cup \Delta_f) = \beta_0(M \times M, M) = 0.$$

$$b_1(M \times M, \Delta \cup \Delta_f) = \beta_1(M \times M, M) + 2r.$$

$$b_i(M \times M, \Delta \cup \Delta_f) = \beta_i(M \times M, M), i \geq 2.$$

This example illustrates the fact that as the number of self-intersections increases, the minimum number of double normals increases.

Corollary 7. *Let $f: M^1 \rightarrow E^2$ be a D -generic immersion of a curve into the plane with r self-intersections. Then*

$$(DN)_{\min} = 2 + r.$$

This result follows from Corollary 1, where we use the fact that $b_0 = 0$, $b_1 = 1 + 2r$, $b_2 = 1$ for $(M \times M, \Delta \cup \Delta_f)$. Corollary 7 was first observed by B. Halpern in [2].

As our last example we look at the following case. Let M^m be a connected compact differentiable manifold and let $A \subset M$ be a connected compact submanifold. Let $f: M \rightarrow E^n$ be an immersion of M into Euclidean n -space. We define

$$\Delta_A = \{(a, m) \in A \times M \mid a = m\} \quad \text{and}$$

$$(\Delta_f)_A = \{(a, m) \in A \times M \mid f(a) = f(m)\}.$$

A double normal of the pair (A, M) is a line segment between two distinct points $f(a)$ and $f(x)$, $a \in A$, $x \in M$ such that it is perpendicular to the tangent planes of $f(A)$ at $f(a)$ and of $f(M)$ at $f(x)$. A procedure analogous to the main case shows that the number of such double normals is related to the betti numbers of $(A \times M, \Delta_A \cup (\Delta_f)_A)$. Here, of course, each double normal is counted only once.

As an example, let A be a curve on a surface M and let $f: M \rightarrow E^3$ be an immersion into Euclidean 3-space. Let r be the number of points in $(\Delta_f)_A$. (This number r is finite for dimensional reasons.) Then the betti numbers of $(A \times M, \Delta \cup (\Delta_f)_A)$ are $b_0 = 0$, $b_1 = d_1 + r$, $b_2 = 1 + d_1$, $b_3 = 1$, where d_1 is the first betti number of M . Hence, the minimum number of double normals,

$$(DN)_{\min} = 2 + 2d_1 + r.$$

Thus, for an immersion of the sphere and a curve on the sphere with the above properties

$$(DN)_{\min} = 2 + r.$$

It is interesting to compare this with Corollary 7.

In closing we mention that the procedures of this paper carry over to immersions into simply connected complete Riemannian manifolds of negative curvature giving analogous results.

REFERENCES

- [1] R. ABRAHAM & J. ROBBIN, *Transversal Mappings and Flows*, Benjamin Press, 1967.
- [2] B. HALPERN, *Global theorems for plane curves*, Bull. Amer. Math. Soc. **76** (1970), 96–100.
- [3] N. H. KUIPER, *Double normals of convex bodies*, Israel J. Math. **2**(1964), 71–80.
- [4] J. MILNOR, *Morse theory*, Ann. Math. Studies, No. 51(1963).
- [5] M. MORSE & S. S. CAIRNS, *Critical Point Theory in Global Analysis and Differential Topology*, Academic Press, New York, 1969.
- [6] A. RIEDE, *Lotgeodätische*, Archiv der Mathematik **19**(1968), 103–112.

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